The Annals of Probability
1985, Vol. 13, No. 2, 492–518

EXCURSION LAWS OF MARKOV PROCESSES IN CLASSICAL DUALITY[1]

BY H. KASPI

Technion, Haifa, Israel

We study excursion laws of two Markov processes X, $\hat{X}$ in duality, out of closed, homogeneous optional sets M, $\hat{M}$ generated by a pair of dual terminal times R and $\hat{R}$. The duality assumptions enable one to compute the laws of (R, X_R) using the pair of exit systems of the two processes. With this, one is able to compute the conditional laws of the excursions, given the boundary process. It turns out that these depend only on the values of the boundary process at the beginning and end of the excursions. We obtain for all x, y $(x \neq y)$ the laws $p^{x,y}$ of the excursions conditioned to start at x and end at y. Under these laws, the excursion process is a homogeneous Markov process. It's transition laws are computed. We use the results above to treat excursions that straddle perfect terminal times.

1. Introduction. During recent years a considerable amount of work has been done on excursions of Markov processes under the assumption of duality.

In [11] and [12], Getoor and Sharpe discuss the problem of excursions of dual processes under the assumption of existence of dual transition densities. This assumption is stronger than the "classical" duality assumptions, in which all one assumes is the existence of dual potential kernel densities. This assumption was used in [11] to compute the laws $P^{x,\ell,y}$ of the excursions known to start at a point x, end at a point y and last for ℓ units of time. These results were used in [11] to obtain the laws of excursions that straddle perfect terminal times, and in [12] to discuss the laws of excursions that straddle general stopping times (for results in this direction without any duality assumptions see also Pitman [24]).

In another direction the laws $P^{x,\ell,y}$ were used to obtain some duality relations between the excursion laws of a process, and the laws of the reversed excursions of its dual [11]. This last result was later obtained without the assumption on existence of dual densities; in [17] using a representation of the restriction of the duality measure ξ to the recurrent classes of the process, and in [23] using the stationary auxiliary process associated with the pair of dual processes.

This last result on the reversal of excursions leads one to believe that although the explicit representation of the laws of excursions whose length is ℓ is possible only if one assumes the existence of dual densities, one can say more about the laws of excursions that start at a point x and end at y, without these strong assumptions. Results in this direction are the main purpose of this work.

This work is organized as follows. Section 2 is devoted to the notation and

Received August 1983; revised June 1984.

[1] Research supported in part by the Technion vice president for research through the L. Edelstein Fund.

AMS 1980 *subject classifications.* Primary 60J60; secondary 60J45.

Key words and phrases. Markov processes in duality, exit systems, excursion laws.

some preliminaries. We recall some results and notation from Getoor and Sharpe [11], Maisonneuve [19] and Jacobs [15] that will be used in the sequel. We shall also set the notation for the use of Mitro's [21] auxiliary stationary process, but we shall not elaborate on its construction. For this we refer the reader to [21], [22], [23], [13].

In Section 3 we obtain the joint Laplace transform-distribution of (R, X_R). In Section 4 we obtain the laws of excursions that "start at x and end at y" for $x \neq y$. We show that under these laws the excursions known to start at x and end at y are homogeneous strong Markov processes. The meaning of excursions that start at x and end at y, as well as information outside the excursion, is different from that of [11] and will be made precise in the sequel. The main tool for this analysis is the theory of excursions via Markov additive processes as presented in [15], [16]. Section 5 is devoted to the use of the theory developed, for excursions straddling perfect terminal times and stopping times. Treatment here is very similar to that of [11] and most proofs will be omitted.

Finally, it was noticed by Getoor and Sharpe [11] that when dual densities exist, one may obtain the laws of excursions conditioned to start at x, end at y and last ℓ units of time via h-path transform on the time-space process stopped at first entrance to M. The same is true in our situation. This fact was noticed in a recent paper by Fitzsimmons [8], where h-path transforms are used to define the laws of excursions known to start at x and end at y.

2. Notations and preliminaries. Let X, $\hat{X}$ be two standard processes in duality relative to a σ-finite measure ξ on their common state space $(E, \mathscr{E})$. We assume that E is Lusinian (homeomorphic to a Borel measurable subset of a compact metric space). In [2] E is assumed LCCB but all arguments given there that will be needed in the sequel will apply for E Lusinian. Here $\mathscr{E}$ are the Borel sets of E. Let $\mathscr{E}^*$ be the universal completion of $\mathscr{E}$. As usual, let Δ be a point not in E, and $\mathscr{E}_\Delta$, $\mathscr{E}^*_\Delta$ the corresponding σ-algebras generated by $\mathscr{E} \vee \{\Delta\}$, $\mathscr{E}^* \vee \{\Delta\}$ respectively. Let $b\mathscr{E}$, $\mathscr{E}_+$, $b\mathscr{E}_+$ denote the $\mathscr{E}$ measurable functions that are bounded, positive, positive and bounded respectively. We shall use similar notations when other σ-algebras are involved. We use for simplicity the canonical realization of the process. Namely Ω is the space of all functions from $\mathbb{R}_+$ into E_Δ that are right continuous, have left limits and admit Δ as a trap. We designate the coordinate random variable by $X_t(\omega) = \omega_t$. We let $\mathscr{F}^0_t$ be the natural σ-algebras of X and $\mathscr{F}^0 = \bigvee_{t \geq 0} \mathscr{F}^0_t$. Let $\mathscr{F}^*$ be the universal completion of $\mathscr{F}^0$. We now suppose that we have two Markov processes X and $\hat{X}$ with semigroups (P_t) and $(\hat{P}_t)$ defined canonically on Ω. For each $x \in E$, the semigroup (P_t) generates a measure P^x on $(\Omega, \mathscr{F}^0)$ and the canonical process $X_t(\omega)$ on $(\Omega, \mathscr{F}^0, P^x)$ is Markov. The same is true for $(\hat{P}_t)$. We let $\mathscr{F}^\mu$ be the completion of $\mathscr{F}^0$ with respect to P^μ, $\mathscr{F} = \bigcap_\mu \mathscr{F}^\mu$ where the intersection is over all probability measures on $(E, \mathscr{E})$ and $\mathscr{F}_t$ is a similar completion of $\mathscr{F}^0_{t+}$ in $\mathscr{F}$. We shall denote by $\hat{\mathscr{F}}$, $\hat{\mathscr{F}}_t$ the corresponding completion with respect to $\hat{P}$ instead of P. Note that we distinguish here the dual process only by its laws $(\hat{P}^\mu)$. We assume that $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ and $(\Omega, \hat{\mathscr{F}}, \hat{\mathscr{F}}_t, \hat{X}_t, \theta_t, \hat{P}^x)$ are standard processes in (classical) duality relative to the σ-finite measure ξ. By classical duality we mean

$$(2.1)\quad \begin{array}{ll} \text{(i)} & U^\alpha(x,\cdot) \ll \xi(\cdot) \text{ for all } x \in E \\ \text{(ii)} & \hat{U}^\alpha(x,\cdot) \ll \xi(\cdot) \text{ for all } x \in E \\ \text{(iii)} & \int \xi(dy)g(y)\hat{U}^\alpha f(y) = \int \xi(dy)f(y)U^\alpha g(y) \text{ for all } f,\ g \in \mathscr{E}_+ \text{ and} \\ & \alpha > 0. \ (U^\alpha \text{ and } \hat{U}^\alpha \text{ are the resolvents of } X \text{ and } \hat{X} \text{ respectively}). \end{array}$$

Let $M \subset \mathbb{R}_+ \times \Omega$. We write $M(\omega)$ for the ω section of M, and when no ambiguity is possible, we write M for $M(\omega)$. We assume that $M(\omega)$ is closed in $\mathbb{R}_+$, optional and $M(\theta_t\omega) \cap (0,\infty) = (M(\omega) - t) \cap (0,\infty)$ for all t, a.s. Let

$$(2.2)\qquad R = \inf\{t > 0: t \in M\} \quad R_t = R(\theta_t); \quad D_t = t + R_t$$

Let $F = \{x: P^x(R = 0) = 1\}$ the points regular for R.

Let M_ℓ be the set of left endpoints of intervals contiguous to M and M_r the set of right endpoints of those intervals. We assume that $M \subset (0, \zeta)$. (Here, as usual, $\zeta = \inf\{t: X_t = \Delta\}$). The following was proved in [11].

THEOREM 2.3. *Let M be a closed, homogeneous optional set contained in $(0, \zeta)$. Then there exists a Borel set $\Gamma \in \mathscr{E} \times \mathscr{E}$ such that M and $\{(X_-, X) \in \Gamma\} \equiv \{(t,\omega): t > 0, (X_{t-}(\omega), X_t(\omega)) \in \Gamma\}$ are indistinguishable. If $\hat{M} \equiv \{(t,\omega): t > 0, (\hat{X}_{t-}(\omega), \hat{X}_t(\omega)) \in \hat{\Gamma}\}$, where $\hat{\Gamma} = \{(x,y): (y,x) \in \Gamma\}$, then $\hat{M}$ is closed in $(0, \hat{\zeta})$. $\hat{R} \equiv \inf\{t > 0: t \in \hat{M}\}$, and R are dual exact terminal times.*

We next cite the main result of [19] on the existence of the optional exit system. We use the notations and statement of [11].

THEOREM 2.4. *There exists an additive functional (AF) B, of X with bounded 1-potential and a kernel $^*P$ from $(E, \mathscr{E})$ into $(\Omega, \mathscr{F}^0)$ such that for all positive optional U (with respect to $(\mathscr{F}_t)$) and all $(\mathscr{R}_+ \times \mathscr{F}^*)_+$ measurable f, and $x \in E$,*

$$(2.5)\qquad E^x \textstyle\sum_{s \in M_\ell; 0<s<\zeta} U_s f_s(\theta_s) = E^x \displaystyle\int_{\mathbb{R}_+} U_s {}^*P^{X_s}(f_s)\, dB_s.$$

For each $x \in E$, $^*P^x$ is σ-finite, $^*P^x(\zeta = 0) = 0$, $^*P^x(R = 0) = 0$, and $^*P^x(1 - e^{-R}) = 1$, B a.e. If $x \notin F$, then $^*P^x(\cdot) = P^x(\cdot)/E^x(1 - e^{-R})$. The continuous part of B is carried by F, and the discontinuous part by $E(R) = E - F$. For each x the process $X = \{X_t: t > 0\}$ is Markov with semigroup (P_t) relative to $^*P^x$; that is, if T is an $\mathscr{F}^0_{t+}$ stopping time with $^*P^x(T = 0) = 0$, then for $H \in b\mathscr{F}^0_{T+}$, $H \geq 0$ and $J \in b\mathscr{F}$, $J \geq 0$

$$(2.6)\qquad {}^*P^x(H \cdot J \circ \theta_T;\ T < \infty) = {}^*P^x(H \cdot E^{X_T}(J);\ T < \infty).$$

We use $(\hat{B}, {}^*\hat{P}^x)$ for the corresponding objects of the dual process. Note that $\hat{B}$ is not the dual additive functional of B as defined in [22], [25]. Let

$$(2.7)\qquad {}^*P^x_t(A) = {}^*P^x(R > t;\ X_t \in A),$$

and $^*\hat{P}^x_t(A)$ be the corresponding object of the dual process.

It was shown in [11], [20] that under the duality assumptions there exists a predictable exit system. Namely, there exists a predictable additive functional C with bounded 1-potential and a kernel $^*Q$ from $(E, \mathscr{E})$ into $(\Omega, \mathscr{F}^0)$ such that for all positive predictable U and all $(\mathscr{R}_+ \times \mathscr{F}^*)_+$ measurable f, and $x \in E$,

$$E^x \textstyle\sum_{s\in M_\ell; 0<s<\zeta} U_s f_s(\theta_s) = E^x \int_{\mathbb{R}_+} U_s \, {}^*Q^{X_s}(f_s) \, dC_s \tag{2.8}$$

The rest of Theorem (2.4) remains valid with $(C, {}^*Q^x)$ replacing $(B, {}^*P^x)$ with the only difference that the continuous part of C need not be carried by F. Let

$${}^*Q_t^x(A) = {}^*Q^x(R > t;\ X_t \in A) \tag{2.9}$$

and $\hat{C}$, ${}^*\hat{Q}^x$, ${}^*\hat{Q}_t^x$ be the corresponding objects for the dual process.

In order to apply the theory of Markov additive process to this situation (as is our intention) one has to construct a local time and a boundary process for the regenerative system M. This was done in considerable detail in [16]; we shall sketch here briefly the idea. We take M_1 to be the perfect kernel of M. M_1 is closed, homogeneous and optional. $M - M_1$ is a countable union of graphs of stopping times. We let $\bar{R}$, $\bar{D}_t$, $\bar{M}_\ell$, $\bar{M}_r$, $\bar{F}$ be defined for M_1 in the same manner that R, D_t, M_ℓ, M_r, F have been defined for M. There exists a natural additive functional $\bar{L}$ such that

$$\begin{aligned} E^x(e^{-\bar{R}}) &= E^x \int_{\mathbb{R}_+} \exp(-t) 1_{M_1}(t)\, dt + E^x \textstyle\sum_{0<s\in \bar{M}_\ell} e^{-s}(1 - \exp(-\bar{R}(\theta_s))) \\ &= E^x \int_{\mathbb{R}_+} e^{-t}\, d\bar{L}_t \end{aligned} \tag{2.10}$$

(the local time of equilibrium of order 1 at M_1). The set of increase of $\bar{L}$ is equal to M_1. $\bar{L}_t$ is continuous if, and only if, $\bar{D}_t$ is quasi-left continuous. Note that $\bar{L}$ jumps at predictable exits from M_1. Let (A, K) be the predictable exit system associated with M_1. It follows from (2.5) that A^c (the continuous part of the NAF A) is absolutely continuous with respect to $\bar{L}^c$ (the continuous part of $\bar{L}$). Further, the jumps of $\bar{L}$ at the predictable exit points $\{T_n\}$ are equal to $\bar{\lambda}(X_{T_n})$, where

$$\bar{\lambda}(x) = E^x(1 - e^{-\bar{R}}). \tag{2.11}$$

Combining all these facts, it follows that there exists a σ-finite kernel $^*H$ from $(E, \mathscr{E})$ into $(\Omega, \mathscr{F}^0)$ such that for all positive predictable U, $f \in (\mathscr{R}_+ \times \mathscr{F}^*)_+$ and $x \in E$,

$$E^x \textstyle\sum_{s\in \bar{M}_\ell; 0<s<\zeta} U_s f_s(\theta_s) = E^x \int_{\mathbb{R}_+} U_s \, {}^*H^{X_s}(f_s)\, d\bar{L}_s \tag{2.12}$$

Note that the continuous part of $\bar{L}$ is concentrated on $\bar{F}$ and for $x \notin \bar{F}$, ${}^*H^x(\cdot) = P^x(\cdot)/E^x(1 - e^{-\bar{R}})$.

In order to construct a boundary process that will be a Markov process, we shall replace all the jumps of $\bar{L}$ by exponentially distributed random variables. We shall also add exponentially distributed random variables at $M - M_1$.

Let $T_0 = 0_{\{R>0\}}$ and $\{T_n\}_{n=1}^\infty$ be stopping times with disjoint graphs that exhaust the jumps of $\bar{L}$ and $M - M_1$. Let

$$\lambda(x) = E^x(1 - e^{-R}) \tag{2.13}$$

$$L_t = \bar{L}_t^c + \textstyle\sum_{T_n \le t} \lambda(X_{T_n}) I_n \tag{2.14}$$

where $\{I_n\}$ is a sequence of i.i.d. exponentially (1) distributed random variables not depending on $\mathscr{F}$. Note that if for some n, $T_n \in M - M_1$ is a right limit of points in $M - M_1$, then $X_{T_n} \in F$, and therefore $\lambda(X_{T_n}) = 0$. One has to enlarge the probability space to include $\{I_n\}$, in such a way that L_t be adapted to the enlarged history $(\mathscr{M}_t)$. The measures $\bar{P}^x$ on the enlarged space will be now $\bar{P}^x = P^x \times P$ where P is the probability measure on the space where the I_n's are defined. One extends the definition of θ_t to the enlarged space in such a way that for any $Z \in \mathscr{M} = \vee_{t>0} \mathscr{M}_t$ and any stopping time $T \in (\mathscr{M}_t)$

$$\bar{E}^x(Z \circ \theta_T \mid \mathscr{M}_t) = \bar{E}^{X_T}(Z) \tag{2.15}$$

(see [16] and Proposition (3.14) of [6]).

Let

$$\tau_s = \inf\{t > 0; L_t > s\} \quad \text{and} \tag{2.16}$$

$$Y_s = X_{\tau_s}. \tag{2.17}$$

One can show that (Y, τ) is a quasi-left continuous Markov additive process (MAP). For the exact statement and specific details see [16].

In order to get now the exit system for M, we define for $x \notin F$, ${}^*H^x(\cdot)$ on $(\Omega, \mathscr{E})$ by

$${}^*H^x(\cdot) = E^x(\cdot)/E^x(1 - e^{-R}) \tag{2.18}$$

The following was proved in [3] in a different setting. We give a proof here for the sake of completeness.

THEOREM 2.19. *For all positive $\mathscr{M}_t$ predictable U and $f \in \mathscr{F}_+^*$*

$$\bar{E}^x \textstyle\sum_{s \in M_{\swarrow}; 0 < s < \zeta} U_s f(\theta_s) = \bar{E}^x \displaystyle\int_{(0,\infty)} U_s \, {}^*H^{X_s}(f) \, dL_s$$

PROOF. Let $L_t^1 = L_t - L_0$. Then L_t^1 is an additive functional relative to the shifts θ_t. All one needs to show is that for all $f \in \mathscr{F}_+^*$

$$A_t = \textstyle\sum_{s \in M_{\swarrow}; 0 < s \le t} (1 - \exp(-R(\theta_s))) f(\theta_s)$$

and

$$B_t = \int_{(0,t]} {}^*H^{X_s}(f \cdot (1 - e^{-R})) \, dL_s$$

have the same dual predictable projection on $(\mathscr{M}_t)$. To do this we use T37 of [7] and (2.15).

$$\bar{E}^x(B_{t+s} - B_t \mid \mathscr{M}_t) = \bar{E}^x\left[\left(\int_{(0,s]} {}^*H^{X_u}(f \cdot (1 - e^{-R}))\, dL_u^1\right) \circ \theta_t \mid \mathscr{M}_t\right]$$

$$= \bar{E}^{X_t} \int_{(0,s]} {}^*H^{X_u}(f \cdot (1 - e^{-R}))\, dL_u^1.$$

Using now the definition of $\bar{P}^x = P^x \times P$, the fact that the expectation of the added exponentially distributed random variables is equal to 1 and the definition of $({}^*H^x)_{x \notin F}$, this last expression is equal to

$$E^{X_t} \int_{(0,s]} {}^*H^{X_u}(f \cdot (1 - e^{-R}))\, d\bar{L}_u^c + E^{X_t} \textstyle\sum_{0<T_n\le s} E^{X_{T_n}}((1 - e^{-R})f)$$

$$= E^{X_t} \textstyle\sum_{u \in M_{\swarrow}; 0<u\le s} (1 - \exp(-R(\theta_u)))f(\theta_u).$$

Again by definition of $\bar{P}^x$ and the fact that the additive functional A doesn't depend on $\{I_n\}$, the last expression is equal to

$$\bar{E}^{X_t} \textstyle\sum_{u \in M_{\swarrow}; 0<u\le s} (1 - \exp(-R(\theta_u)))f(\theta_u) = \bar{E}^{X_t}(A_s) = \bar{E}^x(A_{t+s} - A_t \mid \mathscr{M}_t).$$

This completes the proof.

From now on we shall abuse the notation by using P^x for $\bar{P}^x$ where no ambiguity is possible.

REMARK 2.20. Using methods similar to those used by Glover [14] one can show that Y is a right process. In the present situation, using the fact that ξ is a reference measure for X and arguments that lead to Lemma 2.4 of [1], it is easy to show that L is equivalent to an $\mathscr{M}^0$ measurable process ($\mathscr{M}^0$ the uncompleted σ-algebra generated by $\mathscr{F}^0$ and $\{I_n\}$). Using now the definition of $\bar{P}^x$ it can be easily verified that $x \to \bar{P}^x(f(Y_t))$ is $\mathscr{E}$ measurable for all $f \in \mathscr{E}_+$. This in turn implies that Y is a Borel right process. We shall denote by $\mathscr{E}^e$ the σ-algebra generated by α-excessive ($\alpha \ge 0$) functions with respect to Y. (In the present situation the σ-algebra generated by α-excessive functions with respect to X is equal to $\mathscr{E}$).

Define

$$(2.21) \qquad {}^*H_t^x(A) = {}^*H^x(R > t;\, X_t \in A).$$

We let $\hat{L}$, $\hat{\tau}$, ${}^*\hat{H}$, ${}^*\hat{H}_t$ be the corresponding objects for the dual process.

Turning now to the excursions of X outside of M. Let t be a jump time of $\tau_{\cdot}(\omega)$. Define

$$(2.22) \qquad e_s(t, \omega) = \begin{cases} X_{\tau_{t-}+s}(\omega) & s < \tau_t - \tau_{t-} \\ X_{\tau_t}(\omega) & s \ge \tau_t - \tau_{t-}. \end{cases}$$

This process takes values in V, the collection of all functions from $[0, \infty)$ into $E \cup \{\Delta\}$, that are right continuous, have left limits and are absorbed at first entrance to $M \cup [\zeta, \infty)$. We let $\mathscr{V}$ be the σ-algebra on V induced by the coordinate mappings. Let $\mathscr{H}_s$ be the right continuous complete history generated by Y, $\mathscr{L}_s$ the right continuous complete history generated by (Y, τ), and $\mathscr{G}_s$ the right continuous complete history generated by (Y, τ, e). Completions are taken with respect to $\mathscr{P} = \{P^\mu : \mu \text{ finite measure on } (E, \mathscr{E})\}$.

The following were proved in [15], under some additional assumptions and carry to our case without any changes.

THEOREM 2.23. *There exists a regular version of* $P^y(\cdot \mid \mathscr{H})$ *on* $\mathscr{G} = \bigvee_{s>0} \mathscr{G}_s$ *which is further independent of* $y \in E$.

Let $\{U_j\}$ be the jump times of τ. For each j, U_j is an $\mathscr{L}_t$ stopping time.

THEOREM 2.24. *The random functions* $\{e_{U_j}\}$ *are conditionally independent given* $\mathscr{H}$.

Let $\{l(x)\}_{x \in E}$ be the Motoo derivative of Leb$(M \cap [0, t))$ with respect to $\bar{L}_t^c$. The following is a direct consequence of the quasi-left continuity of τ, results on MAP's [4], and the results of [15], [16].

$$\tau_s = A_s + \tau_s^d + \tau_s^f \tag{2.25}$$

where

(i) $A_s = \int_0^s l(Y_u)\, du$.

(ii) τ_s^d and τ_s^f are conditionally independent given $\mathscr{H}$.

(iii) Let $P_\omega(\cdot) = P^x(\cdot \mid \mathscr{H})$, then under P_ω, τ_s^d is a pure jump, stochastically continuous process. In particular, it has no jump times that coincide with the jump times of Y.

(iv) If U_j is a jump time of τ_s^f, then $Y_{U_j} \neq Y_{U_j -}$.

Let μ be the random counting measure on $(\mathbb{R}_+ \times V, \mathscr{R}_+ \times \mathscr{V})$ so that for $a < b$, $A \in \mathscr{V}$:

$$\mu[[a, b] \times A] = \text{number of } u \in [a, b] \text{ such that } \tau_u \neq \tau_{u-} \text{ and } e(u, \omega) \in A. \tag{2.26}$$

The decomposition of τ_s in (2.24) splits μ into two random measures.

$$\mu = \mu^d + \mu^f. \tag{2.27}$$

Here μ^d counts the excursions that are due to the jumps of τ^d, whereas μ^f counts those excursions that are due to the jumps of τ^f.

Combining the results of [15] and computations carried in [16], we have the following:

THEOREM 2.28. *The random measures* μ^f *and* μ^d *are independent additive random measures over* $(\Omega, \mathscr{G}, P_\omega)$. *Further,* μ^d *is a Poisson random measure with* *mean measure* M_ω *given by*

$$M_\omega([0, t) \times \Gamma) = \int_{[0,t)} {}^*H^{Y_s(\omega)}(\Gamma \cap \{X_R = Y_s(\omega)\})\, ds. \tag{2.29}$$

As for the μ^f part, the existence of a conditional law of e_{U_j}, given Y_{U_j} and Y_{U_j-}, (where $Y_{U_j} \neq Y_{U_j-}$) was proved in [15]. We shall deal with this part in Section 4. Since this result is stated in [15] in a form which is not convenient for our use, and also for the sake of completeness, we shall prove this result again in Section 4.

We end this section with a brief description of the auxiliary stationary process, associated with the two processes X and $\hat{X}$. This is brought for notational purposes only. For a more detailed treatment of the subject we refer the reader to [21], [22], [23], [13].

Let $(\mathbf{\Omega}, \mathbf{F}^0(-\infty, \infty), Q, \mathbf{P}^x, \hat{\mathbf{P}}^x, Z)$ be the auxiliary process associated with the pair of processes X, $\hat{X}$. $\mathbf{\Omega}$, is the space of paths from $\mathbb{R}$ into $E \cup \Delta \cup \hat{\Delta} \equiv \mathbf{E}$, which admit a random birth time $\hat{\zeta}$ and a random death time ζ, that are r.c.l.l. on $(\hat{\zeta}, \zeta)$ and Z is the coordinate process. $Z_t(\omega) = \hat{\Delta}$ for $t \leq \hat{\zeta}$ and $Z_t(\omega) = \Delta$ for $t \geq \zeta$. Shift operators θ_s, killing and birthing operators $\mathbf{k}_s$ and $\hat{\mathbf{k}}_s$ are given by

$$Z_t \circ \theta_s(\omega) = Z_{t+s}(\omega) \tag{2.30}$$

$$Z_t \circ \mathbf{k}_s(\omega) = Z_t(\omega) \quad \text{for} \quad t < s,\ \Delta \quad \text{if} \quad t \geq s \tag{2.31}$$

$$Z_t \circ \hat{\mathbf{k}}_s(\omega) = Z_t(\omega) \quad \text{if} \quad t > s,\ \hat{\Delta} \quad \text{if} \quad t \leq s. \tag{2.32}$$

On $\mathbf{\Omega}$ we define σ-algebras

$$\tilde{\mathscr{F}}^0(I) = \sigma\{Z_u : u \in I\}$$

for every interval I. When completed with respect to the σ-finite measure Q, the 0 superscript is omitted. Define

$$\hat{Z}_t = Z_{(-t)-},\ \hat{\theta}_t : \mathbf{\Omega} \to \mathbf{\Omega} \quad \text{so that} \quad \hat{Z}_{s+t}(\omega) = \hat{Z}_s(\hat{\theta}_t\omega). \tag{2.33}$$

Define mappings π, $\hat{\pi}$: $\mathbf{\Omega} \to \Omega$ by

$$(\pi\omega)_t = Z_t(\omega) \quad \text{if} \quad Z_0(\omega) \in E,\ t \geq 0, \quad \text{and } \Delta \text{ otherwise.} \tag{2.34}$$

$$(\hat{\pi}\omega)_t = \hat{Z}_t(\omega) \quad \text{if} \quad \hat{Z}_0(\omega) \in E,\ t \geq 0, \quad \text{and } \hat{\Delta} \text{ otherwise.} \tag{2.35}$$

We let $\hat{\mathbf{P}}^x$ and $\hat{\mathbf{P}}^x$ be probability measures on $\tilde{\mathscr{F}}^0(0, \infty)$ and $\tilde{\mathscr{F}}^0(-\infty, 0)$, respectively, that embed P^x and $\hat{P}^x$. Namely for any $F \in \mathscr{F}^0$

$$\mathbf{P}^x(F \circ \pi) = P^x(F) \quad \text{and similarly for} \quad \hat{\mathbf{P}}^x. \tag{2.36}$$

The measure Q was shown in [21] to be characterized by

$$Q(F \circ \pi \circ \theta_t \cdot \hat{F} \circ \hat{\pi} \circ \theta_t) = \int \xi(dx) P^x(F) \hat{P}^x(\hat{F}) \tag{2.37}$$

for all positive F, $\hat{F} \in \mathscr{F}^0$, and $t \in \mathbb{R}$. It is clear from (2.37) that Q is shift invariant, i.e., $\theta_u Q = Q$ for all $u \in \mathbb{R}$. We let $\mathscr{O}$, the optional σ-algebra, be the σ-algebra on $\Omega \times \mathbb{R}$ generated by processes adapted to $\tilde{\mathscr{F}}(-\infty,t]$ which are Q a.e. right continuous. $\hat{\mathscr{O}}$ the co-optional σ-algebra generated by processes adapted to $\tilde{\mathscr{F}}[s, \infty)$ which are Q a.e. left continuous. $\mathscr{P}$ the predictable σ-algebra generated by processes adapted to $\tilde{\mathscr{F}}(-\infty, s]$ that are Q a.e. left continuous. $\hat{\mathscr{P}}$ the co-predictable σ-algebra generated by processes adapted to $\tilde{\mathscr{F}}[s, \infty)$ that are Q a.e. right continuous.

Let A be an AF of X, one may regard A as a homogeneous random measure κ on $(\mathbb{R}_+, \mathscr{R}_+)$. The homogeneous embedding $\mathbf{A}$ of A is defined by

$$\mathbf{A}(\omega, C) = \sup_{s\in\mathbb{R}} \kappa(\pi\theta_s\omega, C - s) \quad \text{for all} \quad C \in \mathscr{R}. \tag{2.38}$$

and similarly for an AF $\hat{A}$ of $\hat{X}$ with $\hat{\pi}$ replacing π in (2.38).

To define $\mathbf{M}$ for Z we let

$$\mathbf{M} = \{(t, \omega): (Z_{t-}(\omega), Z_t(\omega)) \in \Gamma\} \quad (\Gamma \text{ as defined in (2.3)}) \tag{2.39}$$

$$\mathbf{D}_t(\omega) = \inf\{s > t: s \in \mathbf{M}(\omega)\}; \ \ell_t(\omega) = \sup\{s < t: s \in \mathbf{M}(\omega)\}. \tag{2.40}$$

Let $\mathbf{M}_\ell$ and $\mathbf{M}_r$ be the set of left and right endpoints of intervals contiguous to $\mathbf{M}$ contained in $(\hat{\zeta}, \zeta)$. For Z, excursions from M can be considered in either direction of time. To get a co-exit system we may consider an *exit* system for the process $\hat{Z}$, replacing Γ by $\hat{\Gamma}$ in (2.39). Following [23], we obtain the optional exit system for Z by defining for $F \in \mathscr{F}^0$, $x \in E$,

$$\begin{aligned} &{}^*\mathbf{P}^x(F \circ \pi) = {}^*P^x(F); \quad \text{this defines a measure on} \quad (\Omega, \tilde{\mathscr{F}}^0[0, \infty)). \\ &\mathbf{B} = \text{the homogeneous embedding of } B. \end{aligned} \tag{2.41}$$

Similarly we define the predictable exit system $({}^*\mathbf{Q}^x, \mathbf{C})$. It was shown in [23] that for all positive $W \in \mathscr{O}$, $H \in \tilde{\mathscr{F}}^0[0, \infty)$

$$Q \sum_{s\in\mathbf{M}_\ell} W_s H(\theta_s) = Q \int_{\mathbb{R}} W_s {}^*\mathbf{P}^{Z_s}(H)\, d\mathbf{B}_s. \tag{2.42}$$

Similar results hold for the predictable exit system with $Z \in \mathscr{P}$ replacing $W \in \mathscr{O}$ in (2.42).

Define $\hat{\mathbf{M}}_r$, $\hat{\mathbf{M}}_\ell$, $\hat{\mathbf{D}}_t$, $\hat{\ell}_t$, for $\hat{\mathbf{M}}$ in the same manner that $\mathbf{M}_\ell$, $\mathbf{M}_r$, $\mathbf{D}_t$, ℓ_t were defined for $\mathbf{M}$. Note that $s \in \hat{\mathbf{M}}_r$ if, and only if, $s \in \mathbf{M}_\ell$. We further define ${}^*\hat{\mathbf{P}}^x$, $\hat{\mathbf{B}}$, ${}^*\hat{\mathbf{Q}}^x$, $\hat{\mathbf{C}}$ for the co-exit system in the same manner that the corresponding objects ${}^*\mathbf{P}^x$, $\mathbf{B}$, ${}^*\mathbf{Q}^x$, $\mathbf{C}$ were defined for the exit system. The proof of the following lemma on time reversal follows along lines similar to those of (4.8) of [23], using the shift invariance of Q, and it is therefore omitted.

LEMMA 2.43. *For* $F \in (\mathscr{E} \times \mathscr{E})_+$, $H \in \tilde{\mathscr{F}}^0[0, \infty)$

$$\begin{aligned} &Q \sum_{0<s<1; s\in\mathbf{M}_\ell} F(Z_{s-}, Z_{\mathbf{D}_0} \circ \theta_s) H(\mathbf{k}_{\mathbf{D}_0} \circ \theta_s) \\ &\qquad = Q \sum_{0<s<1; s\in\hat{\mathbf{M}}_\ell} F(\hat{Z}_{\hat{\mathbf{D}}_0} \circ \theta_s, \hat{Z}_{s-}) H(\rho \circ \hat{\mathbf{k}}_{\hat{\mathbf{D}}_0} \circ \hat{\theta}_s) \end{aligned} \tag{2.44}$$

where $\rho: \Omega \longrightarrow \Omega$ *is defined by*

$$\hat{Z}_t(\rho \circ \hat{\mathbf{k}}_{\hat{\mathbf{D}}_0}) = \begin{cases} \hat{Z}_{\mathbf{D}_0 - t} & 0 < t < \hat{\mathbf{D}}_0 < \infty \\ \Delta & t \leq 0 \\ \hat{\Delta} & \textit{otherwise.} \end{cases} \tag{2.45}$$

The fact that $F \in (\mathscr{E} \times \mathscr{E})_+$ implies that the expressions in (2.44) do not contain contributions of excursions of infinite length. This is essential for the result to hold.

We end this section with results on Revuz measures from [13] adapted to our situation.

Let A be an additive functional, ν_A its Revuz measure and ν_A^d the measure satisfying

$$E^x \int_{\mathbb{R}_+} e^{-\alpha s} f(X_{s-}, X_s)\, dA_s = \int_{E \times E} u^\alpha(x, y) f(y, z) \nu_A^d(dy, dz) \tag{2.46}$$

($u^\alpha(x, y)$ are the α potential densities with respect to ξ). Then

$$\nu_A(f) = Q \int_0^1 f(Z_{t-})\, d\mathbf{A}_t \tag{2.47}$$

$$\nu_A^d(f) = Q \int_0^1 f(Z_{t-}, Z_t)\, d\mathbf{A}_t \tag{2.48}$$

recall that A is integrable (σ-integrable) if ν_A is finite (σ-finite), (this also implies that ν_A^d is finite (σ-finite)).

3. Exit and co-exit systems and the joint law of (R, X_R).

This section is devoted to the computation of the joint Laplace-transform and distribution of (R, X_R).

In what follows we shall adopt the "one hat" notation of Getoor and Sharpe. Namely $\hat{E}^y(f(X_t); R > t)$ will stand for $\hat{E}^y(f(\hat{X}_t); \hat{R} > t)$ and similarly for all other expressions involving objects that correspond to the dual process. For $F \in \mathscr{F}^0 f, g \in \mathscr{E}_+$ let

$$\hat{E}^f(F) = \int \xi(dx) f(x) \hat{E}^x(F) \quad \text{and} \quad E^g(F) = \int \xi(dx) g(x) E^x(F). \tag{3.1}$$

We start with the following:

THEOREM 3.2. *For each* $g \in \mathscr{E}_+$

$$\hat{E}^x(e^{-\lambda R} g(X_R)) = \int_{E \times E} \nu_B^d(dy, dz) g(y) {}^*p_\lambda(z, x) \tag{3.3}$$

where ${}^*p_\lambda(z, x)$ *are the derivatives of*

$$\eta_x(f) = \int_0^\infty e^{-\lambda t}\, {}^*P_t^x(f)\, dt \tag{3.4}$$

with respect to ξ, *and* ν_B^d *is as defined in* (2.48) *for* B *defined in* (2.5).

PROOF. We start with the basic duality relation [2]:

$$\hat{E}^f(e^{-\lambda R}\hat{U}^\lambda g(X_R)) = E^g(e^{-\lambda R}U^\lambda f(X_R)) \quad f, g \in \mathscr{E}_+. \tag{3.5}$$

Let $f \in \mathscr{E}_+$ be zero on $\hat{F}(=\operatorname{Reg}(\hat{R}))$, then using (3.5)

$$\begin{aligned}\hat{E}^f(e^{-\lambda R}\hat{U}^\lambda g(X_R)) &= E^g \int_{\mathbb{R}_+} e^{-\lambda t} 1_M(t) f(X_t)\, dt \\ &\quad + E^g \textstyle\sum_{s\in M_\ell; s>0} e^{-\lambda s}\left(\int_0^R e^{-\lambda u} f(X_u)\, du\right) \circ \theta_s.\end{aligned} \tag{3.6}$$

Using the fact that $\xi(F\Delta\hat{F}) = 0$, it follows that the first summand in (3.6) drops out. As for the second summand using the optional exit system it is easy to see that it is equal to

$$E^g \int_{\mathbb{R}_+} e^{-\lambda t}\left[\int_{\mathbb{R}_+\times E} e^{-\lambda u} {}^*P_u(X_t, dx) f(x)\, du\right] dB_t \tag{3.7}$$

and by (2.50) this expression is equal to

$$\begin{aligned}&\int_{E\times E\times E} \xi(dx) g(x) u^\lambda(x, y) \nu_B^d(dy, dz) \int_{\mathbb{R}_+\times E} e^{-\lambda u} {}^*P_u(z, dx) f(x)\, du \\ &\quad = \int_{E\times E} \hat{U}^\lambda g(y) \nu_B^d(dy, dz) \int_{\mathbb{R}_+\times E} e^{-\lambda u} {}^*P_u(z, dx) f(x)\, du.\end{aligned} \tag{3.8}$$

We note that the measure $\int_{\mathbb{R}_+} e^{-\lambda u} {}^*P_u(z, \cdot)\, du$ is, for each z, λ-excessive for the process X killed at R. It therefore follows that there exists a density ${}^*p_\lambda(z, x)$ of this measure with respect to ξ, so that for each z, $x \to {}^*p_\lambda(z, x)$ is λ-co-excessive (for $\hat{X}$ killed at $\hat{R}$). It follows that

$$\hat{E}^f(e^{-\lambda R}\hat{U}^\lambda g(X_R)) = \int_{E\times E} \nu_B^d(dy, dz) \hat{U}^\lambda g(y) \int_E {}^*p_\lambda(z, x) f(x) \xi(dx). \tag{3.9}$$

Therefore ξ a.e. on $E - \hat{F}$,

$$\hat{E}^x(e^{-\lambda R}\hat{U}^\lambda g(X_R)) = \int_{E\times E} \nu_B^d(dy, dz) \hat{U}^\lambda g(y) {}^*p_\lambda(z, x) \tag{3.10}$$

and since both sides of (3.10) are λ-excessive for $\hat{X}$ killed at $\hat{R}$, they coincide on all $x \in E - \hat{F}$.

Using now the resolvent equation for $\hat{U}^\alpha$, (3.10) implies that for all $\alpha > 0$, $x \in E - \hat{F}$

$$\hat{E}^x(e^{-\lambda R}\hat{U}^\alpha g(X_R)) = \int_{E\times E} \nu_B^d(dy, dz) \hat{U}^\alpha g(y) {}^*p_\lambda(z, x). \tag{3.11}$$

The function 1_E is λ-excessive for $\hat{X}$ and therefore there exists a sequence of functions h_n so that $\hat{U}^\lambda h_n$ increase to 1_E. Using the Monotone Convergence

Theorem and (3.10) it follows that for all $x \in E - \hat{F}$

$$\hat{E}^x(e^{-\lambda R}) = \int_{E\times E} \nu_B^d(dy, dz)^*p_\lambda(z, x). \tag{3.12}$$

This implies that the measure

$$\eta(A) = \int_{A\times E} \nu_B^d(dy, dz)^*p_\lambda(z, x) \tag{3.13}$$

is finite. Let $g \in b\mathscr{E}_+$ be continuous, then $g(x) = \lim_{\alpha\to\infty} \alpha\hat{U}^\alpha g(x)$ and by the Dominated Convergence Theorem and (3.11)

$$\hat{E}^x(e^{-\lambda R}g(X_R)) = \int_{E\times E} \nu_B^d(dy, dz)^*p_\lambda(z, x)g(y). \tag{3.14}$$

It now follows from the Monotone Class Theorem that (3.14) holds for all $g \in b\mathscr{E}$.

REMARK 3.15. Between submission and revision of this paper, P. J. Fitzsimmons has proved that for $K \in (\mathscr{E} \times \mathscr{E})_+$

$$\hat{E}^x(e^{-\lambda R}K(X_{R-}, X_R)) = \int\int \nu_B^d(dy, dz)k(z, y)^*p_\lambda(z, x).$$

Repeating the same arguments that led to (3.14), we obtain

THEOREM 3.16. *Let $g \in \mathscr{E}_+$ and $\lambda \geq 0$, then*

$$\hat{E}^x(e^{-\lambda R}g(X_R)) = \int \nu_C(dy)g(y)^*q_\lambda(y, x) \tag{3.17}$$

where $^*q_\lambda(y, x)$ is the density of $\int_0^\infty e^{-\lambda u}{}^*Q_u(y, \cdot)\, du$ with respect to ξ. For each y, $x \to {}^*q_\lambda(y, x)$ is λ-co-excessive for the process $\hat{X}$ killed at $\hat{R}$.

Our next objective is to show that $\lambda \to {}^*p_\lambda(x, y)$ are Laplace transforms of measures on $(\mathbb{R}_+, \mathscr{R}_+)$. Unlike the case where dual densities exist, these measures are not in general absolutely continuous with respect to the Lebesgue measure. This absolute continuity was the main tool for the analysis carried in [11], which unfortunately doesn't carry to the present situation. However, the joint distribution of (R, X_R) in terms of the exit and co-exit systems has an interest of its own.

Let $u_R^\lambda(x, y)$ be the densities of $U_R^\lambda(x, \cdot) = E^x \int_0^R e^{-\lambda u} 1_{(\cdot)}(X_u)\, du$ with respect to $\xi(\cdot)$. $x \to u_R^\lambda(x, y)$ is λ-excessive for the process X killed at R and is therefore, an increasing limit of $U_R^\lambda h_n(x)$. Using the resolvent equation for $u_R^\alpha(x, y)$ we see that for each y,

$$u_R^\alpha(x, y) = \lim_{n\to\infty} U_R^\alpha h_n(x) \quad h_n \in b\mathscr{E}.$$

Hence for fixed (x, y), $\lambda \to u_R^\lambda(x, y)$ is a limit of Laplace transforms, and it is therefore a Laplace transform of a measure on $(\mathbb{R}_+, \mathscr{R}_+)$.

Now

$$\int_{\mathbb{R}_+} e^{-\lambda u} {}^*P_u(x, f)\, du = \lim_{t\to 0} {}^*P^x(R > t;\, e^{-\lambda t} U_R^\lambda(X_t, f))$$

$$= \lim_{t\to 0} \int_E {}^*P^x(R > t;\, e^{-\lambda t} u_R^\lambda(X_t, y) f(y)) \xi(dy).$$

Using the fact that $x \to u_R^\lambda(x, y)$ is λ-excessive for X killed at R, it is easy to show that

$$t \to {}^*P^x(R > t;\, e^{-\lambda t} u_R^\lambda(X_t, y))$$

increases as t decreases, and thus, using the Monotone Convergence Theorem

$$\int_E \xi(dy) {}^*p_\lambda(x, y) f(y) = \int_{\mathbb{R}+} e^{-\lambda u} {}^*P_u^x(f)\, du$$

$$= \int \xi(dy) f(y) \lim_{t\to 0} {}^*P^x(R > t;\, e^{-\lambda t} u_R^\lambda(X_t, y)).$$

Hence ${}^*p_\lambda(x, y) = \lim_{t\to 0} {}^*P^x(R > t;\, e^{-\lambda t} u_R^\lambda(X_t, y))$ for ξ a.e. y. Since both are λ-co-excessive for $\hat{X}$ killed at $\hat{R}$ they are equal for all $y \in E - \hat{F}$. On the other hand ${}^*P^x(R > t;\, e^{-\lambda t} u_R^\lambda(X_t, y))$ is a Laplace transform of a measure on $(\mathbb{R}_+, \mathscr{R}_+)$ and hence so is ${}^*p_\lambda(x, y)$ as a limit of Laplace transforms.

It therefore follows that

THEOREM 3.18.

$$P^x(R \in dr,\, X_R \in dy) = \int \nu_B^d(dy, dz) {}^*p(dr, z, x)$$

where ${}^*p_\lambda(z, x) = \int_{\mathbb{R}_+} e^{-\lambda r} {}^*p(dr, z, x)$.

We now specialize to the case where R is the hitting time of a Borel Set B, for which $\{t\colon X_t \in B\}$ and $\{t\colon \hat{X}_t \in B\}$ are perfect and unbounded. Let L be the local time of equilibrium of order 1 at B, namely the predictable additive functional whose potential is given by $E^x(e^{-R})$. Let $\{T_n\}$ be the predictable exit points from B and $\{l(x)\}_{x\in B^r}$ the Motoo derivatives of $\mathrm{Leb}(M \cap [0, t))$ with respect to L^c (the continuous part of L). Let $\hat{L}$, $\{\hat{l}(x)\}$ and $\{\hat{T}_n\}$ be the corresponding objects of the dual process.

We then have

$$(3.19)\qquad E^x(e^{-R}) = E^x \int_{\mathbb{R}_+} e^{-t}(l(X_t) + {}^*H^{X_t}(1 - e^{-R}))\, dL_t$$

where ${}^*H$ has been defined in Section 2. It now follows that

$$(3.20)\qquad E^x(e^{-R}) = \int_E u^1(x, y)(l(y) + {}^*H^y(1 - e^{-R}))\nu_L(dy).$$

The measure

$$\pi(dy) = \nu_L(dy)(l(y) + {}^*H^y(1 - e^{-R})) \tag{3.21}$$

was identified in [11] as the 1-capacitary measure of the set B. Define $\hat{\pi}(dy)$ for the dual process by

$$\hat{\pi}(dy) = \nu_{\hat{L}}(dy)(\hat{l}(y) + {}^*\hat{H}^y(1 - e^{-R})). \tag{3.22}$$

Note that by the definition of l, ${}^*H$, $\hat{l}$, ${}^*\hat{H}$, $l(y) + {}^*H^y(1 - e^{-R}) = 1$, ν_L a.e. and similarly for $\hat{l}$, ${}^*\hat{H}$. It therefore follows that $\nu_L = \pi$, $\nu_{\hat{L}} = \hat{\pi}$. The following statement connects π and $\hat{\pi}$ when M and $\hat{M}$ are unbounded.

Theorem 3.23. *Suppose that $\{t: X_t \in B\}$ and $\{t: \hat{X}_t \in B\}$ are perfect and unbounded, then*

$$\pi(dx) = \hat{\pi}(dx)\hat{l}(x) + \int_E \hat{\pi}(dy){}^*\hat{H}^y((1 - e^{-R})1_{\{X_R \in dx\}}). \tag{3.24}$$

Proof. Using the auxiliary process Z we note that

$$\begin{aligned}\int \pi(dx)f(x) \\ &= Q\int_0^1 f(Z_t)1_{\mathbf{M}}(t)\,dt + Q\textstyle\sum_{s\in\mathbf{M}_{\swarrow};0<s<1} f(Z_{s-})(1 - e^{-\mathbf{D}_0(\theta_s)}).\end{aligned} \tag{3.25}$$

The assumptions on the unboundedness of M implies that $\inf\{t: t \in \mathbf{M}) = -\infty$, $\sup\{t: t \in \mathbf{M}\} = +\infty$ Q a.e. We can therefore use (2.43) to show that the right side of (3.25) is equal to

$$\begin{aligned}&Q\int_0^1 f(\hat{Z}_t)1_{\hat{\mathbf{M}}}(t)\,dt + Q\textstyle\sum_{s\in\hat{\mathbf{M}}_{\swarrow};0<s<1} ((1 - e^{-\hat{\mathbf{D}}_0})f(\hat{Z}_{\hat{\mathbf{D}}_0}))\circ\hat{\theta}_s \\ &= \int \hat{\nu}_L(dx)(f(x)\hat{l}(x) + {}^*\hat{H}^x((1 - e^{-R})f(X_R))) \\ &= \int \hat{\pi}(dx)(f(x)\hat{l}(x) + {}^*\hat{H}^x((1 - e^{-R})f(X_R))).\end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned}\pi(E) &= \int_E \hat{\pi}(dx)(\hat{l}(x) + {}^*\hat{H}^x((1 - e^{-R})1_E(X_R))) \\ &= \hat{\pi}(E) \quad \text{as is well known.}\end{aligned} \tag{3.27}$$

We end this section with the following theorem that connects the 1-capacitary measures of R with first entrance results for the dual process. Its proof is identical to that of Theorem (3.2) and is therefore omitted.

THEOREM 3.28.

$$(3.29)\quad \hat{E}^{\xi}(e^{-\lambda R}g(X_R)) = \int \pi(dx)g(x)\left(l(x) + \int_{\mathbb{R}_+} e^{-\lambda t} {}^*H^x(R > t, 1_E(X_t))\, dt\right).$$

In particular

$$(3.30)\qquad \hat{P}^{\xi}(R \in dt,\, X_R \in dx) = \pi(dx){}^*H^x(R > t,\, 1_E(X_t))\, dt.$$

Note that this result doesn't require unboundedness of M or $\hat{M}$. By adding to C the Lebesgue measure of M and adjusting $\{{}^*Q^x\}_{x\in E}$ we can prove a result like (3.30) without the perfectness assumptions. It is interesting when compared with last exit results of [10]; see also [11] for a similar result when dual densities exist.

4. Conditioned excursions. In this section we return to the exit system defined by (2.14), (2.18). We let $Y_s = X_{\tau_s}$, and $(\Omega, \mathscr{M}, \mathscr{M}'_t, Y_t, \tau_t, \theta'_t, P^x)$ be the Markov additive process (MAP) defined in Section 2 and [16]; Y is the boundary process. Let $(\mathscr{K}_t)$, $(\mathscr{L}_t)$, $(\mathscr{G}_t)$ be as defined in Section 2.

Our purpose is to study the laws of the excursions conditioned on $\mathscr{K}$. Let μ be the random measure that counts the excursions as defined in Section 2. Then by (2.27) and the discussion following it, μ is a sum of two conditionally $(\mathscr{K}_t)$ independent measures. The first one is a Poisson random measure that counts the excursions that correspond to jumps of τ that are not jump times of Y. This measure has been completely characterized by its mean measure given in (2.29). The other part μ^f counts excursions that correspond to jumps of τ that are also jumps of Y. Let

$$(4.1)\qquad D = \{(s, \omega)\colon \tau_s(\omega) \neq \tau_{s-}(\omega),\ Y_s(\omega) \neq Y_{s-}(\omega),\ Y_s(\omega) \in E\}.$$

Then $D \subset \bigcup_{n=1}^{\infty} [T_n]$ where $\{T_n\}$ are the jump times of Y. We note that a jump of Y need not always correspond to an excursion. We shall deal in this section with the conditional laws of e_{T_j} given $\mathscr{K}$.

It follows from (2.20) that Y is a Borel right process and we can use the results of [1]. We further note that since Y is quasi-left continuous (in the usual topology of E) a stopping time T for which $Y_T \in E$ a.s. on $\{0 < T < \infty\}$ is totally inaccessible, if and only if $Y_{T-} \neq Y_T$ [9] (Y_{t-} is the left limit of Y in the usual topology). Using this and Lemma 3.3 of [1], there exists a strictly positive $h \in \mathscr{E} \times \mathscr{E}$ so that

$$(4.2)\qquad A_t = \textstyle\sum_{0<s\le t} 1_{\{Y_s \neq Y_{s-}\}} h(Y_{s-}, Y_s)$$

has finite 1-Potential. Further any quasi-left continuous additive functional of Y is of the form

$$(4.3)\qquad \textstyle\sum_{0<s\le t} 1_{\{Y_s\neq Y_{s-}\}} f(Y_{s-}, Y_s) \quad \text{with} \quad f \in \mathscr{E}^e \times \mathscr{E}.$$

Let $\Gamma \in \mathscr{Y}$, using (2.23) define

$$(4.4)\qquad A_t^{\Gamma} = E^{\boldsymbol{\cdot}}\left(\textstyle\sum_{0<s\le t;\, s\in D} 1_{\Gamma}(e_s) \,\middle|\, \mathscr{K}\right).$$

Using the Markov property of (Y, τ, e) it is easy to show (see e.g., [4]) that

$$A_t^\Gamma = E^\bullet(\textstyle\sum_{0<s\le t; s\in D} 1_\Gamma(e_s) \mid \mathscr{K}_t). \tag{4.5}$$

Hence A_t is an additive functional of $(\mathscr{K}_t)$, whose jump times are totally inaccessible. It now follows from (4.3) that there exists a positive function $F_\Gamma \in \mathscr{E}^e \times \mathscr{E}$ so that

$$A_t^\Gamma = \textstyle\sum_{0<s\le t} F_\Gamma(Y_{s-}, Y_s). \tag{4.6}$$

By the special structure of V one can choose F_Γ so that for each (x, y) $\Gamma \to F_\Gamma(x, y)$ is a measure on $(V, \mathscr{V})$ (see e.g. [19]). (Recall that the existence of F was proved in [15] using different techniques.) Note that A_t^Γ need not always be a finite AF but for h defined in (4.2) $A^\Gamma \cdot h$ is a finite additive functional. Let

$$\beta(e_s) = \tau_s - \tau_{s-} \tag{4.7}$$

and

$$F_0 = F_{\{\beta>0\}}. \tag{4.8}$$

Note that for each T_n defined above, $\Gamma \to F_\Gamma(Y_{T_n-}, Y_{T_n})$ gives the conditional law of the excursion e_{T_n} given $\mathscr{K}$. $\{\beta(e_{T_n}) = 0\}$ is the event that the length of the excursion that had occurred at T_n is 0 or equivalently that the jump of Y at T_n was not "caused" by an excursion. Therefore if $F_0(x, y) > 0$, $\Gamma \to F_\Gamma(Y_{T_n-}, Y_{T_n})/F_0(Y_{T_n-}, Y_{T_n})$ is the conditional law of the excursion e_{T_n} given $\mathscr{K}$ and $\{\beta(e_{T_n}) > 0\}$. In view of (2.24) one is justified calling $F_\Gamma(x, y)/F_0(x, y)$ the law of the excursion that starts at x and ends at y. To abbreviate the notations we let

$$F_\Gamma^0(x, y) = F_\Gamma(x, y)/F_0(x, y) \quad \text{(by convention } 0/0 = 0). \tag{4.9}$$

Our objective now is to compute $F_\Gamma^0(x, y)$ using the exit systems defined in Section 2.

Repeating the derivations of Section 4 of [16], one can show that the dual predictable projection of A_t^Γ with respect to $(\mathscr{K}_t)$ is equal to

$$\int_0^t {}^*H^{Y_s}(s_R \in \Gamma;\, X_R \ne Y_s)\, ds \tag{4.10}$$

where the stopping operators s_u are defined by

$$X_t(s_u\omega) = \begin{cases} X_t(\omega) & t < u \\ X_u(\omega) & u \le t. \end{cases} \tag{4.11}$$

For each $\Gamma \in \mathscr{V}$ and $x \in E$, define the measure $\mu_\Gamma^x(\cdot)$ on $(E, \mathscr{E}^*)$ by

$$\mu_\Gamma^x(A) = {}^*H^x(s_R \in \Gamma,\, X_R \in A - \{x\}), \tag{4.12}$$

and for each $x \in E$ define the measure λ^x on $(E, \mathscr{E})$ by

$$\lambda^x(A) = {}^*H^x(X_R \in A - \{x\}). \tag{4.13}$$

It follows from (4.2) and (4.5) that for all x, except possibly on a set of potential

0 with respect to the Y process, λ^x is a σ-finite measure on $(E, \mathscr{E})$. This, of course, implies that so is μ_Γ^x.

Repeating now the derivations carried in Section 4 of [16], using results of [1], [16] it is easy to show that if $F_0(x, y) > 0$, $F_\Gamma^0(x, y)$ is the derivative of μ_Γ^x with respect to λ^x (see also the appendix for more details). The duality assumptions and the results of the previous section enable us to compute the finite dimensional distributions of the excursions conditioned to start at x and end at y, explicitly, in terms of the exit and co-exit systems. This unfortunately cannot be done in general, and in this respect our results extend those given in [15], [16].

Let $U_t(v) = v_t$ be the coordinate mapping on V. Suppose that $F_0(x, y) > 0$, we shall compute $F_\Gamma^0(x, y)$ for Γ of the form $\prod_{j=1}^n 1_{A_j}(U_{t_j})$ for $A_j \in \mathscr{E}$, $y \notin A_j$, $0 < t_1 < t_2 \cdots \le t_n$.

Let $A \in \mathscr{E}$ then

$$\begin{aligned}\lambda^x(A) &= {}^*H^x(X_R \in A - \{x\}) = \lim_{t\to 0}{}^*H^x(R > t;\, X_R \in A - \{x\})\\ &= \lim_{t\to 0}{}^*H^x(R > t;\, P^{X_t}(X_R \in A - \{x\}))\\ &= \lim_{t\to 0}{}^*H^x\left(R > t;\, \int_{A-\{x\}} {}^*\hat{q}_0(z, X_t)\,\nu_{\hat{C}}(dz)\right)\\ &= \lim_{t\to 0} \int_{A-\{x\}} \nu_{\hat{C}}(dz)\,{}^*H^x(R > t;\, {}^*\hat{q}_0(z, X_t)),\end{aligned}$$

where ${}^*\hat{q}_\lambda(z, x)$ are defined for the dual process in the same manner that ${}^*q_\lambda(z, x)$ were defined for the original process. Note that $x \to \hat{q}_0(z, x)$ is for each z, excessive for X killed at R. It therefore follows that $t \to {}^*H^x(R > t;\, {}^*\hat{q}_0(z, X_t))$ increases as $t \to 0$ to a limit that we shall denote by ${}^*q(z, x)$. Using the Monotone Convergence Theorem, it follows that for $A \in \mathscr{E}$

$$\lambda^x(A) = \int_{A-\{x\}} \nu_{\hat{C}}(dz)\,{}^*q(z, x). \tag{4.14}$$

The fact that λ^x is σ-finite L a.e. implies that for L a.e. x, ${}^*q(z, x) < \infty$ $\nu_{\hat{C}}$ a.e.

We next turn to $\mu_\Gamma^x(A)$. Let

$$Q_t(x, B) = P^x(R > t;\, X_t \in B), \quad B \in \mathscr{E}. \tag{4.15}$$

Then for Γ as above

$$\mu_\Gamma^x(A) = \int_{A_1} \cdots \int_{A_n} {}^*H_{t_1}^x(dy_1)\,Q_{t_2-t_1}(y_1, dy_2) \cdots Q_{t_n-t_{n-1}}(y_{n-1}, dy_n) \cdot \int_{A-\{x\}} {}^*\hat{q}_0(z, y_n)\,\nu_{\hat{C}}(dz)$$

and therefore for Γ as above

$$F_\Gamma^0(x, y) = \int_{A_1} \cdots \int_{A_n} {}^*H_{t_1}^x(dy_1)\,Q_{t_2-t_1}(y_1, dy_2) \cdots Q_{t_n-t_{n-1}}(y_{n-1}, dy_n)\,{}^*\hat{q}_0(y, y_n)\,({}^*q(y, x))^{-1}. \tag{4.16}$$

REMARK 4.17. Let (B, K^0) be the Levy system for the jumps of Y. For each $\Gamma \in \mathscr{V}, A \in \mathscr{E}$ let

$$K(x, \Gamma \times A) = \int_A K^0(x, dy) F_\Gamma(x, y).$$

It follows from (4.6) that (B, K) is the Levy system for the jumps of τ that are also jumps of Y, and excursions induced by them. Let

$$H(x, \Gamma \times A) = {}^*H^x(s_R \in \Gamma, X_R \in A - \{x\})$$

then it follows from (4.10) that (t, H) is also a Levy system for the jumps of τ that are also jumps of Y, and excursions induced by them. This implies that

$$\{Y_s \neq Y_{s-}, F_0(Y_{s-}, Y_s) > 0, {}^*q(Y_s, Y_{s-}) = 0 \text{ or } \infty\}$$

is evanescent, and we may define $F_0(x, y) = 0$ if ${}^*q(y, x) \notin (0, \infty)$ without affecting (4.6). Further using (4.6) and (4.10) we see that $\{x: \nu_{\hat{C}}\{y: F_0(x, y) = 0, {}^*q(y, x) > 0\} > 0\}$ has potential 0 with respect to Y.

For each x, y we now define

$$P^{x,y}(\Gamma) = F_\Gamma^0(x, y). \tag{4.18}$$

$P^{x,y}$ is, for all (x, y), either a probability measure, or is equal to 0. If $p^{x,y}$ is not 0 then under $P^{x,y}$ the finite dimensional distributions of $(U_t)_{t>0}$ are given by

$$\begin{aligned} &P^{x,y}(U_{t_1} \in dy_t, \cdots, U_{t_n} \in dy_n) \\ &\quad = H_{t_1}(x, dy_1) Q_{t_2-t_1}(y_1, dy_2) \cdots Q_{t_n-t_{n-1}}(y_{n-1}, dy_n) {}^*\hat{q}_0(y, y_n)({}^*q(y, x))^{-1}. \end{aligned} \tag{4.19}$$

Equivalently if $P^{x,y}$ is not 0, then under $P^{x,y}$, $(U_t)_{t>0}$ is a homogeneous Markov process with entrance laws

$$\eta_t^{x,y}(dz) = \frac{{}^*H_t^x(dz) {}^*\hat{q}_0(y, z)}{{}^*q(y, x)} \tag{4.20}$$

and transition function

$$K_t^y(z, du) = \frac{Q_t(z, du) {}^*\hat{q}_0(y, u)}{{}^*\hat{q}_0(y, z)}. \tag{4.21}$$

Further, using the convention $0/0 = 0$, we can use remark (4.17) to show that for $\nu_{\hat{C}}$ a.e. y, the finite dimensional distributions of $P^{x,y}$ are given by (4.16), for L a.e. x.

We have thus proved

THEOREM 4.22. *For every $(x, y) \in E \times E$ $x \neq y$, there exists a measure $P^{x,y}$ on $(V, \mathscr{V})$ so that*

(i) *$P^{x,y}$ is either a probability measure or is equal to 0.*
(ii) *$(x, y) \to P^{x,y}(\Gamma)$ is $\mathscr{E} \times \mathscr{E}$ measurable for each $\Gamma \in \mathscr{V}$.*
(iii) *Except for a set of L potential 0, for $\nu_{\hat{C}}$ a.e. y, under $P^{x,y}$, $(U_t)_{t>0}$ is a homogeneous Markov process, with entrance laws and transition function given by* (4.20) *and* (4.21) *respectively. If $P^{x,y}$ doesn't satisfy the above it is equal to* 0.

(iv) *If T is a jump time of Y, then with the convention* $0/0 = 0$

$$\frac{E^{\bullet}(f(e_T)\mid \mathscr{H})}{E^{\bullet}(\beta(e_T) > 0 \mid \mathscr{H})} = P^{Y_{T-},Y_T}(f) \quad (\beta \text{ defined in } (4.7)) \tag{4.23}$$

and a.s. on $\{T \in D\}$, $0 < F_0(Y_{T-}, Y_T) < \infty$, $0 < {}^*q(Y_T, Y_{T-}) < \infty$.

Note on Proof. The only thing which we have yet to prove is (ii). Note that $(x, y) \to P^{x,y}$ is $\mathscr{E}^e \times \mathscr{E}$ measurable. The proof that one may choose $F_\Gamma(x, y)$ so as to make them $\mathscr{E} \times \mathscr{E}$ measurable is deferred to the Appendix.

REMARK 4.24. As noted by Fitzsimmons in [8] and by Getoor and Sharpe in [11], (4.21) (or its time space analogue that appears in [11]) implies that conditioned to start at x and end at y (start at x, end at y and have length ℓ in [11]), the excursion process is obtained from the original process killed at R (or its time space analogue in [11]) by an h-path transform. Fitzsimmons constructed the measures $P^{x,y}$ using (4.20) (4.21) and the theory of h-path transforms rather than giving them the interpretation we have. This, of course, makes $(x, y) \to P^{x,y}$ $\mathscr{E} \times \mathscr{E}$ measurable automatically.

LEMMA 4.25 *For $F \in \mathscr{F}^*_+$ let ${}^*H^x_1(F) = {}^*H^x(F; X_R \neq x)$, then for L a.e. x*

$${}^*H^x_1(F \circ k_R \cdot \psi(X_R)) = {}^*H^x_1(P^{x,X_R}(F)\psi(X_R)) \tag{4.26}$$

(*for each t, k_t is the killing operator at time t*).

PROOF. It is enough to prove the lemma for

$$F = f_1(X_{t_1}) \cdots f_k(X_{t_k}) \quad f_i \in b\mathscr{E}_+.$$

Then

$$\begin{aligned} &{}^*H^x_1(F \circ k_R \cdot \psi(X_R)) \\ &= {}^*H^x_1(1_{\{R>t_k\}} f_1(X_{t_1}) \cdots f_k(X_{t_k})\psi(X_R)) \\ &= \int_E \cdots \int_E \int_{E\setminus\{x\}} {}^*H^x_{t_1}(dy_1)Q_{t_2-t_1}(y_1, dy_2) \cdots Q_{t_k-t_{k-1}}(y_{k-1}, dy_k)f_1(y_1) \\ &\qquad \cdots f_k(y_k){}^*\hat{q}_0(u, y_k)\psi(u)\nu_{\hat{C}}(du). \end{aligned} \tag{4.27}$$

Using the definition of $P^{x,y}$ and remark (4.17), the right-hand side of (4.27) is for L a.e. x equal to

$$\int_{E\setminus\{x\}} P^{x,u}(F){}^*q(u, x)\psi(u)\nu_{\hat{C}}(du) = {}^*H^x_1(P^{x,X_R}(F)\psi(X_R)).$$

We let $\hat{P}^{x,y}$ be the conditional excursions laws that correspond to the dual process. Our next objective is to connect $P^{x,y}$ and $\hat{P}^{x,y}$ via time reversal. We shall assume from now on that M (and therefore $\hat{M}$) are almost surely perfect. In this case L is the local time of equilibrium of order 1 at M and

(4.28) $$L_t = C_t + \mathrm{Leb}(M \cap (0, t)), \quad \hat{L}_t = \hat{C}_t + \mathrm{Leb}(\hat{M} \cap (0, t)).$$

Recall from Section 3, that l and $\hat{l}$ are the Motoo derivatives of the Lebesgue measures of M and $\hat{M}$ with respect to L and $\hat{L}$ respectively.

We define the following operation on V

(4.29) $$(\rho v)_t = \begin{cases} v(L(v) - t)- & 0 \le t < L(v) < \infty \\ \Delta & \text{otherwise} \end{cases}$$

where for each v, $L(v)$ is the absorption time at M.

Our next objective is to prove the following.

THEOREM 4.30. *Assume that M and $\hat{M}$ are almost surely perfect then $\rho P^{x,y} = \hat{P}^{y,x}$, $\nu_C \times \nu_{\hat{C}}$ a.e.*

REMARK 4.31. The assumption that M and $\hat{M}$ are perfect is essential for this result. For, in order to be interpreted as the conditional laws of excursions given $\mathscr{K}$, $P^{x,y}$ were defined using a mixture of optional and predictable exit systems. Time reversal results as (4.30) can hold only if one either uses a predictable exit system, and then the points (x, y) correspond to the state of the process before the beginning of an excursion and at absorption, respectively, or using an optional exit system and then (x, y) correspond to the state of the process at the beginning of an excursion and before absorption, respectively. (For the latter see [11], [8].)

PROOF. First note that

(4.32) $$\nu_C(dx) = (1 - l(x))\nu_L(dx), \quad \nu_{\hat{C}}(dx) = (1 - \hat{l}(x))\nu_{\hat{L}}(dx)$$

and ν_L a.e. $l(x) = 1$ implies that ${}^*H^x(1 - e^{-R}) = 0$ and therefore ${}^*q(y, x) = 0$ for all y. As before we shall use $0/0 = 0$.

Let $f, g \in \mathscr{E}_+$. Taking $H \equiv 1$ in (2.44) and using (2.47), it is easy to see that

(4.33)
$$\begin{aligned} &\int\int \nu_C(dx)g(x)\frac{{}^*q(y, x)}{1 - l(x)} f(y)\nu_{\hat{C}}(dy) \\ &= Q \textstyle\sum_{s \in M_{\prime};0<s<1} g(Z_{s-})f(Z_{\mathbf{D}_0} \circ \theta_s) \\ &= Q \textstyle\sum_{s \in \hat{M}_{\prime};0<s<1} f(\hat{Z}_{s-})g(\hat{Z}_{\hat{\mathbf{D}}_0} \circ \hat{\theta}_s) \\ &= \int\int \nu_C(dx)g(x)\frac{{}^*\hat{q}(x, y)}{1 - \hat{l}(y)} f(y)\nu_{\hat{C}}(dy) \end{aligned}$$

where ${}^*\hat{q}(x, y)$ are defined via the co-exit system in the same manner that ${}^*q(y, x)$ were defined using the exit system. It therefore follows that

(4.34) $$\frac{{}^*q(y, x)}{1 - l(x)} = \frac{{}^*\hat{q}(x, y)}{1 - \hat{l}(y)} \quad \nu_C \times \nu_{\hat{C}} \quad \text{a.e.}$$

Let $0 < s_1 < s_2 < \cdots < s_n$ and $f_1, f_2 \cdots f_n \in \mathscr{E}_+$. It follows again from (2.43) that

$$\int\int \nu_C(dx) \frac{g(x)}{1 - l(x)} {}^*H^x(R > s_n; f_1(X_{s_1}) \cdots f_n(X_{s_n}) {}^*\hat{q}_0(y, X_{s_n}))\nu_{\hat{C}}(dy) f(y) \tag{4.35}$$
$$= \int\int \nu_{\hat{C}}(dy) \frac{f(y)}{1 - \hat{l}(y)} {}^*\hat{H}^y(R > s_n; f_1(X_{(R-s_1)-}) \cdots f_n(X_{(R-s_n)-}) g(X_R)).$$

Hence $\nu_{\hat{C}}$ a.e.

$$\int \nu_C(dx) \frac{g(x)}{1 - l(x)} {}^*H^x(R > s_n; f_1(X_{s_1}) \cdots f_n(X_{s_n}) {}^*\hat{q}_0(y, X_{s_n})) \tag{4.36}$$
$$= \frac{1}{1 - \hat{l}(y)} {}^*\hat{H}^y(R > s_n; f_1(X_{(R-s_1)-}) \cdots f_n(X_{(R-s_n)-}) g(X_R)).$$

By (4.16) and (4.17) and the definition of $\hat{P}^{y,x}$ (with the obvious changes of notations to account for the dual objects), it follows that $\nu_C \times \nu_{\hat{C}}$ a.e.

$$\hat{P}^{y,x}(L > s_n, f_1(U_{(L-s_1)-}) \cdots f_n(U_{(L-s_n)-})) \tag{4.37}$$
$$= \frac{{}^*H^x(R > s_n, f_1(X_{s_1}) \cdots f_n(X_{s_n}) {}^*\hat{q}_0(y, X_{s_n}))}{{}^*\hat{q}(x, y)} \cdot \frac{1 - \hat{l}(y)}{1 - l(x)}.$$

It now follows from (4.37), (4.34) and (4.17) that with the convention $0/0 = 0$,

$$\hat{P}^{y,x}(L > s_n, f_1(U_{(L-s_1)-}) \cdots f_n(U_{(L-s_n)-}))$$
$$= P^{x,y}(L > s_n, f_1(U_{s_1}) \cdots f_n(U_{s_n})) \quad \nu_C \times \nu_{\hat{C}} \quad \text{a.e.}$$

Finally note that for $B \in \mathscr{E}^* \times \mathscr{E}^*$ with $\nu_C \times \nu_{\hat{C}}(B) = 0$, the set $\{(t, \omega): t \in D(\omega), (Y_{t-}(\omega), Y_t(\omega)) \in B\}$ is evanescent. Note also that the $\nu_C \times \nu_{\hat{C}}$ negligible set in (4.30) depends on $\Gamma \in \mathscr{Y}$, but using the fact that $\mathscr{Y}$ is countably generated, may be chosen independent of $\Gamma \in \mathscr{Y}$.

5. On excursions straddling terminal times. Although the meaning of $P^{x,y}$ defined in the previous sections as the laws of excursions conditioned to start at x and end at y, is significantly different from that of $P^{x,\ell,y}$ given in [11], they play the same role as the latter, in the analysis of excursions that straddle exact perfect terminal times. The methods and the proofs are very similar, and in most cases will be omitted.

Let T be an exact terminal time, $T = \infty$ if $T \geq \zeta$. We may assume that T is the hitting time of a set $J \in \mathscr{E} \times \mathscr{E}$ by (X_-, X) (see [11]). Let

$$G = G_T = \sup\{s \leq T; s \in M\} \tag{5.1}$$

$$D = D_T = \inf\{s > T; s \in M\}. \tag{5.2}$$

For J defined above let

$${}^*H^x(F \mid 0 < T < R, (X_{0-}, X_0) \notin J) = \frac{{}^*H^x(F; 0 < T < R, (X_{0-}, X_0) \notin J)}{{}^*H^x(0 < T < R, (X_{0-}, X_0) \notin J)} \tag{5.3}$$

where by convention $0/0 = 0$ and ${}^*H^x(X_{0-} \neq x) = 0$. We assume from now on that M is perfect. In this case (4.28) holds. The more general case could be carried out too, but one needs to deal with M_1 (the perfect kernel of M) and $M - M_1$ separately. The following is an obvious extension to the predictable case of Proposition 13.4 of [11].

PROPOSITION 5.4. *For positive predictable Z, $F \in b\mathscr{F}^*$ and $f \in b\mathscr{E}^*_+$*

(i) $E^\mu(Z_G f(X_G) F \circ \theta_G; 0 < G < T < \zeta)$

$= E^\mu \int_{[0,T]} Z_s {}^*H^{X_s}(f(X_0)F; (X_{0-}, X_0) \notin J, 0 < T < R)\, dL_s.$

(ii) $0 < {}^*H^{X_{G-}}((X_{0-}, X_0) \notin J; 0 < T < R) < \infty$ *a.s. on* $\{0 < G < T < \zeta\}$.

(iii) $E^\mu(Z_G F \circ \theta_G; 0 < G < T < \zeta)$

$= E^\mu(Z_G {}^*H^{X_{G-}}(F \mid (X_{0-}, X_0) \notin J, 0 < T < R) G < T < \zeta).$

(iv) *Let* ${}^*H_1^x(F) = {}^*H^x(F; X_R \neq x)$ *then*

$$E^\mu(Z_G F \circ \theta_G, X_D \neq X_{G-}, 0 < G < T < \zeta)$$
$$= E^\mu(Z_G {}^*H_1^{X_{G-}}(F \mid (X_{0-}, X_0) \notin J, 0 < T < R); X_D \neq X_{G-}, 0 < G < T < \zeta),$$

where

$$(5.5)\quad {}^*H_1^x(F \mid (X_{0-}, X_0) \notin J, 0 < T < R) = \frac{{}^*H_1^x(F; (X_{0-}, X_0) \notin J, 0 < T < R)}{{}^*H_1^x((X_{0-}, X_0) \notin J, 0 < T < R)}$$

when the denominator is positive and finite and is equal to 0 *otherwise.*

(v) *Let* ${}^*H_2^x(F) = {}^*H^x(F; X_R = x)$ *then*

$$E^\mu(Z_G F \circ \theta_G; X_D = X_{G-}, 0 < G < T < \zeta)$$
$$= E^\mu(Z_G {}^*H_2^{X_{G-}}(F \mid (X_{0-}, X_0) \notin J, 0 < T < R); X_D = X_{G-}, 0 < G < T < \zeta)$$

where as before

$$H_2^x(F \mid (X_{0-}, X_0) \notin J, 0 < T < R) = \frac{H_2^x(F; (X_{0-}, X_0) \notin J, 0 < T < R)}{{}^*H_2^x((X_{0-}, X_0) \notin J, 0 < T < R)}$$

when the denominator is positive and finite and is equal to 0 *otherwise.*

PROOF. The proof is almost identical to the one given in [11], once we notice that for Z positive predictable

$$(5.6)\quad \begin{aligned} &E^\mu(Z_G F \circ \theta_G; 0 < G < T < \zeta) \\ &\quad = E^\mu \textstyle\sum_{s \in M_\ell} 1_{(0,T]}(s) Z_s F(\theta_s) 1_{\{(X_{s-}, X_0(\theta_s)) \notin J, 0 < T(\theta_s) < R(\theta_s)\}} \end{aligned}$$

Note that X_0 that appears in this theorem is the beginning of the excursion. Note also that $[0, T)$ that appears in [11] with the optional exit system is not predictable. We therefore need the extra condition $(X_{s-}, X_0(\theta_s)) \notin J$.

We now recall the definitions of $\mathcal{F}_{G-}$ and $\mathcal{F}_{\geq D}$. W is in $\mathcal{F}_{G-}$ if $W \in \mathcal{F}$ and for every measure μ there exists a predictable process $(Z_t)_{t\geq 0}$ with $Z = Z_G$, P^μ a.s. on $\{G < \infty\}$. $H \in \mathcal{F}_{\geq D}$ if $H \in \mathcal{F}$ and for each μ there exists $\bar{H} \in \mathcal{F}$ with $H = \bar{H} \circ \theta_D$, P^μ a.s. on $\{D < \infty\}$.

For each $F \in \mathcal{V}$ let

$$(5.7) \qquad P^{x,y}(F \mid (x, U_0) \notin J, 0 < T < L) = \frac{P^{x,y}(F; (x, U_0) \notin J, 0 < T < L)}{P^{x,y}((x, U_0) \notin J, 0 < T < L)}$$

if the denominator is positive, and 0 if the denominator is equal to 0. (L defined below 4.29).

The following is the main result of this section. With Lemma (4.25) and Proposition (5.4) at hand the proof of its first part is identical to that of Theorem 13.7 of [11] and it is therefore omitted. We shall give a proof of its second part.

THEOREM 5.8

(i) *Let* $\Lambda = \{0 < G < T, D - G < \infty\}$ *and* $F \in \mathcal{V}$ *positive. Then almost surely on* $\Lambda \cap \{X_{G-} \neq X_D\}$, $0 < P^{X_{G-}, X_D}((X_{G-}, U_0) \notin J, 0 < T < L) < \infty$, *and*

$$E^\mu(F \circ s_R \circ \theta_G \mid \mathcal{F}_{G-}, \mathcal{F}_{\geq D}) = P^{X_{G-}, X_D}(F \mid (X_{G-}, U_0) \notin J; 0 < T < L).$$

(ii) *Let* Λ *be as in* (i), *then almost surely on* $\Lambda \cap \{X_{G-} = X_D\}$,

$0 < {}^*H_2^{X_{G-}}((X_{0-}, X_0) \notin J, 0 < T < R) < \infty$ *and for* $F \in \mathcal{V}$ *positive*

$$E^\mu(F \circ s_R \circ \theta_G \mid \mathcal{F}_{G-}, \mathcal{F}_{\geq D}) = {}^*H_2^{X_{G-}}(F \circ s_R \mid (X_{0-}, X_0) \notin J, 0 < T < R).$$

PROOF OF (ii). Let $N_\infty = \{x: {}^*H_2^x((X_{0-}, X_0) \notin J, 0 < T < R) = \infty\}$ then

$$1 \geq P^\mu(X_{G-} \in N_\infty, 0 < G < T < \zeta, X_{G-} = X_D)$$

$$= E^\mu \int_{[0,T]} 1_{N_\infty}(X_s) {}^*H_2^{X_s}((X_{0-}, X_0) \notin J, 0 < T < R)\, dL_s.$$

But the integrand on the right side is either zero or infinity, and therefore

$$P^\mu(X_{G-} \in N_\infty, 0 < G < T < \zeta, X_{G-} = X_D) = 0.$$

The same argument with $N_0 = \{x: {}^*H_2^x((X_{0-}, X_0) \notin J, 0 < T < R) = 0\}$ shows that $P^\mu(X_{G-} \in N_0, 0 < G < T < \zeta, X_{G-} = X_D) = 0$.

Let $F \in \mathcal{V}$, $Y \in \mathcal{F}$ be positive. Then

$${}^*H_2^x(F \circ s_R \cdot Y \circ \theta_R; (X_{0-}, X_0) \notin J, 0 < T < R < \infty)$$

$$= {}^*H_2^x(F \circ s_R; (X_{0-}, X_0) \notin J, 0 < T < R < \infty E^{X_R}(Y))$$

$$= E^x(Y) {}^*H_2^x(F \circ s_R; (X_{0-}, X_0) \notin J, 0 < T < R).$$

Note that for $x \in E$, $H_2^x(R = \infty) = 0$ because $X_R = \Delta$ on $\{R = \infty\}$. Now let

$(Z_t)_{t\geq 0}$ be a predictable process and $Y \in b\mathscr{F}^*_+$,

$$E^\mu(Z_G F \circ s_R \circ \theta_G \cdot Y \circ \theta_D; \Lambda \cap \{X_{G-} = X_D\})$$

$$= E^\mu\left(Z_G \frac{{}^*H_2^{X_{G-}}(F \circ s_R; (X_{0-}, X_0) \notin J, 0 < T < R)}{{}^*H_2^{X_{G-}}((X_{0-}, X_0) \notin J, 0 < T < R)} E^{X_{G-}}(Y); \Lambda \cap \{X_{G-} = X_D\}\right)$$

$$= E^\mu(Z_G {}^*H_2^{X_{G-}}(F \circ s_R \mid (X_{0-}, X_0) \notin J, 0 < T < R) E^{X_{G-}}(Y);\ \Lambda \cap \{X_{G-} = X_D\})$$

$$= E^\mu[Z_G {}^*H_2^{X_{G-}}(F \circ s_R \mid (X_{0-}, X_0) \notin J, 0 < T < R) E^{X_D}(Y);\ \Lambda \cap \{X_{G-} = X_D\}]$$

$$= E^\mu[Z_G {}^*H_2^{X_{G-}}(F \circ s_R \mid (X_{0-}, X_0) \notin J, 0 < T < R) Y(\theta_D);\ \Lambda \cap \{X_{G-} = X_D\}]$$

where the last equality follows by the strong Markov property at D.

REMARK 5.9. The same methods can now be applied to treat excursions that straddle $(\mathscr{F}_t)$ stopping times in the same manner it was done in [12]. As this is very similar to the treatment above and proofs in [12], we shall not elaborate on the matter.

APPENDIX

The objective of this appendix is to show that the measures $F_\Gamma(x, y)$ defined in Section 4, can be chosen so that $(x, y) \to F_\Gamma(x, y)$ are $\mathscr{E} \times \mathscr{E}$ measurable and when $F_0(x, y) > 0$, then $F_\Gamma^0(x, y)$ is the derivative of the measure μ_Γ^x with respect to λ^x, defined in Section 4.

We shall split the construction of F_Γ into two parts. We first deal with points $x \in F$.

Let h be the function defined in (4.2) and $f \in \mathscr{V}_+$. Define

$$C_t = \textstyle\sum_{s\leq t} 1_{\{Y_s \neq Y_{s-}\}} h(Y_{s-}, Y_s) \tag{A.1}$$

$$D_t^f = E^\cdot(\textstyle\sum_{s\leq t} 1_{\{Y_s \neq Y_{s-}\}} f(e_s) h(Y_{s-}, Y_s) \mid \mathscr{H}). \tag{A.2}$$

Let $\bar{C}_t$ and $\bar{D}_t^f$ be the dual predictable projections of C and D^f respectively (with respect to $(\mathscr{H}_t)$). Since Y is quasi-left continuous, both $\bar{C}$ and $\bar{D}^f$ are CAF's of Y, and for each t, $\bar{C}_t < \bar{D}_t^f$. Hence

$$\int_0^{L_t} 1_F(Y_s)\, d\bar{D}_s^f \leq \int_0^{L_t} 1_F(Y_s)\, d\bar{C}_s \tag{A.3}$$

and using the time change $s = L_t$

$$\int_0^t 1_F(X_s)\, d\bar{D}^f \circ L_s \leq \int_0^t 1_F(X_s)\, d\bar{C} \circ L_s. \tag{A.4}$$

Since the jumps of L are concentrated on $\{X \in F^c\}$ both sides of (A.4) define CAF's of X. (The condition of $1_F(X_s)$ further implies that they do not depend on $\{I_n\}$ defined in Section 2.) Since ξ is a reference measure for X, there exists a function $N^f \in \mathscr{E}_+$ so that

$$\text{(A.5)} \qquad \int_0^u 1_F(X_s)\, d\bar{D}^f \circ L_s = \int_0^u 1_F(X_s) N^f(X_s)\, d\bar{C} \circ L_s.$$

Therefore for $u = \tau_t$, and after a time change we obtain

$$\text{(A.6)} \qquad \int_0^{L_{\tau_t}} 1_F(Y_s)\, d\bar{D}_s^f = \int_0^{L_{\tau_t}} 1_F(Y_s) N^f(Y_s)\, d\bar{C}_s.$$

Now again, since all jumps of L are concentrated on $\{X \in F^c\}$, it follows that

$$\text{(A.7)} \qquad \int_0^t 1_F(Y_s)\, d\bar{D}_s^f = \int_0^{L_{\tau_t}} 1_F(Y_s)\, d\bar{D}_s^f = \int_0^t 1_F(Y_s) N^f(Y_s)\, d\bar{C}_s.$$

Let (B, K^0) be the Levy system for the jumps of Y. One may take here $B = \bar{C}$ defined above. Further using the argument that led to the fact that $1_F \cdot N^f \in \mathscr{E}$, it is easy to see that $1_F \cdot K^0$ is a kernel from $(E, \mathscr{E})$ into $(E, \mathscr{E})$. Let

$$\text{(A.8)} \qquad N = \{x: {}^*H^x((1 - e^{-R}); X_r \neq x) > 0\}, \quad \text{then} \quad N \in \mathscr{E}.$$

It follows from the proof of Theorem 4.2 of [1], (4.10), and reasoning similar to what has led to $N^f \in \mathscr{E}$, that there exists a function $f_1 \in \mathscr{E}$ so that

$$\text{(A.9)} \qquad \int_{[0,t)} 1_{N \cap F}(Y_s)\, ds = \int_{[0,t)} 1_F(Y_s) f_1(Y_s)\, dB_s$$

and that there exists a function $\bar{F}_f \in \mathscr{E} \times \mathscr{E}$ so that for each x, $\bar{F}_f(x, y)$ is a density of $1_F(x) f_1(x) {}^*H^x(f(s_R), X_R \in A - \{x\})$ with respect to $K^0(x, A)$. One may, of course, define $\bar{F}_f(x, y) = 0$ for all y if $f_1(x) = 0$. This, using a classical argument, defines a kernel $1_F(x) F_\Gamma(x, y)$ from $\mathscr{E} \times \mathscr{E}$ into $(V, \mathscr{V})$ such that for all $f \in \mathscr{V}$, $1_F(x) F_f(x, y) = 1_F(x) \bar{F}_f(x, y)$, K^0 a.e. Now if $1_F(x) F_0(x, y) = 1_F(x) F_{\{\beta > 0\}}(x, y) > 0$, (4.16) follows from the above, and the computations of Section 4.

We now turn to points in F^c. We start with the following.

LEMMA A.10. *$\{(t, \omega): X_{t-}(\omega) \in F^c, X_t(\omega) \in F\} \cap (M - M_r)$ is evanescent.*

PROOF. Since M_r is the set of right endpoints of intervals contiguous to M, and M is closed and optional, it follows from [7] that $M - M_r$ is predictable. Let $(A, \bar{K})$ be the Lévy system for the jumps of X, and $h \in \mathscr{E}_+$ then

$$E^\bullet \sum_s e^{-s} 1_{M - M_r}(s) 1_{\{X_{s-} \in F^c, X_s \in F\}} h(X_s)$$
$$= E^\bullet \int_0^\infty e^{-s} 1_{F^c}(X_s) 1_{M - M_r}(s) \int_F \bar{K}(X_s, dy) h(y)\, dA_s = 0$$

because A is a CAF and $\{(s, \omega): X_s(\omega) \in F^c, s \in M(\omega) - M_r(\omega)\}$ has countable sections.

Having established this lemma, it follows that all jumps of Y from $x \in F^c$ are "caused" by excursions. Further, since points in F^c, are holding points for Y it follows from the construction of F_Γ in [1] that in this case

$$\text{(A.11)}\quad \begin{array}{l} 1_{F^c}(x)F_\Gamma(x, y) \text{ is the derivative of } 1_{F^c}(x)E^x(s_R \in \Gamma, X_R \in A - \{x\}) \\ \text{with respect to } 1_{F^c}(x)E^x(X_R \in A - \{x\}). \end{array}$$

The fact that $(x, y) \to 1_{F^c}(x)F_\Gamma(x, y)$ are $\mathscr{E} \times \mathscr{E}$ measurable follows from the fact that for $\Gamma \in \mathscr{Y}$, $g \in \mathscr{E}_+$, $x \to E^x(s_R \in \Gamma; g(X_R))$ is $\mathscr{E}$ measurable, Doob's lemma and the fact that $F^c \in \mathscr{E}$. Formula (4.16) in this case is a direct consequence of (A.11) and the results of Section 3.

Recalling Remark 4.17

$$\{(s, \omega): Y_{s-}(\omega) \neq Y_s(\omega), {}^*q(Y_s(\omega), Y_{s-}(\omega)) = 0 \text{ or } \infty, F_0(Y_{s-}(\omega), Y_s(\omega)) > 0\}$$

is evanescent, hence by redefining $F_0(x, y) = F_\Gamma(x, y) = 0$ for all (x, y) with ${}^*q(y, x) = 0$ or ∞, we do not affect either the $\mathscr{E} \times \mathscr{E}$ measurability of F_Γ or any of the above.

Acknowledgment. I wish to thank Prof. B. Maisonneuve for many helpful comments on presentation and content of an earlier version of this work.

REFERENCES

[1] BENVENISTE, A. and JACOD, J. (1973). Systemes de Lévy des processus de Markov. *Invent. Math.* **21** 183–198.

[2] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). Markov Processes and Potential Theory. Academic, New York.

[3] CANETTI, J. and EL-KAROUI, N. (1977). Processus de Markov associe a un fonctionelle additive gauche. Preprint.

[4] ÇINLAR, E. (1972). Markov additive processes II. *Z. Wahrsch. verw. Gebiete* **24** 95–121.

[5] ÇINLAR, E., JACOD, J., PROTTER, P. and SHARPE, M. J. (1980). Semimartingales and Markov processes. *Z. Wahrsch. verw. Gebiete* **54** 161–219.

[6] ÇINLAR, E. (1980). Jump structure of Markov processes. Preprint.

[7] DELLACHERIE, C. (1972). *Capacités et Processus Stochastiques.* Springer-Verlag. New York.

[8] FITZSIMMONS, P. J. (1983). On the excursions of Markov processes in classical duality. Preprint.

[9] GETOOR, R. K. (1975). "Markov Processes, Ray Processes, and Right Processes". *Lecture Notes in Mathematics.* No. 440. Springer-Verlag. Berlin.

[10] GETOOR, R. K. and SHARPE, M. J. (1973). Last exit decompositions and distributions. *Indiana Univ. Math. J.* **23** 377–404.

[11] GETOOR, R. K. and SHARPE, M. J. (1982). Excursions of dual processes. *Adv. in Math.* Vol. *45*, No. 3, 259–309.

[12] GETOOR, R. K. and SHARPE, M. J. (1981). Two results on dual excursions. *Seminar on Stochastic Processes.* Birkhäuser, Boston, 31–52.

[13] GETOOR, R. K. and SHARPE, M. J. (1983). Naturality, standardness and weak duality for Markov processes. *Z. Wahrsch. verw. Gebiete* **67** 1–62.

[14] GLOVER, J. (1983). Discontinuous time changes of semiregenerative processes and balayage theorems. Preprint.

[15] JACOBS, P. A. (1978). Excursions of a Markov process induced by continuous additive functionals. *Z. Wahrsch. verw. Gebiete* **44** 325–336.

[16] KASPI, H. (1983). Excursions of Markov processes: an approach via Markov additive processes. *Z. Wahrsch. verw. Gebiete* **64** 251–268.

[17] KASPI, H. (1984). On invariant measures and dual excursions of Markov processes. *Z. Wahrsch. verw. Gebiete.* **66** 185–204.

[18] MAISSONNEUVE, B. (1974). Systêmes Régêneratifs. Asterisque No. 15, S.M.F.
[19] MAISSONNEUVE, B. (1975). Exit systems. *Ann. Probab.* **3** 399–411.
[20] MAISSONNEUVE, B. (1979). On the structure of certain excursions of a Markov process. *Z. Wahrsch. verw. Gebiete* **47** 61–67.
[21] MITRO, J. B. (1979). Dual Markov processes: construction of a useful auxiliary process. *Z. Wahrsch. verw. Gebiete* **47** 139–156.
[22] MITRO, J. B. (1979). Dual Markov processes: applications of a useful auxiliary process. *Z. Wahrsch. verw. Gebiete* **48** 97–114.
[23] MITRO, J. B. (1982). Exit systems for dual Markov processes. *Z. Wahrsch. verw. Gebiete* **66** 259–267.
[24] PITMAN, J. W. (1982). Levy systems and path decompositions. *Seminar on Stochastic Processes*, 79–110 (1981). Birkhäuser, Boston.
[25] WALSH, J. B. (1972). Markov processes and their functionals in duality. *Z. Wahrsch. verw. Gebiete* **24** 229–246.

FACULTY OF INDUSTRIAL
ENGINEERING AND MANAGEMENT
TECHNION
HAIFA 32000 ISRAEL