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AN UPPER BOUND ON THE CRITICAL PERCOLATION PROBABILITY FOR THE THREE-DIMENSIONAL CUBIC LATTICE

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We prove that the critical probability for site percolation on the three-dimensional cubic lattice satisfies the inequality $p_c^{(3)} < \frac{1}{2}$. An application to the three-dimensional Ising model is given.

1. Introduction. Considerable progress has been obtained in the last few years on the problem of two-dimensional percolation. For the case of Bernoulli measures most of the recent results are contained in [8]. Percolation has also been studied as a tool for the understanding of the two-dimensional Ising model ([9], [1], [5]) and other two-dimensional models in statistical mechanics. Very few rigorous results, on the other hand, have been obtained for more than two dimensions (see [2]).

The aim of this paper is a rigorous proof of the bound $p_c^{(3)} < \frac{1}{2}$ for the critical percolation probability in the site problem on the three-dimensional cubic lattice. This result was expected on the basis of numerical investigations, but we think it is interesting for the following reasons:

1. This bound proves that at least in a small interval surrounding ½ there is, for $d = 3$, coexistence of infinite clusters both of "occupied" and "empty" sites. This phenomenon, which is an essential qualitative difference between two-dimensional and higher dimensional percolation, was a long-standing conjecture since Harris proved his Theorem ([4]).

2. Our result can be easily extended to the three-dimensional Ising model in order to prove that at low external field and large temperature there is coexistence of infinite clusters of both types. At low external field, coexistence of infinite clusters of both types is expected to persist even below the critical temperature (see for example [3]). It is believed that possible extensions of this result in the region $h = 0$, $\beta > \beta_c$, could be useful to understand the "roughening transition" ([2], [3]).

Our proof is based on two points. The first point, exposed in Section 2, is essentially a simple remark concerning the comparison of percolation on a given graph $\mathscr{G}$ and on a graph obtained from $\mathscr{G}$ by identification of sites. This remark, in particular, can be applied to the comparison between the triangular planar

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graph $\mathscr{T}$ (for which the critical percolation probability is known ([8]) to be ½) and the three-dimensional cubic graph $\mathscr{S}$ and it easily yields the weak inequality $p_c^{(3)} \leq ½$.

In order to prove the strict inequality $p_c^{(3)} < ½$ we use a technique recently devised by H. Kesten ([8], see in particular Chapter 10) which can be successfully used in many cases in the proof of strict inequalities between the critical percolation probabilities of pairs of planar graphs. Unfortunately Kesten's method (as almost all of the techniques so far developed) seems to be essentially confined to planar graphs. In our particular case we can overcome this difficulty by introducing (in Section 3) a third graph $\mathscr{T}'$ (nonplanar but still essentially two-dimensional) chosen in such a way that the method of Section 2 is still applicable to the pair $(\mathscr{T}', \mathscr{S})$ and Kesten's technique can be extended to the pair $(\mathscr{T}, \mathscr{T}')$. This last point is exposed in Section 4. Section 5 contains the extension to the three-dimensional Ising model.

2. Notation, definitions and proof of the inequality $p_c^{(3)} \leq ½$. Given a measurable space Ω, i.e. a set endowed with a σ-algebra, we shall denote by $M_1(\Omega)$ the set of the probability measures on Ω. For $\mu \in M_1(\Omega)$ $E_\mu \xi$ will denote the expectation of the random variable ξ with respect to μ. Given two measurable spaces Ω_1 and Ω_2 and a measurable map $\chi\colon \Omega_1 \to \Omega_2$, χ^* will denote the map $\chi^*\colon M_1(\Omega_1) \to M_1(\Omega_2)$ defined by $(\chi^*\mu)(A) = \mu(\chi^{-1}(A))$ for every measurable $A \subset \Omega_2$.

Let Λ be a finite or countable set. We shall consider the set $\Omega_\Lambda = \{-1, 1\}^\Lambda$ as a configuration space on Λ. For $\omega \in \Omega_\Lambda$ and $x \in \Lambda$, ω_x will denote the value taken by ω at the site x of Λ. If $\Lambda_1 \subset \Lambda$ and $\omega \in \Omega_\Lambda$, $\omega\,|_{\Lambda_1}$ is the configuration of Ω_{Λ_1} obtained by restricting ω to Λ_1. Given a finite subset C of Λ, the positive cylinder set corresponding to C is defined by

$$E_+(C) = \{\omega \in \Omega_\Lambda \mid \forall x \in C \;\; \omega_x = 1\}. \tag{2.1}$$

Ω_Λ will always be considered as a measurable space with the σ-algebra $\mathscr{B}_\Lambda$ generated by the sets $E_+(C)$, as C varies among the finite subsets of Λ. If $\Lambda_1 \subset \Lambda$ the sub σ-algebra of $\mathscr{B}_\Lambda$ generated by the sets $E_+(C)$ with $C \subset \Lambda_1$ will be denoted by $\mathscr{B}_{\Lambda_1}^{(\Lambda)}$. The Bernoulli probability measure $\mu_p^{(\Lambda)} \in M_1(\Omega_\Lambda)$ is such that the random variables ω_x, $x \in \Lambda$, are independent and identically distributed w.r.t. $\mu_p^{(\Lambda)}$ and $\mu_p^{(\Lambda)}(E_+(\{x\})) = p$, $\forall x \in \Lambda$.

We shall usually consider Λ as the support of some graph $\mathscr{L} = (\Lambda, \tilde{\Lambda})$, where $\tilde{\Lambda}$ is a family of subsets of Λ, each containing two elements. $\tilde{\Lambda}$ defines a notion of connection between points of Λ: two points $x, y \in \Lambda$, $x \neq y$, are connected if $\{x, y\} \in \tilde{\Lambda}$. Given a subset A of Λ the internal and external boundaries of A are defined by

$$\partial^i A = \{x \in A \mid \exists y \in \Lambda \backslash A \text{ such that } \{x, y\} \in \tilde{\Lambda}\}, \quad \partial^e A = \partial^i(\Lambda \backslash A). \tag{2.2}$$

A chain $c = (x^{(i)})_{i \in I}$ is a finite or infinite sequence of points of Λ with $I = \{0, \cdots, n\}$ or $I = \mathbb{N}$ such that $x^{(i-1)}$ and $x^{(i)}$ are connected for $i \in I \backslash \{0\}$. The support of c is the set $\operatorname{supp}(c) = \bigcup_{i \in I} \{x^{(i)}\}$. The chain c is called self-avoiding if $x^{(i)} \neq x^{(j)}$ for $i \neq j$. Given $\omega \in \Omega_\Lambda$ and a chain c, we say that c is a +chain (respectively a $-$chain) in ω if $\forall x \in \text{supp}(c)$ $\omega_x = 1$ (resp. $\omega_x = -1$). The union $W_+(x)$ (resp. $W_-(x)$) of the supports of all the +chains (resp. $-$chains) starting from x in the configuration ω is called the +cluster (resp. the $-$cluster) to which x belongs in the configuration ω. $W_+(x)$ and $W_-(x)$ are sets depending on $\omega \in \Omega_\Lambda$. They are empty if $\omega_x = -1$, $\omega_x = 1$ respectively. It is clear that in the definition of $W_+(x)$ and $W_-(x)$ it is enough to consider self-avoiding chains.

We define now some measures which can be thought of as Bernoulli measures "with identification of some points". Theorem 2.1 will allow us to compare the probabilities of some increasing events evaluated with two such measures. Let $\mathscr{F}$ be a partition of Λ, π be the canonical map $\pi: \Lambda \to \mathscr{F}$, $\pi(x) = F$ if $x \in F$ and χ be the map $\chi: \Omega_{\mathscr{F}} \to \Omega_\Lambda$, $(\chi\omega)_x = \omega_{\pi(x)}$. We define $\mu_{p,\mathscr{F}}^{(\Lambda)} \in M_1(\Omega_\Lambda)$ by

$$\mu_{p,\mathscr{F}}^{(\Lambda)} = \chi^* \mu_p^{(\mathscr{F})}. \tag{2.3}$$

In the following we shall omit the superscript Λ of $\mu_p^{(\Lambda)}$, $\mu_{p,\mathscr{F}}^{(\Lambda)}$ when no confusion can arise. $\#(A)$ will denote the number of elements belonging to the set A.

We have the following

THEOREM 2.1. *Let Λ be finite or countable, $\mathscr{F}$, $\mathscr{F}'$ two partitions of Λ with $\mathscr{F}'$ finer than $\mathscr{F}$. Let $\mathscr{C}$ be a finite family of finite subsets of Λ such that*

$$\forall F \in \mathscr{F}, \quad \forall C \in \mathscr{C} \quad \text{the set } F \cap C \text{ intersects at most one element of } \mathscr{F}'. \tag{2.4}$$

Then

$$\mu_{p,\mathscr{F}}^{(\Lambda)}(\cup_{C\in\mathscr{C}} E_+(C)) \le \mu_{p,\mathscr{F}'}^{(\Lambda)}(\cup_{C\in\mathscr{C}} E_+(C)). \tag{2.5}$$

If $\#(F \cap C) \le 1$ $\forall F \in \mathscr{F}$, $\forall C \in \mathscr{C}$, then

$$\mu_{p,\mathscr{F}}^{(\Lambda)}(\cup_{C\in\mathscr{C}} E_+(C)) \le \mu_p^{(\Lambda)}(\cup_{C\in\mathscr{C}} E_+(C)). \tag{2.6}$$

PROOF. It is clear that, even if Λ is infinite, the only relevant set is the finite set $\Lambda' = \cup_{C\in\mathscr{C}} C$ with the partitions induced in it by $\mathscr{F}$ and $\mathscr{F}'$. So we can assume that Λ is finite. We can also assume that $\mathscr{F}'$ is obtained from $\mathscr{F}$ by splitting some element $F \in \mathscr{F}$ into two nonempty subsets, say F_1 and F_2, because any partition of Λ finer than $\mathscr{F}$ can be obtained by iterating this procedure a finite number of times and condition (2.4) is preserved at each step.

Let us put

$$\begin{aligned} A &= \cup_{C\in\mathscr{C}} E_+(C), \quad A_0 = \cup_{C\in\mathscr{C};C\cap(F_1\cup F_2)=\varnothing} E_+(C), \\ A_i &= \cup_{C\in\mathscr{C};C\cap F_i\neq\varnothing} E_+(C) \quad \text{for} \quad i = 1, 2. \end{aligned} \tag{2.7}$$

A is the union of A_0, A_1 and A_2. So we have

$$\begin{aligned} \mu_{p,\mathscr{F}}(A) = \mu_{p,\mathscr{F}}(A_0) &+ \mu_{p,\mathscr{F}}(A_1) + \mu_{p,\mathscr{F}}(A_2) \\ &- \mu_{p,\mathscr{F}}(A_0 \cap A_1) - \mu_{p,\mathscr{F}}(A_0 \cap A_2) - \mu_{p,\mathscr{F}}((A_1 \cap A_2)\setminus A_0) \end{aligned} \tag{2.8}$$

and a similar expression for $\mu_{p,\mathscr{F}'}(A)$. The definition of $\mu_{p,\mathscr{F}}$ and $\mu_{p,\mathscr{F}'}$ implies, as it is easy to see, that

$$\mu_{p,\mathscr{F}}(E) = \mu_{p,\mathscr{F}'}(E) \quad \forall E \in \mathscr{B}^{(\Lambda)}_{\Lambda\setminus F_1} \cup \mathscr{B}^{(\Lambda)}_{\Lambda\setminus F_2}. \tag{2.9}$$

It is clear that $A_0 \in \mathscr{B}^{(\Lambda)}_{\Lambda\setminus F_1} \cup \mathscr{B}^{(\Lambda)}_{\Lambda\setminus F_2}$. On the other hand the condition (2.4) implies that $A_1, A_0 \cap A_1 \in \mathscr{B}^{(\Lambda)}_{\Lambda\setminus F_2}$ and $A_2, A_0 \cap A_2 \in \mathscr{B}^{(\Lambda)}_{\Lambda\setminus F_1}$. Hence the first five terms on the r.h.s. of equation (2.8) are unchanged if we replace the measure $\mu_{p,\mathscr{F}}$ with $\mu_{p,\mathscr{F}'}$. Let us consider the last term on the r.h.s. of (2.8). The following formulae are a straightforward consequence of the definition of $\mu_{p,\mathscr{F}}$ and $\mu_{p,\mathscr{F}'}$:

$$\mu_{p,\mathscr{F}}((A_1 \cap A_2)\setminus A_0) = p\mu_{p,\mathscr{F}'}((A_1 \cap A_2)\setminus A_0 \mid G), \tag{2.10}$$

$$\mu_{p,\mathscr{F}'}((A_1 \cap A_2)\setminus A_0) = p^2\mu_{p,\mathscr{F}'}((A_1 \cap A_2)\setminus A_0 \mid G), \tag{2.11}$$

where $G = \{\omega \in \Omega_\Lambda \mid \forall x \in F, \omega_x = 1\}$. (2.10) and (2.11) imply that

$$\mu_{p,\mathscr{F}}((A_1 \cap A_2)\setminus A_0) \geq \mu_{p,\mathscr{F}'}((A_1 \cap A_2)\setminus A_0). \tag{2.12}$$

So that (2.5) is proved. (2.6) is a particular case of (2.5) when $\mathscr{F}'$ is the total partition of Λ i.e. $\#(F) = 1$, $\forall F \in \mathscr{F}'$. Indeed in this case $\mu^{(\Lambda)}_{p,\mathscr{F}'} = \mu^{(\Lambda)}_p$. □

We exhibit now a first easy application of Theorem 2.1, by obtaining the weak inequality $p_c^{(3)} \leq 1/2$ for the critical probability for site percolation on the three-dimensional cubic graph $\mathscr{G}$. $\mathscr{G}$ is defined by introducing in $\mathbb{Z}^3$ the following notion of connection: $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are said to be connected if

$$|x - y| = \textstyle\sum_{i=1}^3 |x_i - y_i| = 1. \tag{2.13}$$

The triangular graph $\mathscr{T}$ can be defined by introducing in $\mathbb{Z}^2$ the following notion of connection: $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are said to be connected if

$$\begin{aligned} &\text{either} \quad |x - y| = |x_1 - y_1| + |x_2 - y_2| = 1 \\ &\text{or} \qquad x_1 - y_1 = x_2 - y_2 = \pm 1. \end{aligned} \tag{2.14}$$

We can think of $\mathscr{T}$ as obtained from $\mathscr{G}$ via the map

$$\phi\colon \mathbb{Z}^3 \to \mathbb{Z}^2, \quad \phi(x_1, x_2, x_3) = (x_1 - x_3, x_2 - x_3).$$

Indeed it is easy to see that two points $x, y \in \mathbb{Z}^2$ satisfy (2.14) if and only if there are $x', y' \in \mathbb{Z}^3$ such that $x = \phi(x')$, $y = \phi(y')$ and $|x' - y'| = 1$, i.e. x' and y' are connected in $\mathscr{G}$.

In the future we shall exploit the following easy lemma:

LEMMA 2.2. *For every self-avoiding chain c in $\mathscr{T}$ with the origin of $\mathbb{Z}^2$ as initial point there exists a unique self-avoiding chain $d(c)$ in $\mathscr{G}$ with the origin of $\mathbb{Z}^3$ as initial point such that*:

$$\phi(\operatorname{supp}(d(c))) = \operatorname{supp}(c). \tag{2.15}$$

Furthermore $\phi\,|_{\operatorname{supp}(d(c))}$ is a bijection beween $\operatorname{supp}(d(c))$ *and* $\operatorname{supp}(c)$.

PROOF. If $c = (x^{(i)})_{i\in I}$, then the chain $d(c) = (x'^{(i)})_{i\in I}$ is obtained by induction by putting $x'^{(0)} = (0, 0, 0)$ and

$$x'^{(i+1)} = (x_1'^{(i)} \pm 1, x_2'^{(i)}, x_3'^{(i)}) \quad \text{if} \quad x^{(i+1)} = (x_1^{(i)} \pm 1, x_2),$$

$$x'^{(i+1)} = (x_1'^{(i)}, x_2'^{(i)} \pm 1, x_3'^{(i)}) \quad \text{if} \quad x^{(i+1)} = (x_1^{(i)}, x_2^{(i)} \pm 1)$$

and

$$x'^{(i+1)} = (x_1'^{(i)}, x_2'^{(i)}, x_3'^{(i)} \mp 1) \quad \text{if} \quad x^{(i+1)} = (x_1^{(i)} \pm 1, x_2^{(i)} \pm 1).$$

It is easy to check that $d(c)$ has all the required properties. □

Now we are in the condition to prove the following:

THEOREM 2.3. *Let $p_c^{(3)}$ be the critical probability for site percolation on the cubic lattice*:

$$p_c^{(3)} = \inf\{p \mid \mu_p(\#(W_+(0)) = \infty) > 0\}. \tag{2.16}$$

Then

$$p_c^{(3)} \leq 1/2. \tag{2.17}$$

PROOF. (2.17) is equivalent to the fact that for any $p > 1/2$ the μ_p-probability that the origin is connected to the internal boundary of the cube $\Lambda'(L) = [-L, L]^3$ by a +chain is bounded from below by a strictly positive constant a uniformly in L. It is known (see [8] Chapter 3, Section 3, Application i) that the critical probability for site percolation on the triangular lattice $\mathscr{T}$ is $1/2$. Let us consider the square $\Lambda(2L) = [-2L, 2L]^2$ in $\mathbb{Z}^2$ and let $\mathscr{C}_{2L}$ be the family of all the self-avoiding chains of $\mathscr{T}$ in $\Lambda(2L)$ starting from the origin and ending on $\partial^i\Lambda(2L)$. Let $\mathscr{C}'_{2L}$ be the family of all the chains $d(c)$ for $c \in \mathscr{C}_{2L}$, where d is the map constructed in Lemma 2.2. It is easy to see that each chain of $\mathscr{C}'_{2L}$ connects the origin with $\partial^i\Lambda'(L)$, since it must end either on $\partial^i\Lambda'(L)$ or on $\mathbb{Z}^3\backslash\Lambda'(L)$.

Let $\mathscr{F}$ be the partition of $\mathbb{Z}^3$ induced by the map $\phi\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ (two points of $\mathbb{Z}^3$ belong to the same element of $\mathscr{F}$ iff their image under ϕ is equal). The definition of $\mathscr{C}'_{2L}$ and Lemma 2.2 imply that if $c \in \mathscr{C}'_{2L}$ and $F \in \mathscr{F}$ then $\#(\mathrm{supp}(c) \cap F) \leq 1$ so that by Theorem 2.1:

$$\mu_p^{(\mathbb{Z}^3)}(\cup_{c\in\mathscr{C}'_{2L}} E_+(\mathrm{supp}(c))) \geq \mu_{p,\mathscr{F}}^{(\mathbb{Z}^3)}(\cup_{c\in\mathscr{C}'_{2L}} E_+(\mathrm{supp}(c))). \tag{2.18}$$

On the other hand, by the definitions of the partition $\mathscr{F}$ and of $\mathscr{C}'_{2L}$

$$\mu_{p,\mathscr{F}}^{(\mathbb{Z}^3)}(\cup_{c\in\mathscr{C}'_{2L}} E_+(\mathrm{supp}(c))) = \mu_p^{(\mathbb{Z}^2)}(\cup_{c\in\mathscr{C}_{2L}} E_+(\mathrm{supp}(c))) \geq a \tag{2.19}$$

for some positive a independent of L, since the critical probability on $\mathscr{T}$ is $1/2$. But if there is a +chain of $\mathscr{C}'_{2L}$ then the origin is connected by a +chain to $\partial^i\Lambda'(L)$, so that (2.18) and (2.19) imply the Theorem. □

3. Percolation in some intermediate graphs between $\mathscr{T}$ and $\mathscr{G}$. We consider the graph $\mathscr{T}'$ defined by introducing on the set $K = \mathbb{Z}^2 \times \{0, 1\}$ the following notion of connection: $x = (x_1, x_2, \varepsilon)$ and $y = (y_1, y_2, \varepsilon')$ are connected iff

$$\begin{aligned}&\text{either} \quad |x_1 - y_1| + |x_2 - y_2| = 1 \quad \text{and} \quad \varepsilon = \varepsilon' \\ &\text{or} \quad x_1 - y_1 = x_2 - y_2 = \pm 1 \quad \text{and} \quad \varepsilon = 1 - \varepsilon'.\end{aligned} \tag{3.1}$$

Similarly to the case of $\mathscr{T}$, we can think of $\mathscr{T}'$ as the image of $\mathscr{G}$ via the map $\phi' : \mathbb{Z}^3 \to K$ defined by

$$\phi'(x_1, x_2, x_3) = \begin{cases} (x_1 - x_3, x_2 - x_3, 0) & \text{if } x_3 \text{ is even} \\ (x_1 - x_3, x_2 - x_3, 1) & \text{if } x_3 \text{ is odd.} \end{cases} \tag{3.2}$$

Two points $x, y \in K$ satisfy (3.1) if and only if there are x', y' such that x' and y' are $\mathscr{G}$-connected and $x = \phi'(x')$, $y = \phi'(y')$.

We need some more definitions. Given an integer q (we shall take $q \geq 3$), we put

$$T_q = \{(x_1, x_2) \in \mathbb{Z}^2 \mid x_1 = (2k_1 + 1)q,\ x_2 = (2k_2 + 1)q,\ k_1, k_2 \in \mathbb{Z}\}. \tag{3.3}$$

For $x \in \mathbb{Z}^2$ we define the sets

$$F_x = F_{(x_1,x_2)} = \{(x_1, x_2, 0), (x_1, x_2, 1)\} \tag{3.4}$$

and

$$K_q = \cup_{x \in T_q} F_x. \tag{3.5}$$

Given $x = (x_1, x_2) \in \mathbb{Z}^2$, χ_x will be the indicator of the event $E(x)$ of $\mathscr{B}_K$

$$E(x) = \{\omega \in \Omega_K \mid \omega_y = 1,\ \forall y \in \tilde{N}_x\} \tag{3.6}$$

where

$$N_x = \{(x_1 + 1, x_2), (x_1 + 1, x_2 + 1), (x_1 - 1, x_2), (x_1 - 1, x_2 - 1)\},$$

$$\tilde{N}_x = N_x \times \{0, 1\}.$$

For $0 \leq p \leq 1$ we define the probability measures $\tilde{\mu}_{p,i}$, $i = 0, \cdots, 3$ by

$$\tilde{\mu}_{p,0} = \mu_p^{(K)}, \tag{3.7}$$

$$\tilde{\mu}_{p,1} = \mu_{p,\mathscr{F}_1}^{(K)}, \tag{3.8}$$

where $\mathscr{F}_1 = \{F_x \mid x \in \mathbb{Z}^2 \backslash T_q\} \cup \{\{y\} \mid y \in K_q\}$.

$$\tilde{\mu}_{p,2}(A) = \tilde{\mu}_{p,1}(A) \quad \text{for} \quad A \subset \Omega_K, \quad A \in \mathscr{B}_{K \backslash K_q}^{(K)} \tag{3.9i}$$

$$\tilde{\mu}_{p,2}(\ \mid \mathscr{B}_{K\backslash K_q}^{(K)}) \mid_{\Omega_{K_q}} = \mu_{p,\mathscr{F}(\omega)}^{(K_q)}, \tag{3.9ii}$$

where the partition $\mathscr{F}(\omega)$ is defined by

$$\mathscr{F}(\omega) = \{F_x \mid x \in T_q,\ \chi_x(\omega) = 0\} \cup \{\{y\} \mid y \in F_x,\ x \in T_q,\ \chi_x(\omega) = 1\}.$$

Note that the functions χ_x for $x \in T_q$ are measurable with respect to $\mathscr{B}_{K\backslash K_q}^{(K)}$, so that (3.9ii) makes sense.

$$\tilde{\mu}_{p,3} = \mu_{p,\mathscr{F}_3}^{(K)} \quad \text{where} \quad \mathscr{F}_3 = \{F_x \mid x \in \mathbb{Z}^2\}. \tag{3.10}$$

A chain $c = (y^{(i)})_{i \in I}$ in $\mathscr{T}'$ is called totally self-avoiding (t.s.a.) if for $i, j \in I$, $i \neq j$, $y^{(i)}$ and $y^{(j)}$ belong to different elements of the partition $\mathscr{F}_3$.

Let E be the event

$$E = \{\omega \in \Omega_K \mid \text{in } \omega \text{ there exists an infinite totally self-avoiding +chain of } \mathscr{T}' \text{ starting from } (0, 0, 0)\}.$$

We define the critical percolation probabilities $\tilde{p}_{c,i}$ $i = 0, \cdots, 3$ as

$$\tilde{p}_{c,i} = \inf\{p \mid \tilde{\mu}_{p,i}(E) > 0\}. \tag{3.11}$$

We have the following:

THEOREM 3.1. *The critical percolation probabilities satisfy the inequalities*

$$p_c^{(3)} \leq \tilde{p}_{c,0}, \quad \tilde{p}_{c,i} \leq \tilde{p}_{c,i+1} \quad \text{for} \quad i = 0, 1, 2. \tag{3.12}$$

PROOF. The proof of the inequalities $p_c^{(3)} \leq \tilde{p}_{c,0}$ and $\tilde{p}_{c,p} \leq \tilde{p}_{c,1}$ is analogous to the proof of Theorem 2.3. We only need the remark that Lemma 2.2 still holds (in particular for t.s.a. chains) if we replace $\mathbb{Z}^2$, $\mathscr{T}$, ϕ respectively with K, $\mathscr{T}'$, ϕ'. Furthermore, since the definitions of $\tilde{p}_{c,i}$ involve t.s.a. chains, Theorem 2.1 can be applied here in the same way as in the proof of Theorem 2.3.

Also the inequalities $\tilde{p}_{c,1} \leq \tilde{p}_{c,2}$ and $\tilde{p}_{c,2} \leq \tilde{p}_{c,3}$ can be obtained using the method of the proof of Theorem 2.3. One observes that the measures $\tilde{\mu}_{p,1}$, $\tilde{\mu}_{p,2}$ and $\tilde{\mu}_{p,3}$ are equal when restricted to the σ-algebra $\mathscr{B}^{(K)}_{K \setminus K_q}$. On the other hand for each condition in $\Omega_{K \setminus K_q}$, the restrictions of their conditional measures to Ω_{K_q} can be written as $\mu^{(K_q)}_{p,\tilde{\mathscr{F}}_i}$, $i = 1, 2, 3$, where $\tilde{\mathscr{F}}_1$ is the total partition of K_q, $\tilde{\mathscr{F}}_2 = \mathscr{F}(\omega)$ (see 3.9ii) and $\tilde{\mathscr{F}}_3$ is the partition of K_q into the sets F_x, $x \in T_q$. For any condition in $\Omega_{K \setminus K_q}$, $\tilde{\mathscr{F}}_1$ is finer than $\tilde{\mathscr{F}}_2$ and $\tilde{\mathscr{F}}_2$ is finer than $\tilde{\mathscr{F}}_3$ so that the method of the proof of Theorem 2.3 applies to the conditional measures for any condition. The result is then readily obtained by taking the average with respect to the common restriction of the three measures to $\Omega_{K \setminus K_q}$. □

From now on we shall concentrate our attention on the measures $\tilde{\mu}_{p,2}$ and $\tilde{\mu}_{p,3}$. We shall reduce the problem to one involving the planar graph $\mathscr{T}$. First we need some more definitions.

We define the map ψ: $\Omega_K \to \Omega_{\mathbb{Z}^2}$ by

$$(\psi\omega)_{(x_1,x_2)} = \max(\omega_{(x_1,x_0,0)}, \omega_{(x_1,x_2,1)}). \tag{3.13}$$

Let $\tilde{\Lambda}(L)$ and $\Lambda(L)$ be the sets:

$$\tilde{\Lambda}(L) = \{(x_1, x_2, \varepsilon) \in K \mid |x_1| \leq L, |x_2| \leq L\},$$

$$\Lambda(L) = \{(x_1, x_2) \in \mathbb{Z}^2 \mid |x_1| \leq L, |x_2| \leq L\}.$$

$\tilde{\tau}_L$ (resp. τ_L) is the event that there exists a t.s.a. +chain of $\mathscr{T}'$ in $\tilde{\Lambda}(L)$ (resp. a self-avoiding +chain of $\mathscr{T}$ in $\Lambda(L)$) starting from the origin (0, 0, 0) of K (resp. from the origin (0, 0) of $\mathbb{Z}^2$) and ending on $\partial^i\tilde{\Lambda}(L)$ (resp. $\partial^i\Lambda(L)$). We explicitly remark that the word "self-avoiding" can be dropped in the definition of τ_L.

LEMMA 3.2. *Let L be such that*

$$\partial^i\tilde{\Lambda}(L) \cap K_q = \varnothing. \tag{3.14}$$

Then

$$\tilde{\mu}_{p,2}(\tilde{\tau}_L) = (\psi^*\tilde{\mu}_{p,2})(\tau_L), \tag{3.15}$$

$$\tilde{\mu}_{p,3}(\tilde{\tau}_L) = (\psi^*\tilde{\mu}_{p,3})(\tau_L) = \mu_p^{(\mathbb{Z}^2)}(\tau_L). \tag{3.16}$$

PROOF. The proof of (3.16) is immediate. Indeed the definition of $\tilde{\mu}_{p,3}$ implies that $\psi^*\tilde{\mu}_{p,3} = \mu_p^{(\mathbb{Z}^2)}$. On the other hand for the measure $\tilde{\mu}_{p,3}$, $\omega_{(x_1,x_2,0)} = \omega_{(x_1,x_2,1)}$ for every $(x_1, x_2) \in \mathbb{Z}^2$ with probability 1. If this condition is verified in the configuration $\omega \in \Omega_K$, then it is easy to verify that in ω there exists a t.s.a. +chain connecting the origin to $\partial^i\tilde{\Lambda}(L)$ if and only if a self-avoiding +chain of $\mathscr{T}$ connects the origin to $\partial^i\Lambda(L)$ in the configuration $\psi\omega \in \Omega_{\mathbb{Z}^2}$. This fact implies (3.16).

In order to prove (3.15), we shall show that for $\tilde{\mu}_{p,2}$-a.e. ω if in $\psi\omega$ a self-avoiding +chain connects the origin to $\partial^i\Lambda(L)$, then a t.s.a. +chain in ω connects the origin to $\partial^i\tilde{\Lambda}(L)$ (the other implication is trivial).

Let

$$T(\omega) = \{x \in T_q \mid (\psi\omega)_x = 1,\ \chi_x(\omega) = 1\}.$$

We shall denote by $D(\omega)$ the set of all the chains $c = (y^{(i)})_{i\in I}$ of $\mathscr{T}$ such that for any $x \in T(\omega)$ the set

$$\{i \in I \mid y^{(i)} \in \{x\} \cup N_x\}$$

is an interval of consecutive integers.

We remark that if for a given configuration $\omega \in \Omega_K$ there is a self-avoiding +chain $c = (y^{(i)})$ of $\mathscr{T}$ in $\psi\omega$ joining (0, 0) to $\partial^i\Lambda(L)$, then for the same configuration ω there is also a chain $c' = (y'^{(i)}) \in D(\omega)$ with the same property. Indeed, since for any $x \in T(\omega)$ the set $\{x\} \cup N_x$ is a connected set included in $(\psi\omega)^{-1}(1)$, the chain c' can be easily constructed by induction on $x \in T(\omega)$ by cutting the part of the chain c between the first and the last i for which $y^{(i)} \in \{x\} \cup N_x$ (and possibly inserting the site x).

Now we construct the chain $c'' = (y''^{(i)})_{i\in I}$ of $\mathscr{T}'$ by induction by putting $y''^{(0)} = (0, 0, 0)$ and by choosing $y''^{(i+1)}$ as the only site in K such that:

(i) $y''^{(i+1)}$ is connected in $\mathscr{T}'$ with $y''^{(i)}$,
(ii) the projection of $y''^{(i+1)}$ on $\mathbb{Z}^2$ is $y'^{(i+1)}$.

We remark that c'' is a t.s.a. chain but, in general, it is not a +chain in ω. However a t.s.a. +chain in ω can be easily obtained by modifying c'' in some of the sets $\{(x_1, x_2, 0), (x_1, x_2, 1)\} \cup \tilde{N}_{(x_1,x_2)}$, $(x_1, x_2) \in T_q$. We only need the remark that if $y''^{(i)} = (x_1, x_2, \varepsilon)$, $\omega(y''^{(i)}) = -1$, then $\tilde{\mu}_{p,2}$-a.s. $(x_1, x_2) \in T(\omega)$ and $y''^{(i-1)}$ and $y''^{(i+1)}$ are both connected with the connected set

$$\tilde{N}_{(x_1,x_2)} \cup \{(x_1, x_2, 1 - \varepsilon)\} \subset \omega^{-1}(1). \quad \square$$

4. Proof of the inequality $p_c^{(3)} < ½$. In this section we shall prove the following:

THEOREM 4.1. $p_c^{(3)} < ½$.

Theorem 4.1 will be an immediate consequence of Theorem 3.1 and of the following proposition:

PROPOSITION 4.2. $\tilde{p}_{c,2} < \tilde{p}_{c,3} = ½$.

The equality $\tilde{p}_{c,3} = ½$ follows from (3.16) and the fact that the critical percolation probability on the triangular lattice is ½. The remainder of this section is devoted to the proof of the inequality $\tilde{p}_{c,2} < \tilde{p}_{c,3}$.

We decompose K as $K_0 \cup K_1$, where $K_0 = \mathbb{Z}^2 \times \{0\}$ and $K_1 = \mathbb{Z}^2 \times \{1\}$, so that Ω_K can be identified with $\Omega_{K_0} \times \Omega_{K_1}$. The map $\sigma: \Omega_K = \Omega_{K_0} \times \Omega_{K_1} \to \Omega_{\mathbb{Z}^2}$ is defined by $\sigma(\omega^{(0)}, \omega^{(1)})_x = \max(\omega_x^{(0)}, \omega_x^{(1)})$ if $x \in T_q$ and $\omega_y^{(0)} = 1$ for $y \in N_x$, $\sigma(\omega^{(0)}, \omega^{(1)})_x = \omega_x^{(0)}$ otherwise.

REMARK 4.3. If we define the measures ν_p^0, ν_p^1 on $\Omega_K = \Omega_{K_0} \times \Omega_{K_1}$ by

$$\nu_p^0 = \mu_p^{(K_0)} \times \mu_0^{(K_1)}, \quad \nu_p^1 = \mu_p^{(K_0)} \times \mu_p^{(K_1)},$$

then

$$\sigma^*\nu_p^0 = \psi^*\tilde{\mu}_{p,3} = \mu_p^{(\mathbb{Z}^2)}, \quad \sigma^*\nu_p^1 = \psi^*\tilde{\mu}_{p,2}. \tag{4.1}$$

This remark, which is an immediate consequence of the definitions, together with the proof of Lemma 3.2 allows us to consider only Bernoulli measures in this section. It will be useful to interpolate between the measures ν_p^0 and ν_p^1 by considering for $\lambda \in [0, 1]$ the measures $\nu_p^\lambda = \mu_p^{(K_0)} \times \mu_{\lambda p}^{(K_1)}$.

Let $R(L, M)$ be the rectangle $[0, L] \times [0, M] \subset \mathbb{Z}^2$. For $L \neq M$ let ℓ_1, ℓ_2 (resp. $\bar{\ell}_1, \bar{\ell}_2$) be the two shorter (resp. longer) sides of $R(L, M)$ and let $G_{L,M}$ and G_∞ be the events:

$$G_{L,M} = \{\omega \in \Omega_K \mid \text{in } \sigma(\omega) \text{ there is a +chain of } \mathcal{T} \text{ in } R(L, M) \text{ connecting the sides } \ell_1 \text{ and } \ell_2 \text{ of } R(L, M)\},$$
$$G_\infty = \{\omega \in \Omega_K \mid \text{in } \sigma(\omega) \text{ there is an infinite +chain of } \mathcal{T}\}.$$

The following lemma establishes a connection between the events $G_{L,M}$ and G_∞.

LEMMA 4.4. *There exists a constant $C > 0$ such that if for some λ, p, L*

$$\nu_p^\lambda(G_{L,2L}) \geq 1 - C \quad \textit{and} \quad \nu_p^\lambda(G_{2L,L}) \geq 1 - C, \quad \textit{then} \quad \nu_p^\lambda(G_\infty) = 1.$$

PROOF. This lemma can be proved exactly in the same way as the analogous results in [10]. We only need to remark that the events $G_{L,2L}$, $G_{2L,L}$ and G_∞ are increasing, so that F.K.G. inequalities can be applied. □

In the remainder of this section we shall use freely the notation introduced in [10] (see also [8]). In particular we shall denote by $\delta_x G$, $x \in K$, the event: "x is a pivotal site for the event G". Furthermore the number of pivotal sites for the event $G_{L,M}$ in the configuration $\omega = (\omega^{(0)}, \omega^{(1)}) \in \Omega_K$ will be denoted simply by $n_{L,M}(\omega^{(0)}, \omega^{(1)})$. Throughout this section the constant q which appears in the definition of T_q is assumed to take some fixed value with the condition $q \geq 3$.

The following lemma provides a lower bound of the ν_p^λ-probability for a site of the type $(x, 1)$ with $x \in T_q$ to be a pivotal site for the event $G_{L,M}$.

LEMMA 4.5. *Let $0 < p < 1$, $0 < \bar{\lambda}$ be fixed. There exists a constant $A(\bar{\lambda}, p)$ such that for every $L, M > 4q$, $L \neq M$, every $x \in R(L, M)$ and every $\lambda > \bar{\lambda}$ there exists $y \in T_q \cap R(L, M)$ such that*

$$|x - y| \leq 4q \tag{4.2}$$

$$\nu_p^\lambda(\delta_{(x,0)} G_{L,M}) < A(\bar{\lambda}, p)\nu_p^\lambda(\delta_{(y,1)} G_{L,M}). \tag{4.3}$$

PROOF. For $x \in R(L, M)$, let Q_x be the square:

$$Q_x = \{z \in \mathbb{Z}^2 \mid |z_1 - x_1| \leq 4q, |z_2 - x_2| < 4q\}.$$

We put $Q_x' = Q_x \backslash \partial^i Q_x$, $Q_x'' = Q_x' \backslash \partial^i Q_x'$, $Q_x''' = Q_x'' \backslash \partial^i Q_x''$.

We choose $y = (y_1, y_2) \in Q_x''' \cap T_q$. Such a y exists because of the definition of Q_x and the choice $q \geq 3$. From now on we shall deal with the case $Q_x \subset R(L, M) \backslash \partial^i R(L, M)$ (the other possible cases do not present any additional difficulties and can be dealt with by the same method with obvious changes).

We consider the following event:

$\Delta_x = \{\omega \in \Omega_K \mid$ in the configuration $\sigma(\omega)|_{R(L,M)\backslash Q_x}$ there exist +chains c_1, c_2 connecting ℓ_1 and ℓ_2 to $\partial^e Q_x$ and −chains $\bar{c}_1$ and $\bar{c}_2$ connecting $\bar{\ell}_1$ and $\bar{\ell}_2$ to $\partial^e Q_x\}$.

It is easy to check that $\Delta_x \supset \delta_{(x,0)} G_{L,M}$. Hence, if we choose $\bar{y} = (y, 1)$ with $y \in Q_x''' \cap T_q$, we have

$$\nu_p^\lambda(\delta_{\bar{y}} G_{L,M}) \geq \nu_p^\lambda(\Delta_x)\nu_p^\lambda(\delta_{\bar{y}} G_{L,M} \mid \Delta_x) \geq \nu_p^\lambda(\delta_{(x,0)} G_{L,M})\nu_p^\lambda(\delta_{\bar{y}} G_{L,M} \mid \Delta_x); \tag{4.4}$$

on the other hand it is not difficult to get a lower bound independent from L and M for the conditional probability appearing on the r.h.s. of (4.4). Namely it is true that:

$$\nu_p^\lambda(\delta_{\bar{y}} G_{L,M} \mid \Delta_x) \geq \min(p, 1 - p)^{(8q+1)^2} \min(\bar{\lambda}p, 1 - p)^{25}. \tag{4.5}$$

Indeed for $\omega \in \Delta_x$, let $a, b \in \partial^e Q_x$ be two points connected by a −chain in the configuration $\sigma(\omega)|_{R(L,M)\backslash Q_x}$ respectively to $\bar{\ell}_1$ and $\bar{\ell}_2$. We remark that, by the definition of Δ_x, a and b are not connected. Then we can find $a', b' \in \partial^i Q_x$ such that a and a' are connected, b and b' are connected and a' and b' are not connected.

We put

$$\gamma = \{(t_1, t_2) \in \mathbb{Z}^2 \mid t_1 = y_1\} \cap Q_x''. \tag{4.6}$$

Then we can find two integers m, n and a self-avoiding chain $c = (z^{(0)}, \cdots, z^{(N)})$, $N = 8q + m + n - 2$ such that:

(i) $$z^{(0)} = a, \quad z^{(1)} = a', \quad z^{(N-1)} = b', \quad z^{(N)} = b$$

(ii) $$\{z^{(2)}, \cdots, z^{(n+1)}\} \cup \{z^{(N-m-1)}, \cdots, z^{(N-2)}\} \subset \partial^i Q'_x$$

(iii) $$\{z^{(n+2)}, \cdots, z^{(N-m-2)}\} = \gamma.$$

Now we define a configuration $\omega' \in \Omega_K$ by putting:

$$\begin{array}{lll} \forall t \in Q_x \backslash \mathrm{supp}(c) & \omega_t'^{(0)} = 1, & \omega_t'^{(1)} = \omega_t^{(1)} \\ \forall t \in \mathrm{supp}(c) \backslash T_q & \omega_t'^{(0)} = -1, & \omega_t'^{(1)} = \omega_t^{(1)} \\ \forall t \in \mathbb{Z}^2 \backslash Q_x & \omega_t'^{(0)} = \omega_t^{(0)}, & \omega_t'^{(1)} = \omega_t^{(1)} \\ \forall t \in \mathrm{supp}(c) \cap T_q & \omega_t'^{(0)} = -1, & \omega_t'^{(1)} = -1. \end{array}$$

It is clear that our construction implies $\omega' \notin G_{L,M}$.

We note that ω' is such that $\sigma\omega'$ contains two +clusters connected respectively to ℓ_1 and ℓ_2 and separated in Q''_x by γ. Moreover since $y \in Q'''_x$, $(\sigma\omega')_t = 1$ for every $t \in N_y$ and this implies by the definition of σ that $(\sigma s_{\bar{y}}\omega')_y = 1$, where $s_{\bar{y}}\omega'$ is the configuration obtained from ω' by changing the value in $\bar{y}$ from -1 to 1. The two previous remarks imply that $s_{\bar{y}}\omega' \in G_{L,M}$ and hence $\omega' \in \delta_{\bar{y}} G_{L,M}$. The estimate (4.5) follows from the fact that ω' is obtained by changing the values of ω only in $\tilde{Q}_x = (Q_x \times \{0\}) \cup ((Q_x \cap T_q) \times \{1\})$ and the quantity appearing on the r.h.s. of (4.5) is a lower bound for the ν_p^λ-probabilities of any configuration in $\tilde{Q}_x$. □

By using the estimates (4.2) and (4.3), we are able to prove the following:

LEMMA 4.6. *For every* $\tilde{\lambda} > 0$

(4.7i) $$\lim_{L\to\infty} \nu_{1/2}^{\tilde{\lambda}}(G_{L,2L}) = 1$$

(4.7ii) $$\lim_{L\to\infty} \nu_{1/2}^{\tilde{\lambda}}(G_{2L,L}) = 1.$$

PROOF. In the same way as in [7] (see also [10] and [8]) we can prove that for every N there exists L_0 such that for $L > L_0$ and for every $\lambda \geq 0$

(4.8) $$E_{\nu_{1/2}^\lambda}(n_{L,2L} \mid \Omega_K \backslash G_{L,2L}) \geq N.$$

Note that even if the measure $\sigma^* \nu_{1/2}^\lambda$ on $\Omega_{\mathbb{Z}^2}$ is not a Bernoulli measure, it has finite range of dependence, so that only minor changes in the proofs are required.

Let λ be such that $\tilde{\lambda}/2 \leq \lambda \leq \tilde{\lambda}$. (4.8) implies:

(4.9) $$E_{\nu_{1/2}^\lambda}(n_{L,2L}) \geq N(1 - \nu_{1/2}^\lambda(G_{L,2L}))$$

for $L > L_0$. On the other hand, if we put

$$Z_1 = (R(L, 2L) \times \{0\}) \cup ((R(L, 2L) \cap T_q) \times \{1\})$$

and

$$Z_2 = (R(L, 2L) \cap T_q) \times \{1\},$$

we get

$$\begin{aligned} E_{\nu_{1/2}^{\lambda}}(n_{L,2L}) &= \textstyle\sum_{z\in Z_1} \nu_{1/2}^{\lambda}(\delta_z G_{L,2L}) \\ &\le (A(\tilde{\lambda}/2, 1/2)(8q+1)^2 + 1) \textstyle\sum_{z\in Z_2} \nu_{1/2}^{\lambda}(\delta_z G_{L,2L}) \\ &= 2(A(\tilde{\lambda}/2, 1/2)(8q+1)^2 + 1)(\partial/\partial\lambda)\nu_{1/2}^{\lambda}(G_{L,2L}), \end{aligned} \tag{4.10}$$

where, in the last step, we have used Lemma 4.5 and the formula (4.18) of [8] page 78 (see also [10]) to express the derivative of $\nu_{1/2}^{\lambda}(G_{L,2L})$ w.r.t. λ in terms of the expectation of the number of pivotal sites.

By combining (4.9) and (4.10) we get

$$(\partial/\partial\lambda)(1 - \nu_{1/2}^{\lambda}(G_{L,2L})) \le -NB(\tilde{\lambda})^{-1}(1 - \nu_{1/2}^{\lambda}(G_{L,2L})), \tag{4.11}$$

where $B(\tilde{\lambda}) = 2A(\tilde{\lambda}/2, 1/2)(8q+1)^2 + 2$. Integrating (4.11) from $\tilde{\lambda}/2$ to $\tilde{\lambda}$ we get

$$1 - \nu_{1/2}^{\tilde{\lambda}}(G_{L,2L}) \le \exp(-N\tilde{\lambda}/(2B(\tilde{\lambda}))). \tag{4.12}$$

As $L \to \infty$, N can be chosen arbitrarily large, so that (4.12) gives (4.7i). The proof of (4.7ii) is identical. □

Now we can prove Proposition 4.2.

PROOF OF PROPOSITION 4.2. As remarked before, it is enough to prove the inequality $\tilde{p}_{c,2} < 1/2$ and, by Remark 4.3, this will be achieved if some $p < 1/2$ is found such that $\nu_p^1(G_\infty) = 1$. By Lemma 4.6, for L large enough,

$$\nu_{1/2}^1(G_{L,2L}) \ge 1 - C/2, \quad \nu_{1/2}^1(G_{2L,L}) \ge 1 - C/2, \tag{4.13}$$

where C is the constant introduced in the statement of Lemma 4.4. Since the events $G_{L,2L}$ and $G_{2L,L}$ are local, we get by continuity that for some $p < 1/2$

$$\nu_p^1(G_{L,2L}) \ge 1 - C, \quad \nu_p^1(G_{2L,L}) \ge 1 - C. \tag{4.14}$$

By virtue of Lemma 4.4, for the same p

$$\nu_p^1(G_\infty) = 1. \quad □ \tag{4.15}$$

5. An application to the three-dimensional Ising model. In this section we prove that our inequality $p_c^{(3)} < 1/2$ implies percolation of both signs for the three-dimensional Ising model if the parameter β is small enough in dependence with h. Higuchi applied a completely analogous method to a two-dimensional problem ([6]).

Let $P_{\beta,h}$ be a Gibbs measure for the three-dimensional Ising model. That means that $P_{\beta,h}$ is a probability measure on $\Omega_{\mathbb{Z}^3}$ such that, for every finite subset Λ of $\mathbb{Z}^3$, the conditional distribution on Ω_Λ of $P_{\beta,h}$ given the configuration $\bar{\omega}$ in

$\mathbb{Z}^3\backslash\Lambda$ is given by

$$P^{\Lambda}_{\beta,h}(\omega \mid \bar{\omega}) = Z(\Lambda \mid \bar{\omega})^{-1}\exp(-\beta H(\omega \mid \bar{\omega})) \tag{5.1}$$

with

$$H(\omega \mid \bar{\omega}) = h \textstyle\sum_{x\in\Lambda} \omega_x + \frac{1}{2} \sum_{x,y\in\Lambda; x\neq y} J_{x-y}\omega_x\omega_y + \sum_{x\in\Lambda; y\in\mathbb{Z}^3\backslash\Lambda} J_{x-y}\omega_x\bar{\omega}_y$$

and

$$Z(\Lambda \mid \bar{\omega}) = \textstyle\sum_{\omega\in\Omega_\Lambda} \exp(-\beta H(\omega \mid \bar{\omega})).$$

Here $\beta > 0$ and h are real parameters and J is a map $J\colon \mathbb{Z}^3\backslash\{0\} \to \mathbb{R}$ that we assume to verify $J_x = J_{-x}$ and $\| J \| = \sum_{x\neq 0} | J_x | < \infty$.

It is a consequence of our result of the last section that in a specified range of the parameters h and β an infinite +cluster exists in the three-dimensional Ising model and for h and β sufficiently small infinite + and − clusters coexist. The result is contained in the following

THEOREM 5.1. *Let $P_{\beta,h}$ be a Gibbs measure defined as above. If*

$$(1 + \exp(2\beta(\| J \| - h)))^{-1} > p_c^{(3)} \tag{5.2}$$

then

$$P_{\beta h}(\#(W_+(0)) = \infty) > 0. \tag{5.3}$$

Similarly, if

$$(1 + \exp(2\beta(\| J \| + h)))^{-1} > p_c^{(3)} \tag{5.4}$$

then

$$P_{\beta,h}(\#(W_-(0)) = \infty) > 0. \tag{5.5}$$

(5.3) *and* (5.5) *are true, in particular, for any given h for β small enough depending on h.*

PROOF. It is enough to prove (5.3), since (5.5) follows from (5.3) by means of the transformation $\xi\colon \Omega_{\mathbb{Z}^3} \to \Omega_{\mathbb{Z}^3}$, $(\xi\omega)_x = -\omega_x$.

Let $p = (1 + \exp(2\beta(\| J \| - h)))^{-1}$. The conditional $P_{\beta,h}$-probability that $\omega_x = 1$ for some $x \in \mathbb{Z}^3$, given the configuration $\bar{\omega}$ in $\mathbb{Z}^3\backslash\{x\}$ satisfies:

$$\begin{aligned} &P^{\{x\}}_{\beta,h}(\omega_x = 1 \mid \bar{\omega}) \\ &\quad = \exp(\beta(h - \textstyle\sum_{x\neq 0} J_x))(\exp(\beta - \sum_{x\neq 0} J_x)) + \exp(\beta(\sum_{x\neq 0} J_x - h))^{-1} \\ &\quad = (1 + \exp(2\beta(\textstyle\sum_{x\neq 0} J_x - h)))^{-1} > (1 + \exp(2\beta(\| J \| - h)))^{-1} = p. \end{aligned} \tag{5.6}$$

Then it is easy to see (see e.g. Lemma 1, Section 3 of [11] in the particular, simpler case $\varepsilon = 0$) that the measure $P_{\beta,h}$ is bounded from below in the F.K.G. sense by the Bernoulli measure $\mu_p^{(\mathbb{Z}^3)}$. That means that we can find a joint representation ν in $\Omega_{\mathbb{Z}^3} \times \Omega_{\mathbb{Z}^3}$ of $P_{\beta,h}$ and $\mu_p^{(\mathbb{Z}^3)}$ such that

$$\nu(\{(s, t) \mid s_x \geq t_x \ \forall x \in \mathbb{Z}^3\}) = 1. \tag{5.7}$$

In particular (5.7) implies

$$P_{\beta,h}(\#(W_+(0)) = \infty) \geq \mu_p^{(\mathbb{Z}^3)}(\#(W_+(0)) = \infty) > 0 \tag{5.8}$$

since $(W_+(0)) = \infty$ is an increasing event.

The last statement of the theorem follows from Theorem 4.1. □

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REFERENCES

[1] AIZENMAN, M. (1980). Translational invariance and instability of phase coexistence in the two-dimensional Ising system. *Comm. Math. Phys.* **73** 83–94.

[2] AIZENMAN, M., CHAYES, J., CHAYES, L., FRÖHLICH, J. and RUSSO, L. (1983). On a sharp transition from area law to perimeter law in a system of random surfaces. *Comm. Math. Phys.* **92** 19–69.

[3] BRICMONT, J., FONTAINE, J. R. and LEBOWITZ J. L. (1982). Surface tension, percolation and roughening. *J. Statist. Phys.* **29** 193–203.

[4] HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Phil. Soc.* **56** 13–20.

[5] HIGUCHI, Y. (1981). On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model. In *Random Fields: Rigorous Results in Statistical Mechanics and Quantum Field Theory.* (J. Fritz, J. L. Lebowitz and D. Szasz, eds.) North Holland, Amsterdam.

[6] HIGUCHI, Y. (1983). Applications of a stochastic inequality to two-dimensional Ising and Widom–Rowlinson models. Preprint.

[7] KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals ½. *Comm. Math. Phys.* **74** 41–79.

[8] KESTEN, H. (1982). Percolation theory for mathematicians. In *Progress in Probability and Statistics, Vol. 2.* Birkhäuser, Boston.

[9] RUSSO, L. (1979). The infinite cluster method in the two-dimensional Ising model. *Comm. Math. Phys.* **67** 251–266.

[10] RUSSO, L. (1981). On the critical percolation probabilities. *Z. Wahrsch. verw. Gebiete* **56** 229–237.

[11] RUSSO, L. (1982). An approximate zero-one law. *Z. Wahrsch. verw. Gebiete* **61** 129–139.

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