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CHARACTERIZATION AND DOMAINS OF ATTRACTION OF p-STABLE RANDOM COMPACT SETS

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Let $(\mathscr{K}(\mathbb{B}), \delta)$ denote the nonempty compact subsets of a separable Banach space $\mathbb{B}$ topologized by the Hausdorff metric. Let K, K_1, K_2 be i.i.d. random compact convex sets in $\mathbb{B}$. K is called p-stable if for each $\alpha, \beta \geq 0$ there exist compact convex sets C and D such that

$$\mathscr{L}(\alpha K_1 + \beta K_2 + C) = \mathscr{L}((\alpha^p + \beta^p)^{1/p} K + D)$$

where + refers to Minkowski sum. A characterization of the support function for a compact convex set is provided and then utilized to determine all p-stable random compact convex sets. If $1 \leq p \leq 2$, they are trivial, merely translates of a fixed compact convex set by a p-stable $\mathbb{B}$-valued random variable. For $0 < p < 1$, they are translates of stochastic integrals with respect to nonnegative independently scattered p-stable measures on the unit ball of co $\mathscr{K}(\mathbb{B})$. Deconvexification is also discussed. The domains of attraction of p-stable random compact convex sets with $0 < p < 1$ are completely characterized. The case $1 < p \leq 2$ is considered in Giné, Hahn and Zinn (1983). Precedents: Lyashenko (1983) and Vitale (1983) characterize the Gaussian random compact sets in $\mathbb{R}^d$.

1. Introduction. Studies by Kendall (1974) and Matheron (1975) have revitalized the area of geometric probability by establishing a rigorous mathematical theory of random sets. Quite naturally, the setwise average of random sets under Minkowski addition has received a substantial amount of attention. Already established are strong laws of large numbers (Artstein and Hansen (1984), Artstein and Hart (1981), Artstein and Vitale (1975), Cressie (1975), Hess (1979), Puri and Ralescu (1982)), various central limit theorems (Cressie (1979), Giné, Hahn and Zinn (1983), Lyashenko (1982), Trader and Eddy (1981), Vitale (1977, 1981) and Weil (1982)), a version of the law of the iterated logarithm (Giné, Hahn and Zinn, 1983) together with some first statistical considerations (Lyashenko, 1983) and some applications (e.g. to an optimization problem arising in allocation under uncertainty, Artstein and Hart, 1981).

The role and utility of asymptotics in statistics is usually dictated by the tractability of the limit distributions. Lyashenko (1983) and Vitale (1983) describe two notions of Gaussian random set which apply respectively to random compact convex and random compact sets in $\mathbb{R}^d$. In this article we provide a more intrinsic definition of Gaussian as well as p-stable compact convex sets. This new definition is shown to be equivalent to the previous two notions after they are

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appropriately modified. We construct all p-stable compact convex sets in separable Banach spaces. The construction justifies the versions of the central limit theorems established in Giné, Hahn and Zinn (1983) for $1 < p \le 2$ and allows us to extend the theory to stable limits with $0 < p < 1$. Previously, Mase (1979) considered p-stable compact convex subsets of $\mathbb{R}^d$ containing the origin. His techniques require the sets to be finite-dimensional and to contain the origin while our approach is unrestricted. The compact nonconvex case is also considered. Our main tools are the representation of stable laws in Banach spaces (see e.g. Araujo and Giné, 1980) and an exact characterization of the support functions of compact convex subsets of $\mathbb{B}$ (see Section 2 below).

Let $\mathbb{B}$ denote a separable Banach space with norm $\|\cdot\|$. $\mathscr{K}(\mathbb{B})$ will be the collection of *nonempty* compact subsets of $\mathbb{B}$. Two relevant operations defined on $\mathscr{K}(\mathbb{B})$ are:

$$A + B := \{a + b\colon a \in A,\ b \in B\} \quad \text{(Minkowski addition)}$$

$$\alpha A := \{\alpha a\colon a \in A\} \quad \text{(positive homothetics)}$$

for $A, B \in \mathscr{K}(\mathbb{B})$. $\mathscr{K}(\mathbb{B})$ is not a vector space since $(A - B) + B$ need not equal A. However, the Hausdorff distance δ,

$$\delta(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|,\ \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$$

$$= \inf\{\varepsilon > 0\colon A \subset B_\varepsilon,\ B \subset A_\varepsilon\}$$

(where $A_\varepsilon := \{x \in \mathbb{B}\colon \inf_{y\in A} \|y - x\| \le \varepsilon\}$ and likewise for B_ε) induces a complete, separable metric topology on $\mathscr{K}(\mathbb{B})$. We will also let

$$\|A\| := \delta(A, \{0\}) = \sup\{\|a\|\colon a \in A\}$$

for $A \in \mathscr{K}(\mathbb{B})$. An important subset of $\mathscr{K}(\mathbb{B})$ is co $\mathscr{K}(\mathbb{B})$: $= \{\text{co } A\colon A \in \mathscr{K}(\mathbb{B})\}$ where co A denotes the closed convex hull of A.

A convenient representation of compact (convex) sets as elements of a concrete Banach space involves using support functions. Let $\mathbb{B}^*$ denote the dual of $\mathbb{B}$ and $\mathbb{B}_1^* = \{x \in \mathbb{B}^*\colon \|x\| \le 1\}$.

1.1 DEFINITION. The *support function* of a subset $A \subset \mathbb{B}$ is the function s_A defined on $\mathbb{B}_1^*$ by the equation

$$(1.2) \qquad s_A(f) := \sup_{x\in A} f(x), \quad f \in \mathbb{B}_1^*.$$

In general, s_A maps $\mathbb{B}_1^*$ into $(-\infty, \infty]$. However, if A is compact, then $s_A(f) < \infty$ for all $f \in \mathbb{B}_1^*$. By e.g. Lemma 1.1 in Giné, Hahn and Zinn (1983),

$$(1.3) \qquad \delta(A, B) = \|s_A - s_B\|_\infty \quad \text{for all} \quad A, B \in \text{co } \mathscr{K}(\mathbb{B}).$$

In particular, $\|A\| = \|s_A\|_\infty$. (However, note that if $A, B \in \mathscr{K}(\mathbb{B})$, then only $\delta(A, B) \ge \|s_A - s_B\|_\infty$.) Hausdorff distance satisfies a subadditivity property that will be useful later

$$(1.4) \quad \delta(A + B, A' + B') \le \delta(A, A') + \delta(B, B'), \quad A, A', B, B' \in \mathscr{K}(\mathbb{B}).$$

(Note that $\delta(A, B) < \varepsilon$, $\delta(A', B') < \eta$ imply $A \subset B_\varepsilon$ and $A' \subset B'_\eta$ so $A + A' \subset B_\varepsilon + B'_\eta \subset (B + B')_{\varepsilon+\eta}$. Similarly, $B + B' \subset (A + A')_{\varepsilon+\eta}$.)

A *random compact set* is a Borel measurable function K from an abstract probability space (Ω, Σ, P) into $\mathscr{K}(\mathbb{B})$.

1.5 DEFINITION. Let $K(\omega)$ be a random compact set. The *support process* $K(\cdot) := K(\cdot, \omega)$ is defined by

$$K(f, \omega) := S_{K(\omega)}(f), \quad f \in \mathbb{B}_1^*.$$

If $K \in \text{co}\,\mathscr{K}(\mathbb{B})$ a.s., K is called a *random compact convex set.* In this case the support process $K(\cdot)$ uniquely determines K. Moreover, the correspondence is isometric by (1.3) and preserves both addition and multiplication by positive scalars. In a separable Banach space the support process takes values in the space of weak-star continuous functions on $\mathbb{B}_1^*$, $C(\mathbb{B}_1^*, \text{w}^*)$ (see Theorem 2.3).

The intrinsic definition of a p-stable compact set should be analogous to that of a stable random variable with values in $\mathbb{R}$ or any other Banach space, allowing at most for the inadequacies of subtraction.

1.6 DEFINITION. A random compact *convex* set is called *p-stable*, $0 < p \le 2$, if for any K_1, K_2 i.i.d. with the same law as K and for all $\alpha, \beta \ge 0$, there exist sets $C, D \in \text{co}\,\mathscr{K}(\mathbb{B})$ such that

$$\mathscr{L}(\alpha K_1 + \beta K_2 + C) = \mathscr{L}((\alpha^p + \beta^p)^{1/p} K + D). \tag{1.7}$$

K is *strictly stable* if C and D can be chosen to be $\{0\}$. If $p = 2$, K is called *Gaussian.*

In actuality, D may be chosen to be $\{0\}$ for $0 < p < 1$ and C may be chosen to be $\{0\}$ for $1 \le p \le 2$. However, to see that for all p a single nonrandom set cannot always be chosen to appear on the same side in (1.7), consider a constant random set $K(\omega) = M$ where $M \in \text{co}\,\mathscr{K}(\mathbb{B})$. Since

$$\begin{aligned}(\alpha^p + \beta^p)^{1/p} - (\alpha + \beta) &\ge 0 \quad \text{for} \quad 0 < p < 1\\ &\le 0 \quad \text{for} \quad 1 \le p \le 2,\end{aligned}$$

it is only valid that for all $\alpha, \beta \ge 0$,

$$\alpha M + \beta M + [(\alpha^p + \beta^p)^{1/p} - (\alpha + \beta)]M = (\alpha^p + \beta^p)^{1/p} M \quad \text{for} \quad 0 < p < 1$$

$$\alpha M + \beta M = (\alpha^p + \beta^p)^{1/p} M + [(\alpha + \beta) - (\alpha^p + \beta^p)^{1/p}]M \quad \text{for} \quad 1 \le p \le 2.$$

The reason for restricting Definition 1.6 to convex sets stems from the fact that any $M \in \mathscr{K}(\mathbb{B}) \setminus \text{co}\,\mathscr{K}(\mathbb{B})$ has the property that $\alpha M + \beta M \subsetneq (\alpha + \beta)M$ for some $\alpha, \beta > 0$, and therefore M may not satisfy (1.7).

Other notions of a p-stable compact convex set are possible and useful. To formulate them we require the following definition.

1.8 DEFINITION. A function $\phi: \mathscr{K}(\mathbb{B}) \to \mathbb{R}^d$, $d \in \mathbb{N}$, is in $L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^d)$ if for all $A_1, A_2 \in \mathscr{K}(\mathbb{B})$

(a) $\phi(\alpha A_1 + \beta A_2) = \alpha\phi(A_1) + \beta\phi(A_2)$ whenever $\alpha, \beta \geq 0$;

(b) there exists $c = c(\phi) < \infty$ such that

$$|\phi(A_1) - \phi(A_2)| \leq c\delta(A_1, A_2)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

In other words, $L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^d)$ is the set of "positively linear" Lipschitzian functions on $\mathscr{K}(\mathbb{B})$ with values in $\mathbb{R}^d$. $L^+(\mathscr{K}(\mathbb{B}), \mathbb{R})$ is defined in Vitale (1984).

1.9 THEOREM. *The following are equivalent for a random compact convex set $K(\omega)$ in $\mathbb{B}$:*

(a) *$K(\omega)$ is p-stable;*

(b) *the support process $K(\cdot)$ is a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable;*

(c) *$\phi(K)$ is p-stable for all $\phi \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^2)$;*

(d) *$\phi(K)$ is p-stable for all $\phi \in \bigcup_{d=1}^{\infty} L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^d)$.*

1.10 REMARK. (1) The above equivalences are also true if "p-stable" is replaced by "strictly p-stable", and then $\mathbb{R}$ may replace $\mathbb{R}^2$ in condition (c).

(2) If $p > 1$, $\mathbb{R}^2$ can also be replaced by $\mathbb{R}$ in condition (c).

These two observations reflect the fact that a random vector need not be stable even though all its 1-dimensional projections are stable. Only if the 1-dimensional projections are all either strictly stable or all of index $p > 1$ can stability of the random vector be deduced. The case $p = 1$ is unknown. In general, stability of the 2-dimensional projections is needed. (See e.g. Giné and Hahn, 1983, and Marcus, 1983.)

(3) in $\mathbb{R}^d$ with $p = 2$, (b) coincides with Lyashenko's (1983) definition and (c) with $\mathbb{R}$ instead of $\mathbb{R}^2$ is Vitale's (1983) definition.

Although Definition 1.6 seems inappropriate for nonconvex compact sets, statements (b)–(d) provide reasonable definitions of p-stable (nonconvex) compact sets. These notions are related as follows:

1.11 THEOREM.

(i) *For a general (not necessarily convex) compact random set $K(\omega)$ of $\mathbb{B}$, conditions* (b)–(d) *are each equivalent to*

(e) co *K is a p-stable compact convex set.*

(ii) *For every $p \in (0, 2]$, there exist random compact sets which satisfy* (b) *(hence* (c)–(e)*), but not* (1.7).

Our construction of p-stable compact convex sets in an arbitrary separable Banach space relies heavily on characterizations of support functions of compact convex sets (Corollary 2.5) and of support functions of single points (Corollary 2.6). The representation theorem in Section 2 identifies the set of support functions of compact convex sets as a closed cone $\mathscr{V}$ of $C(\mathbb{B}_1^*, \mathrm{w}^*)$ which contains

$\mathbb{B}$ (acting as $x(f) = f(x)$) as the largest linear subspace. This has the effect that a Gaussian measure, or more generally a p-stable measure, with $1 \leq p \leq 2$, whose support is contained in $\mathscr{V}$, has its support in effect reduced to $s_M + \mathbb{B}$ for some $M \in \operatorname{co} \mathscr{K}(\mathbb{B})$ (see Proposition 4.5). For $p < 1$, no such restriction is necessary essentially because there exist real p-stable measures supported by half lines. (Proposition 4.7 contains the representation for $0 < p < 1$.) Once the representation of p-stable measures with support in $\mathscr{V}$ is obtained, the general form of p-stable random sets follows immediately using Theorem 1.9 ((a) $\Leftrightarrow$ (b)).

Thus, for $1 \leq p \leq 2$ we obtain

1.12 THEOREM. *$K(\omega)$ is a p-stable compact convex set in $\mathbb{B}$ with $1 \leq p \leq 2$ if and only if*

$$K = M + \{\xi\} \quad a.s. \tag{1.13}$$

for some $M \in \operatorname{co} \mathscr{K}(\mathbb{B})$ and a p-stable $\mathbb{B}$-valued random variable ξ.

For $0 < p < 1$, an example of a nondegenerate p-stable compact convex set is θM where θ is a positive real p-stable random variable and M is a nonrandom set in $\operatorname{co} \mathscr{K}(\mathbb{B})$. These are the building blocks for a stochastic integral construction which yields all of the p-stable compact convex sets for $0 < p < 1$. Let

$$\mathscr{K}_1 = \{A \in \operatorname{co} \mathscr{K}(\mathbb{B}): \| A \| = 1\}.$$

1.14 THEOREM. *$K(\omega)$ is a p-stable compact convex set in $\mathbb{B}$ with $0 < p < 1$ if and only if*

$$K = M + \int_{\mathscr{K}_1} x \, dL \quad a.s. \tag{1.15}$$

where $M \in \operatorname{co} \mathscr{K}(\mathbb{B})$ and L is a positive p-stable independently scattered random measure on $\mathscr{K}_1$ with finite spectral measure σ.

The stochastic integral will be constructed in Section 3, where positive p-stable independently scattered random measures are also discussed.

For deconvexification, Theorems 1.12 and 1.14 when combined with Theorem 1.11 yield

1.16 COROLLARY. *A random compact set $K(\omega)$ of $\mathbb{B}$ satisfies any one (hence all) of the conditions* (b)–(e) *of Theorems* 1.9 *and* 1.11 *if and only if* co K *admits the representation in Theorem* 1.12 *if* $1 \leq p \leq 2$, *or in Theorem* 1.14 *if* $0 < p < 1$.

Giné, Hahn and Zinn (1983) study the domain of attraction problem for compact convex random sets with $1 < p \leq 2$, only to the point of giving sufficient conditions for statements of the form

$$\lim_{n \to \infty} \mathscr{L}[n a_n^{-1} \delta((K_1 + \cdots + K_n)/a_n, EK)] = \mathscr{L}(\| Y \|_\infty)$$

(where K_i are i.i.d. with law K, + denotes Minkowski addition, and Y is a p-stable $C(\mathbb{B}_1^*, \text{w}^*)$-valued random variable). These results provide a rate of convergence for the law of large numbers, but they do not provide the typical approximation of the laws of normalized sums by stable laws. A posteriori, this is a natural fact in view of the degeneracy arising in Theorem 1.12 (note that Y does not correspond to a stable random set). The case $p < 1$ is different. In this case there are enough stable laws (Theorem 1.14) and there is no need for centering in the CLT, therefore limit theorems approximating the laws of $(K_1 + \cdots + K_n)/a_n$ by the laws of p-stable random sets, $p < 1$, are possible. Let us recall a definition.

1.17 DEFINITION. Let $K, K_1, K_2, \cdots$ be i.i.d. compact convex random subsets of $\mathbb{B}$, and let ν be a p-stable law on co $\mathscr{K}(\mathbb{B})$ with $0 < p < 1$. Let $S_n = K_1 + \cdots + K_n$, $n \in \mathbb{N}$ (Minkowski sum). Then K belongs to the *domain of attraction of* ν *with norming constants* $\{a_n\}$ ($K \in \mathrm{DA}(\nu, \{a_n\})$ or $K \in \mathrm{DA}(\{a_n\})$) if

$$\lim_{n\to\infty} \mathscr{L}(S_n/a_n) = \nu$$

in the sense of weak convergence of measures on the metric space (co $\mathscr{K}(\mathbb{B}), \delta$). If $a_n = n^{1/p}$ then K is in the *domain of normal attraction of* ν, ($K \in \mathrm{DNA}(\nu)$ or $K \in \mathrm{DNA}$).

Generally, $a_n = n^{1/p}\ell(n)$, where ℓ is a slowly varying function.

The isometry between compact convex sets and support functions implies a correspondence between DA for random compact convex sets K and DA for their support processes $K(\cdot)$. Namely,

$$\begin{aligned} &\mathscr{L}\left(\textstyle\sum_{i=1}^n K_i/a_n\right) \to_w \mathscr{L}(L) \quad \text{in } (\text{co } \mathscr{K}(\mathbb{B}), \delta) \quad \text{iff} \\ &\mathscr{L}\left(\textstyle\sum_{i=1}^n K_i(\cdot)/a_n\right) \to_w \mathscr{L}(L(\cdot)) \quad \text{in } C(\mathbb{B}_1^*, \mathrm{w}^*). \end{aligned} \tag{1.18}$$

It is convenient to remark that for a random compact set to have a nonvoid domain of attraction it must verify Definition 1.6. Hence, by Theorem 1.11(ii) there are nonconvex random sets which are "stable" in the sense of (b)–(d) in 1.10 but have a void domain of attraction. So, we will restrict our attention to domains of attraction of random compact convex sets.

The following is a direct translation to random sets of one of the main results on domains of attraction in Banach spaces (Corollary 3.6.19 and Theorem 3.7.10 in Araujo and Giné, 1980). If $A \subset \mathscr{K}(\mathbb{B})$, we let $\delta(K, A) = \inf\{\delta(K, L): L \in A\}$.

1.19 THEOREM. *Let K be a random compact convex set in $\mathbb{B}$. Then K is in the $DA\{a_n\}$ of a p-stable random set, $0 < p < 1$, if and only if both:*

(a) *$\phi(K)$ is in the $DA\{a_n\}$ of a real p-stable random variable for all $\phi \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R})$, and*

(b) *there exists a sequence of compact convex sets M_i, with $M_1 = \{0\}$, such that if $F_m^+ = \{\sum_{i=1}^m \lambda_i M_i, \lambda_i \geq 0\}$, then*

$$\lim_{m\to\infty} \limsup_n nP\{\delta(K, F_m^+) > a_n\} = 0,$$

with all the lim sup$_n$ *finite.*

If this is the case, the measures on $\mathscr{K}_1$,

$$A \to pnP\{K/\|K\| \in A, \|K\| > a_n\}, \quad A \in \mathscr{B}(\mathscr{K}_1),$$

converge weakly to a finite measure σ. *Moreover, the p-stable limit law of* $\sum_{i=1}^n K_i/a_n$, K_i *independent copies of K, is* $\mathscr{L}(\int_{\mathscr{K}_1} x\, dL)$, *where L is a positive p-stable independently scattered measure with spectral measure* σ.

Section 5 contains a proof of this theorem together with several examples concerning domains of attraction.

2. Support functions of compact convex sets in Banach spaces. In this section we characterize the support functions of compact convex subsets of a Banach space $\mathbb{B}$ (not necessarily separable). This is the only section in which $\mathbb{B}$ is not assumed to be separable. A crucial step in the proof of the main theorem relies on the proof of a similar theorem of Hormander (1954) characterizing support functions of closed convex sets. To be consistent with Hormander, in this section we first consider the support function to be defined on all of $\mathbb{B}^*$.

2.1 DEFINITION. The *extended support function* of a subset $A \subset \mathbb{B}$ is the function $\tilde{s}_A$ defined on $\mathbb{B}^*$ by the equation

$$\tilde{s}_A(f) = \sup_{x \in A} f(x), \quad f \in \mathbb{B}^*. \tag{2.2}$$

If A is compact, then $\tilde{s}_A(f) < +\infty$ for all $f \in \mathbb{B}^*$. Note that support functions are positively homogeneous, i.e. $\tilde{s}_A(\lambda f) = \lambda \tilde{s}_A(f)$ for all $\lambda > 0$. Consequently, $\tilde{s}_A$ is really determined by its values on the unit ball $S \equiv \{f: \|f\| \le 1\}$ (in fact, by its values on ∂S).

Prior to proving the main theorem of this section, we recall a few facts about the bounded-weak-star (bw*) topology of $\mathbb{B}^*$ (see e.g. Dunford and Schwartz (1958), V.5.3, 4, 5 and V.3.9). This is the strongest topology of $\mathbb{B}^*$ which coincides with the w*-topology of $\mathbb{B}^*$ on every ball $B_r^* = \{f \in \mathbb{B}^*: \|f\| \le r\}$, $0 < r < \infty$; that is, a subset $F \subset \mathbb{B}^*$ is bw*-closed iff $F \cap \mathbb{B}_r^*$ is w*-closed for all $0 < r < \infty$. A neighborhood base at the origin for the bw*-topology consists of all sets of the form

$$\{f \in \mathbb{B}^*: |f(x_k)| < 1, k = 1, 2, \cdots\}$$

where $\{x_k, k \ge 1\}$ is any sequence in $\mathbb{B}$ such that $\lim_k \|x_k\| = 0$. Two important facts about this topology are that $(\mathbb{B}^*, \text{bw}^*)$ is a locally convex topological vector space and that its dual is $\mathbb{B}$, i.e. $(\mathbb{B}^*, \text{bw}^*)^* = \mathbb{B}$.

2.3 THEOREM. *Let* $\mathbb{B}$ *be any Banach space and let* $H: \mathbb{B}^* \to \mathbb{R}$. *Then H is the extended support function of a compact convex set if and only if*

(1) *H is* bw*-*continuous*;
(2) *H is subadditive, i.e.* $H(f_1 + f_2) \le H(f_1) + H(f_2)$, $f_1, f_2 \in \mathbb{B}^*$; *and*
(3) *H is positively homogeneous, i.e.* $H(\lambda f) = \lambda H(f)$, $\lambda > 0$, $f \in \mathbb{B}^*$.

PROOF. Let A be a compact convex (nonrandom) subset of $\mathbb{B}$. From the definition of extended support function, $\tilde{s}_A$ is subadditive, positively homogeneous and

$$|\tilde{s}_A(f) - \tilde{s}_A(g)| \le \sup_{x\in A} |f(x) - g(x)|, \quad f, g \in \mathbb{B}^*.$$

On $\mathbb{B}_r^*$, $0 < r < \infty$, the weak-star topology coincides with the topology of uniform convergence on compact subsets of $\mathbb{B}$. Hence, as a consequence of the above inequality $\tilde{s}_A|_{\mathbb{B}_r^*}$ is w*-continuous for each $0 < r < \infty$. Therefore, $\tilde{s}_A$ is bw*-continuous as desired.

For the converse, let H be a function from $\mathbb{B}^* \to \mathbb{R}$ satisfying (1)–(3). The proof of Hormander's Theorem (1954, pages 182–184) applies, modulo two simple changes, to produce a *closed* convex set A such that $H = \tilde{s}_A$. The two changes in Hormander's proof are that his cone D is now bw*-closed (rather than w*-closed) and his final argument requires the fact that $(\mathbb{B}^*, \text{bw}^*)^* = \mathbb{B}$ instead of $(\mathbb{B}, \text{w}^*)^* = \mathbb{B}$.

It remains to establish that A is compact. For this we utilize both the bw*-continuity of $\tilde{s}_A$ and the fact that A is closed. Note that bw*-continuity of $\tilde{s}_A$ at 0 implies the existence of a sequence $x_i \to 0$, $x_i \in \mathbb{B}$, such that

$$\sup_i |f(x_i)| < 1 \Rightarrow \tilde{s}_A(f) < 1, \quad f \in \mathbb{B}^*. \tag{2.4}$$

Let $J = \text{co}\{\pm x_i\}$, which is compact since $x_i \to 0$. Then, $\tilde{s}_J(f) = \sup_i f(x_i)$. Hence (2.4) implies

$$\tilde{s}_A \le \tilde{s}_J$$

by homogeneity. Therefore, $A \subseteq J$. As a closed subset of a compact set, A must be compact. □

The support functions of compact convex subsets of $\mathbb{B}$ can also be characterized as a particular closed cone $\mathscr{V}$ of $C(\mathbb{B}_1^*, \text{w}^*)$, using Theorem 2.3.

Let H: $\mathbb{B}^* \to \mathbb{R}$ satisfy (1)–(3) in the above theorem. Then $H|_{\mathbb{B}_1^*}$ satisfies:

(1)′ $H|_{\mathbb{B}_1^*}$ is w*-continuous,

(2)′ $H|_{\mathbb{B}_1^*}(f+g) \le H|_{\mathbb{B}_1^*}(f) + H|_{\mathbb{B}_1^*}(g), \quad f, g, f+g \in \mathbb{B}_1^*$,

(3)′ $H|_{\mathbb{B}_1^*}(\lambda f) = \lambda H|_{\mathbb{B}_1^*}(f), \quad \lambda > 0, \quad f, \lambda f \in \mathbb{B}_1^*$.

Conversely, if L: $\mathbb{B}_1^* \to \mathbb{R}$ satisfies (1)′–(3)′, then the function $\tilde{L}$: $\mathbb{B}^* \to \mathbb{R}$ defined by

$$\tilde{L}(f) = \|f\| L(f/\|f\|), \quad f \in \mathbb{B}^*,$$

satisfies (1)–(3) in Theorem 2.3.

Thus the support functions s_A defined in Definition 1.1 are characterized. For convenience, a function on $\mathbb{B}_1^*$ satisfying (2)′ or (3)′ will be also called respectively subadditive and positively homogeneous. Observe that if it satisfies (1)′, (3)′, and $s(f+g) = s(f) + s(g)$ for all $f, g \in \mathbb{B}_1^*$ with $f+g \in \mathbb{B}_1^*$, then s is the restriction to $\mathbb{B}_1^*$ of a linear function; hence we will simply call it linear. We adhere to these three conventions throughout the remainder of the paper.

2.5 COROLLARY. *A function $H: \mathbb{B}_1^* \to \mathbb{R}$ is the support function of a compact convex subset of $\mathbb{B}$ if and only if $H \in C(\mathbb{B}_1^*, \mathrm{w}^*)$, H is subadditive, and H is positively homogeneous. The set of all these functions is a closed cone $\mathscr{V}$ of $C(\mathbb{B}_1^*, \mathrm{w}^*)$.*

(The last statement holds because properties (2)′ and (3)′ are preserved upon taking limits on $C(\mathbb{B}_1^*, \mathrm{w}^*)$.)

This latter characterization of support functions is convenient in our situation because $\mathbb{B}$ is a separable Banach space: in this case $(\mathbb{B}_1^*, \mathrm{w}^*)$ is compact metric and as a consequence $C(\mathbb{B}_1^*, \mathrm{w}^*)$ is a separable Banach space. For this reason we define support functions on $\mathbb{B}_1^*$.

A set consisting of a single point $x \in \mathbb{B}$ has as its support function the function $x(f) = f(x)$, i.e. x itself, which is linear. Conversely, if $H: \mathbb{B}_1^* \to \mathbb{R}$ is linear then H is the support function of a single point because H extends uniquely to an element of $(\mathbb{B}^*, \mathrm{bw}^*)^* = \mathbb{B}$. Let us record this simple fact:

2.6 COROLLARY. *A function $H: \mathbb{B}_1^* \to \mathbb{R}$ is the support function of a set consisting of a single point iff H is linear. In fact H is the support function of $\{x\}$ iff $H(f) = f(x)$, $f \in \mathbb{B}_1^*$. The identification of $x \in \mathbb{B}$ with the function $f \to f(x)$ is a linear isometry of $\mathbb{B}$ onto the linear functions of $C(\mathbb{B}_1^*, \mathrm{w}^*)$ and it is in this sense that $\mathbb{B} \subset C(\mathbb{B}_1^*, \mathrm{w}^*)$.*

These two corollaries will be used in a crucial way to obtain a characterization of p-stable compact convex sets. Our method of proof will also exploit the fact that the map from $(\mathrm{co}\,\mathscr{K}(\mathbb{B}), \delta)$ to $C(\mathbb{B}_1^*, \mathrm{w}^*)$ given by

$$A \to s_A$$

is an isometry onto the image $\mathscr{V}$ of $\mathrm{co}\,\mathscr{K}(\mathbb{B})$ which converts Minkowski addition of sets into addition of functions, and preserves positive scalar multiplication. We refer e.g. to Giné, Hahn and Zinn (1983) for details on this and other generalities about compact convex sets (random or not) and support functions.

(Several corrections to Giné, Hahn and Zinn (1983) should be noted. First, Artstein and Vitale (1975) defined the expectation of a random compact set using the Aumann integral. Second, on page 112 line 7, L^1 should replace L^2. Finally, $EX \neq E(\mathrm{co}\,X)$ in a general Banach space. Artstein and Vitale supplied us with the following example: Let $F: [0, 1] \to L^2[0, 1]$ be defined by $F(t) = \{0(\cdot), I_{[0,1]}(\cdot)\}$. If $\phi(s) = 1 - s$, then $\tfrac{1}{2}\phi \in \int \mathrm{co}\,F$ but $\tfrac{1}{2}\phi \notin \int F$. These corrections do not affect the validity of any of the results in that paper.)

3. Construction of stochastic integrals. Our characterization of p-stable compact convex sets for $0 < p < 1$ requires a set-valued stochastic integral while the proof also utilizes a stochastic integral in $C(\mathbb{B}_1^*, \mathrm{w}^*)$. The method is the same in both cases so we supply the details only in $\mathrm{co}\,\mathscr{K}(\mathbb{B})$. (An alternative, more general and more complicated construction in Banach spaces may be found in Woyczynski, 1978.)

Let $\mathscr{K}_1 = \{A \in \mathrm{co}\,\mathscr{K}(\mathbb{B}): \|A\| = 1\}$ and let σ be a finite Borel measure on $\mathscr{K}_1$.

Without loss of generality we may assume $\sigma(\mathscr{K}_1) = 1$. A standard real positive p-stable random variable, θ, $0 < p < 1$, has characteristic function

$$E \exp(it\theta) = \exp(-c_1 |t|^p + ic_2 t^p) \tag{3.1}$$

where $c_1 = \int_0^\infty (1 - \cos u)u^{-1-p}\, du$, $c_2 = \int_0^\infty (\sin u)u^{-1-p}\, du$ and $t^p = |t|^p \operatorname{sgn} t$. Given σ and θ, we define on the Borel sets of $\mathscr{K}_1$ an independently scattered random measure M_p by

(i) $M_p(A) =_{\mathscr{L}} (\sigma(A))^{1/p}\theta$;
(ii) if $\{A_i\}$ are disjoint then

$$M_p(\cup_{i=1}^n A_i) = \sum_{i=1}^n M_p(A_i) \quad \text{a.s.,} \quad n < \infty,$$

(3.2) and

$$M_p(\cup_{i=1}^\infty A_i) = \text{pr} - \lim_{n\to\infty} \sum_{i=1}^n M_p(A_i);$$

(iii) for all $n < \infty$, $M_p(A_1), \cdots, M_p(A_n)$ are independent if $\{A_i\}$ are disjoint.

The Kolmogorov consistency theorem implies that M_p exists. We call M_p a *positive p-stable independently scattered random measure on* $\mathscr{K}_1$.

Although a general theory of integration with respect to M_p can be developed, it suffices for our purposes to construct only

$$\int_{\mathscr{K}_1} x\, dM_p(x).$$

It is easy to construct a sequence of simple functions $x_n : \mathscr{K}_1 \to \mathscr{K}_1$ such that

$$\int_{\mathscr{K}_1} \delta^p(x_n, x)\, d\sigma(x) \to 0. \tag{3.3}$$

(Simply choose compact sets $Q_n \subset \mathscr{K}_1$ such that $\sigma(Q_n) \ge 1 - 2^{-n}$. Q_n can be partitioned into disjoint collections of sets A_{nj}, $1 \le j \le r_n$ so that the δ-diameter of $A_{nj} \le 2^{-n}$. Fix $x_{n0} \in A_{n,0} := Q_n^c$ and $x_{nj} \in A_{nj}$ and let $x_n = \sum_{j=0}^{r_n} x_{nj} I_{A_{nj}}$.) For simple functions the meaning of $\int_{\mathscr{K}_1} x\, dM_p$ is obvious: $M_p(A_{nj}) = \theta_{nj}(\sigma(A_{nj}))^{1/p}$, with θ_{nj} i.i.d., $1 \le j \le r_n$, and we have

$$\int_{\mathscr{K}_1} x_n\, dM_p = \sum_{j=0}^{r_n} \theta_{nj}(\sigma(A_{nj}))^{1/p} x_{nj}. \tag{3.4}$$

Using the inequality

$$\Lambda_p\left(\sum_{k=1}^n |a_k\theta_k|\right) \le \alpha\left(\frac{2-p}{1-p}\right)^{1/p} \left(\sum_{k=1}^n |a_k|^p\right)^{1/p}$$

where $\Lambda_p(\eta) := (\sup_{t>0} t^p P(\eta > t))^{1/p}$ and $\alpha := \Lambda_p(\theta)$ (proved analogously to inequality (22) in Lemma 2.1 of Giné and Marcus, 1982; see e.g. Remark 4.7 in

Giné and Marcus (1983)) we obtain

$$\begin{aligned}
&\Lambda_p\left(\delta\left(\int_{\mathscr{K}_1} x_n\,dM_p, \int_{\mathscr{K}_1} x_m\,dM_p\right)\right)\\
&\quad\le \Lambda_p\left(\int_{\mathscr{K}_1} \delta(x_n, x_m)\,dM_p\right) \qquad (3.5)\\
&\quad\le \alpha\left(\frac{2-p}{1-p}\right)^{1/p}\left(\int_{\mathscr{K}_1} \delta^p(x_n, x_m)\,d\sigma\right)^{1/p} \to 0 \quad \text{by (3.3).}
\end{aligned}$$

(The first inequality uses (1.4).) This shows that the sequence

$$\left\{\int_{\mathscr{K}_1} x_n\,dM_p\right\}$$

is Cauchy in probability and must therefore converge. Call the limit

$$\int_{\mathscr{K}_1} x\,dM_p.$$

It is easy to see, using computations similar to (3.5), that this definition does not depend on the sequence $x_n \to x$ in $L_p(\sigma)$, so it is well-defined. In particular $\int_{\mathscr{K}_1} x\,dM_p$ has a very concrete meaning: there exist sets $\{\{x_{nj}\}, n \ge 1, 1 \le j \le r_n\}$ and rowwise independent random variables $\{\theta_{nj}, n \ge 1, 1 \le j \le r_n\}$ with law θ such that

$$\sum_{j=1}^{r_n} \theta_{nj} x_{nj} \to \int_{\mathscr{K}_1} x\,dM_p \quad \text{in} \quad \Lambda_p \quad \text{and a.s.} \qquad (3.6)$$

Note that $\int_{\mathscr{K}_1} x\,dM_p$ is a p-stable compact convex set (e.g. because the random sets on the left side of (3.6) are p-stable and limits of p-stable compact convex sets are also p-stable). Previously, Mase (1979) recognized as p-stable the random sets of the form $\sum_{i=1}^{n} \theta_i A_i$ where θ_i are i.i.d. real nonnegative p-stable random variables and $A_i \in \operatorname{co}\mathscr{K}(\mathbb{R}^d)$.

The above construction of a stochastic integral for compact convex sets does not seem to extend to the nonconvex case.

4. Proofs of results on stable random sets. Our first objective is to obtain a constructive representation for the support process $K(\cdot)$ of any p-stable random set. Different representations will arise depending upon whether $0 < p < 1$ or $1 \le p \le 2$. These results will then be used to prove Theorems 1.9, 1.11, 1.12, and 1.14.

So, assume $K(\cdot)$ is a p-stable $C(\mathbb{B}_1^*, \text{w}^*)$-valued random variable. Probability in Banach space theory identifies $K(\cdot)$ as follows:

(4.1) (i) For $p = 2$, $K(\cdot)$ is Gaussian and for all $t \in \mathbb{R}$ and $h \in (C(\mathbb{B}_1^*, \mathrm{w}^*))^*$,

$$E\exp(ith(K)) = \exp\{ith(a) - t^2\Phi_\gamma(h, h)/2\}$$

where $\Phi_\gamma(h, h) = \int h^2(x)d\gamma(x)$, $a \in C(\mathbb{B}_1^*, \mathrm{w}^*)$ and γ is a centered Gaussian measure on $C(\mathbb{B}_1^*, \mathrm{w}^*)$.

For the non-Gaussian case, let σ' denote a finite Borel measure on the unit sphere $U = \{f \in C(\mathbb{B}_1^*, \mathrm{w}^*): \|f\| = 1\}$. Then,

(4.1) (ii) For $0 < p < 1$, $\mathscr{L}(K(\cdot)) = \delta_a * \mathrm{Pois}(d\sigma' \times r^{-1-p}dr)$ where for any $t \in \mathbb{R}$ and $h \in (C(\mathbb{B}_1^*, \mathrm{w}^*))^*$

$$\begin{aligned} E&\exp(ith(K)) \\ &= \exp\left\{ith(a) + \int_U\int_0^\infty (\exp(itrh(u)) - 1)r^{-1-p}\,dr\,d\sigma'(u)\right\} \\ &= \exp\left\{ith(a) - c_1|t|^p\int_U |h(u)|^p\,d\sigma'(u) + ic_2\int_U (th(u))^p\,d\sigma(u)\right\}. \end{aligned}$$

(4.1) (iii) For $1 \le p < 2$, $\mathscr{L}(K(\cdot)) = \delta_a * c_1\mathrm{Pois}(d\sigma' \times r^{-1-p}\,dr)$ where for any $t \in \mathbb{R}$ and $h \in (C(\mathbb{B}^*, \mathrm{w}^*))^*$

$$\begin{aligned} E&\exp(ith(K)) \\ &= \exp\left\{ith(a) + \int_U\int_0^\infty (\exp(itrh(u)) - 1 - itrh(u)I_{r\le 1})r^{-1-p}\,dr\,d\sigma'(u)\right\} \\ &= \exp\left\{ith(a) - c_1|t|^p\int_U |h(u)|^p\,d\sigma'(u)\right. \\ &\qquad\left. + i\int_U\int_0^\infty (\sin(trh(u)) - trh(u)I_{r\le 1})r^{-1-p}\,dr\,d\sigma'(u)\right\} \end{aligned}$$

(see e.g. Araujo and Giné, 1980). σ' is called the *spectral measure* of $K(\cdot)$.

Since $K(\cdot)$ is both positively homogeneous and subadditive, an appropriate choice of linear functionals yields pertinent information. For $x \in C(\mathbb{B}_1^*, \mathrm{w}^*)$,

(4.2)
(i) given $\lambda > 0$ and $f \in \mathbb{B}_1^*$ with $\lambda f \in \mathbb{B}_1^*$ set

$$h_1(x) := h_1(\lambda, f, x) := x(\lambda f) - \lambda x(f);$$

(ii) given $f, g \in \mathbb{B}_1^*$ with $f + g \in \mathbb{B}_1^*$ set

$$h_2(x) := h_2(f, g, x) := x(f) + x(g) - x(f + g).$$

Positive homogeneity and subadditivity of $K(\cdot)$ then imply

$$h_1(K(\cdot)) = 0 \quad\text{and}\quad h_2(K(\cdot)) \ge 0. \tag{4.3}$$

For $1 \le p \le 2$, there do not exist any real positive p-stable random variables other than the degenerate ones. Consequently,

$$h_2(K(\cdot)) = c \geq 0 \quad \text{if} \quad 1 \leq p \leq 2, \tag{4.4}$$

where c is a constant.

For $1 \leq p \leq 2$, the desired support process representation is

4.5 PROPOSITION. *Let $K(\cdot)$ be a support process. Then $K(\cdot)$ is a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable with $1 \leq p \leq 2$ iff there exist a fixed set $M \in \mathrm{co}\mathscr{K}(\mathbb{B})$ and a p-stable $\mathbb{B}$-valued random variable ξ such that*

$$K(\cdot) = s_M + \{\xi\}(\cdot). \tag{4.6}$$

PROOF. Sufficiency is trivial. Necessity requires separate, though related, arguments for the Gaussian and non-Gaussian cases.

(a) $1 \leq p < 2$. $K(\cdot)$ is assumed to be a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable and thus takes the form given in (4.1)(iii). It suffices to show that: *the spectral measure σ' is supported by linear functions*, i.e. by $\mathbb{B}$. Then since the linear functions form a closed subspace of $C(\mathbb{B}_1^*, \mathrm{w}^*)$, they actually support the measure $c_1\mathrm{Pois}(d\sigma' \times r^{-1-p}\, dr)$. Invoking Corollary 2.6, $c_1\mathrm{Pois}(d\sigma' \times r^{-1-p}\, dr)$ must then be the law of a singleton, i.e. a p-stable $\mathbb{B}$-valued random variable ξ. So for some such ξ, $K(f) = a(f) + f(\xi)$ or $a(f) = K(f) - f(\xi)$ which is the support function of $K + \{-\xi\}$. However, $a(\cdot)$ is nonrandom; thus, $a(\cdot)$ is the support function of some $M \in \mathrm{co}\mathscr{K}(\mathbb{B})$. So $K(\cdot) = s_M + \{\xi\}(\cdot)$ a.s.

To verify that σ' is supported by linear functions, combine (4.1)(iii), (4.3) and (4.4) to obtain

$$\int_U |u(\lambda f) - \lambda u(f)|^p \, d\sigma'(u) = 0 \quad \text{for} \quad \lambda > 0,\ f,\ \lambda f \in \mathbb{B}_1^*$$

and

$$\int_U |u(f) + u(g) - u(f+g)|^p \, d\sigma'(u) = 0 \quad \text{for} \quad f,\ g,\ f+g \in \mathbb{B}_1^*.$$

Varying λ, f, g over countable dense sets and using the continuity of u allows us to conclude that

$$\sigma'(U \cap \{\text{linear functions}\}) = \sigma'(U).$$

The proof for $1 \leq p < 2$ is now complete.

(b) $p = 2$. $K(\cdot)$ is assumed to be a Gaussian $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable and thus takes the form in (4.1)(i) with $\gamma = \mathscr{L}(K(\cdot) - a(\cdot))$. Combining (4.1)(i), (4.3) and (4.4),

$$0 = \Phi_\gamma(h_i, h_i) = \int h_i^2(x)\, d\gamma(x) \quad \text{for} \quad i = 1, 2.$$

As in part (a),

$$\gamma(C(\mathbb{B}_1^*, \mathrm{w}^*) \cap \{\text{linear functions}\}) = \gamma(C(\mathbb{B}_1^*, \mathrm{w}^*)).$$

By Corollary 2.6, γ is supported on linear functions, i.e. γ is a Gaussian law on $\mathbb{B}$. Hence there exists a $\mathbb{B}$-valued Gaussian random variable ξ with $\mathscr{L}(\xi) = \gamma$ such that $K(f) - a(f) = f(\xi)$. Again, $a(\cdot)$ nonrandom implies $K(\cdot) = s_M + \{\xi\}(\cdot)$ a.s. for some $M \in \operatorname{co} \mathscr{K}(\mathbb{B})$. □

Next we obtain the support process representation for $0 < p < 1$. As above, $\mathscr{V}$ denotes the closed cone of $C(\mathbb{B}_1^*, \mathrm{w}^*)$ consisting of positively homogeneous subadditive functions.

4.7 PROPOSITION. *Let $K(\cdot)$ be a support process. Then $K(\cdot)$ is a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable for $0 < p < 1$ iff there exist $M \in \operatorname{co}\mathscr{K}(\mathbb{B})$ and a positive p-stable independently scattered random measure L' whose spectral measure σ' is supported by the closed set $S := U \cap \mathscr{V}$ such that*

$$K(\cdot) = s_M + \int_S x \, dL'. \tag{4.8}$$

PROOF. Computing characteristic functions shows that (4.8) verifies (4.1)(ii), which gives sufficiency. Thus, we turn to necessity. Let $K(\cdot)$ be a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable. Then there exists a finite Borel measure σ' on U with $K(\cdot)$ satisfying (4.1)(ii). To establish (4.8) it suffices to show that

(4.9) (i) $a(\cdot) \in \mathscr{V}$;
(ii) the support of σ' is contained in S.

Then by Corollary 2.5 there exists $M \in \operatorname{co}\mathscr{K}(\mathbb{B})$ with $s_M = a(\cdot)$. Also, if L' is defined to be a positive p-stable independently scattered random measure with spectral measure σ', then the characteristic function of

$$a + \int_S x \, dL'$$

coincides with that of $K(\cdot)$. So (4.8) follows.

To establish (4.9)(i) and (4.9)(ii), combine (4.3) for h_1 and (4.1)(ii) to obtain that for $t \in \mathbb{R}$,

$$0 = ith_1(a) - c_1|t|^p \int_U |h_1(u)|^p \, d\sigma'(u) + ic_2 \int_U (th_1(u))^p \, d\sigma'(u).$$

Thus, $\int_U |h_1(u)|^p \, d\sigma'(u) = 0$. Consequently, both $\int_U (h_1(u))^p \, d\sigma'(u) = 0$ and $h_1(a) = 0$ or

$$a(\lambda f) = \lambda a(f),$$
$$u(\lambda f) = \lambda u(f) \quad \sigma' \text{ a.e.}$$

Taking countable dense sets of λ's and f's and using continuity, we conclude

$$(4.10)\qquad \begin{cases} a(\lambda f) = \lambda a(f) \quad \text{for all} \quad \lambda \geq 0, \quad f \in \mathbb{B}_1^* \\ \sigma'(U \cap \{\text{positively homogeneous functions}\}) = \sigma'(U). \end{cases}$$

Now employing (4.3) for h_2, $h_2(K(\cdot))$ is a real *positive* (or degenerate) p-stable random variable, i.e. its characteristic function is of the form

$$(4.11)\qquad \exp\{it\alpha - c_1\beta \,|\, t\,|^p + ic_2\beta t^p\}$$

where $\alpha, \beta \geq 0$. Equating (4.11) with (4.1)(ii) yields

$$h_2(a) \geq 0$$

and

$$\int_U |\, h_2(u)\,|^p \, d\sigma'(u) = \int_U (h_2(u))^p \, d\sigma'(u);$$

so $|\, h_2(u)\,| = h_2(u)$ σ' a.e.

Therefore,

$$a(f + g) \leq a(f) + a(g)$$

and

$$u(f + g) \leq u(f) + u(g) \quad \sigma' \text{ a.e.}$$

Again taking a countable dense set of f's and g's we conclude:

$$(4.12)\qquad \begin{cases} a \text{ is subadditive} \\ \sigma'(U \cap \{\text{subadditive functions}\}) = \sigma'(U). \end{cases}$$

Conditions (i) and (ii) now follow from (4.10) and (4.12); whence the proof is complete. □

PROOF OF THEOREM 1.9. Obviously, (a) ⟹ (d) and (d) ⟹ (c). So we are left with the implications (c) ⟹ (b) and (b) ⟹ (a).

(c) ⟹ (b): If F: $C(\mathbb{B}_1^*, \mathrm{w}^*) \to \mathbb{R}^2$ is a continuous linear function, then $\phi_F(A) := F(s_A)$ for $A \in \mathscr{K}(\mathbb{B})$ satisfies

$$|\, \phi_F(A_1) - \phi_F(A_2)\,| \leq \|\, F \,\| \, \|\, s_{A_1} - s_{A_2} \|_\infty \leq \|\, F \,\| \, \delta(A_1, A_2).$$

Hence $\phi_F \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^2)$. By (c), $\phi_F(K(\omega)) = F(K(\omega))$ is a p-stable $\mathbb{R}^2$-valued random variable. Therefore the support process is a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable (Giné and Hahn, 1983, Theorem 2).

(b) ⟹ (a): If the support process $K(\cdot)$ is a p-stable $C(\mathbb{B}_1^*, \mathrm{w}^*)$-valued random variable then it has a representation of the form (4.6) if $1 \leq p \leq 2$ or (4.8) if $0 < p < 1$. Hence, by passing to sets, $K(\omega)$ has the representation given in (1.13) for $1 \leq p \leq 2$ or in (1.15) for $0 < p < 1$. The positive p-stable independently scattered random measure L on $\mathscr{K}_1(\mathbb{B})$ has spectral measure σ defined by

$$\sigma(G) = \sigma'\{s_A: A \in G\}.$$

But it is obvious that both random sets $M + \{\xi\}$ and $M + \int_{\mathscr{K}_1} x\, dL$ satisfy (1.7) respectively for $1 \le p \le 2$ and for $0 < p < 1$ (see end of Section 2 for the latter). □

PROOF OF THEOREMS 1.12 AND 1.14. Obvious from Propositions 4.4 and 4.7 and from Theorem 1.9, by use of the representation theorems of Section 2 (Corollaries 2.5 and 2.6). □

Turning attention to the nonconvex case we first require several simple lemmas.

4.13 LEMMA (Vitale). *If* $A \in \mathscr{K}(\mathbb{B})$, $\lim_{n\to\infty}\delta(A/n + \cdots^{n)} + A/n, \text{co } A) = 0$.

PROOF. Let $\varepsilon > 0$ be given. Denote by A_ε an ε-net of A ($x_1, \cdots, x_n \in A$ form an ε-net of A if $\sup_{x\in A}\min_{i<n}\| x - x_i\| < \varepsilon$). Then

$$(1/n)(A_\varepsilon + \cdots + A_\varepsilon) \subseteq (1/n)(A + \cdots + A) \subseteq \text{co } A$$

so

$$\delta((1/n)(A + \cdots + A), \text{co } A) \le \delta((1/n)(A_\varepsilon + \cdots + A_\varepsilon), \text{co } A).$$

The finite-dimensional case (Matheron, 1975) together with the fact that the convex hull of A_ε and co A_ε are at distance 0 apart imply

$$\lim_{n\to\infty}\delta((1/n)(A_\varepsilon + \cdots + A_\varepsilon), \text{co } A_\varepsilon) = 0.$$

Hence

$$\limsup_{n\to\infty}\delta((1/n)(A + \cdots + A), \text{co } A) \le \delta(\text{co } A_\varepsilon, \text{co } A) = \varepsilon. \quad □$$

4.14 LEMMA. (a) $s_A = s_{\text{co}A}$, $A \in \mathscr{K}(\mathbb{B})$. (b) If $\phi \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^d)$, *then* $\phi(A) = \phi(\text{co}A)$, $A \in \mathscr{K}(\mathbb{B})$.

PROOF. (a) Obviously $s_A(f) \le s_{\text{co}A}(f)$ for every $f \in \mathbb{B}_1^*$. On the other hand, the points of the form $\{\sum_{i=1}^j \lambda_i x_i: \sum_{i=1}^j \lambda_i = 1, \lambda_i \ge 0, j \text{ finite}, x_i \in A\}$ are dense in co A. For such points $f(\sum_{i=1}^j \lambda_i x_i) \le \sup f(x_i)$. Hence $s_{\text{co}A}(f) \le s_A(f)$.

(b) It follows immediately from $\phi(A) = \phi(A/n + \cdots + A/n) \to \phi(\text{co } A)$ as $n \to \infty$. □

PROOF OF THEOREM 1.11.

(i) (d) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (b) follows exactly as in Theorem 1.9 ((c) $\Rightarrow$ (b)).

(b) $\Leftrightarrow$ (e) follows from Lemma 4.14 (a) and Theorem 1.9 ((b) $\Leftrightarrow$ (a)).

(b) $\Rightarrow$ (d): Let $\phi = (\phi_1, \cdots, \phi_d) \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R}^d)$. The Hahn-Banach theorem applied componentwise to ϕ determines a continuous linear map F:

$C(\mathbb{B}^*, \mathrm{w}^*) \to \mathbb{R}^d$ with $\phi(K) = F(K(\cdot))$ by Lemma 4.14(b). Since $F(K(\cdot))$ is p-stable by (b), so is $\phi(K)$.

(ii) To produce examples of random sets K which are not p-stable but such that $\operatorname{co} K$ is stable we consider two cases. For $p < 1$, let θ be a positive p-stable real random variable. Consider the random set $K = \{0, \theta\}$. Then $\operatorname{co} K = [0, \theta] = [0, 1]\theta$ which is p-stable. Let K_1, K_2 be independent copies of K. Then

$$\alpha K_1 + \beta K_2 = \{0, \alpha\theta_1, \beta\theta_2, \alpha\theta_1 + \beta\theta_2\},$$

is a set with a.s. four different points, whereas $(\alpha^p + \beta^p)^{1/p}K$ has at most two different points. Therefore, K cannot satisfy Definition 1.6.

For $p \geq 1$, let η denote a p-stable real symmetric random variable. $K := \{\eta, \eta + 1\} = \{0, 1\} + \{\eta\}$ provides another example by the same reasoning as above. □

Vitale (1984) also handles deconvexification for Gaussian compact sets in $\mathbb{R}^d$ by using the identity $\phi(K) = \phi(\operatorname{co} K)$ for $\phi \in L^+(\mathscr{K}(\mathbb{R}^d), \mathbb{R})$.

5. Remarks on domains of attraction. Fix $p \geq 1$. Let K, K_i be i.i.d. random compact convex sets and let ξ be a p-stable $\mathbb{B}$-valued random variable. Suppose there exists a slowly varying function ℓ and a sequence of nonrandom sets H_n such that

$$\mathscr{L}(\textstyle\sum_{i=1}^n K_i/a_n + H_n) \to \mathscr{L}(M + \{\xi\})$$

with $a_n = n^{1/p}\,\ell(n)$. Then it is easy to show (using arguments involving the functions $h(K) = K(f+g) - K(f) - K(g)$ $f, g, \in \mathbb{B}^*$, similar to those used in previous proofs) that for $p > 1$, K is a random singleton, and that for $p = 1$, K is the sum of a fixed set and a random singleton. So, as mentioned in the introduction, the domain of attraction problem for $p \geq 1$ is not interesting if posed in these terms. (See Giné, Hahn and Zinn, 1983, for a "different domain of attraction problem".) This is not the situation for $0 < p < 1$. This section contains a proof of Theorem 1.19 together with several examples. (For simplicity, we will only construct examples of random sets in domains of normal attraction.)

PROOF OF THEOREM 1.19. The result follows from Theorem 3.7.10 and Corollary 3.6.19 in Araujo and Giné (1980) upon making the following observations:

(i) By the Hahn-Banach theorem, hypothesis (a) in Theorem 1.19 holds if and only if $h(K(\cdot)) \in \mathrm{DA}\{a_n\}$ for all $h \in (C(\mathbb{B}_1^*, \mathrm{w}^*))^*$. Due to the absence of centering, the Cramér-Wold theorem implies that this holds if and only if the $\mathbb{R}^d$-valued random variable $(h_1(K(\cdot)), \cdots, h_d(K(\cdot)))$ is in $\mathrm{DA}\{a_n\}$ for all $d \geq 1$ and all $h_i \in (C(\mathbb{B}_1^*, \mathrm{w}^*))^*$.

(ii) All separable Banach spaces, in particular $C(\mathbb{B}_1^*, \mathrm{w}^*)$, are of Rademacher type 1. (See e.g. page 158 of Araujo and Giné, 1980.)

(iii) If F_m is the linear span of $M_1(\cdot), \cdots, M_m(\cdot)$, then $\delta(X, F_m^+) \geq d(X(\cdot), F_m)$ ($\equiv \inf\{\|X(\cdot) - x(\cdot)\|_\infty: x \in F_m\}$). Hence sufficiency of (1.19)(a) and (b) follows from Theorem 3.7.10, loc. cit., by the previous observations.

(iv) Necessity of 1.19(a) and (b) can be deduced exactly as in Corollary 3.6.19, loc. cit., by passing to support processes. The only changes required in the proof are that F_m be replaced by the image of F_m^+ and the requirement that $\overline{\cup F_m^+} = \text{co}\,\mathscr{K}(\mathbb{B})$ (which is possible because $\text{co}\,\mathscr{K}(\mathbb{B})$ is separable). □

5.1 EXAMPLE. Let σ be a probability measure on $\mathscr{K}_1$ and let μ be a probability measure on $\mathbb{R}^+$ such that $\lim_{t\to\infty} t^p\mu[t, \infty) = \lim_{t\to\infty} t^p P\{\theta > t\}$, where θ is the standard positive p-stable random variable. Let $\mathscr{L}(K) = \sigma$ and $\mathscr{L}(\xi) = \mu$ with K and ξ independent. Then ξK is in the domain of normal attraction of $\int_{\mathscr{K}_1} x\, dL$, where L is a nonnegative p-stable independently scattered random measure with spectral measure σ.

PROOF. Fubini's theorem implies that for $\phi \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R})$,

$$\lim_{t\to\infty} t^p P\{\xi\phi(K) > t\} = E(0 \vee \phi(K))^p \lim_{t\to\infty} t^p P\{\theta > t\}.$$

$$\lim_{t\to\infty} t^p P\{\xi\phi(K) < -t\} = E\,|\,0 \wedge \phi(K)\,|^p \lim_{t\to\infty} t^p P\{\theta > t\}.$$

Consequently the 1-dimensional random variables $\xi\phi(K)$ are in the DNA of the corresponding law.

For any sequence F_m^+ such that $\overline{\cup F_m^+} = \mathscr{Y}$, we have

$$\lim_m \lim_{t\to\infty} t^p P\{\delta(\xi K, F_m^+) > t\} = \lim_m \lim_{t\to\infty} t^p P\{\xi\delta(K, F_m^+) > t\}$$
$$= (\lim_m E(\delta(K, F_m^+))^p)(\lim_{t\to\infty} t^p P\{\theta > t\}) = 0.$$

Hence conditions (a) and (b) of Theorem 1.19 are verified. □

The following two examples are based on examples 3.7 and 3.9 in Giné and Marcus (1983).

5.2 EXAMPLE. Fix $0 < p < 1$ and let θ denote a standard positive p-stable random variable on $\mathbb{R}$. Let μ be a positive measure on $\mathbb{R}^+$ satisfying

$$\sup_{t>0} t^p \mu(0, t)^c < \infty, \quad \lim_{t\to\infty} t^p \mu(0, t)^c = \alpha$$

for some fixed $p \in (0, 1)$, where $\alpha = \lim_{t\to\infty} t^p P\{\theta > t\}$. Let σ be a finite measure on $\mathscr{K}_1(\mathbb{B})$. Define an independently scattered nonnegative random measure on $\mathscr{K}_1$ by the equation

$$E \exp(itM(A)) = \exp\left(\sigma(A) \int_0^\infty (e^{itu} - 1)\, d\mu(u)\right),$$
$$t \in \mathbb{R}, \quad A \in \mathscr{B}(\mathscr{K}_1),$$

where $\mathscr{B}(\mathscr{K}_1)$ denotes the Borel sets of $\mathscr{K}_1$. (Such a measure exists by the

Kolmogorov consistency theorem.) The arguments on pages 66 and 67 of Giné and Marcus (1983) (but using inequality (4.16) there instead of their Lemma 2.1) give:

$$M(A) \in \mathrm{DNA}((\sigma(A))^{1/p}\theta), \quad A \in \mathscr{B}(\mathscr{K}_1) \tag{5.3}$$

and

$$\sup_{A \in \mathscr{B}(\mathscr{K}_1)} \Lambda_p(M(A)/(\sigma(A))^{1/p}) \le c \quad \text{for some} \quad c < \infty. \tag{5.4}$$

Inequality (5.4) allows us to define, as in Section 3, the stochastic integral

$$\int_{\mathscr{K}_1} x \, dM(x).$$

Let M_p be an independently scattered positive p-stable random measure with spectral measure σ. Then

$$\int_{\mathscr{K}_1} x \, dM(x) \in \mathrm{DNA}\left(\int_{\mathscr{K}_1} x \, dM_p(x)\right). \tag{5.5}$$

PROOF. Let M, M^i be i.i.d. Another application of inequality (4.16), loc. cit., in conjunction with (5.4), implies that for any $f \in L_p(\mathscr{K}_1, \mathscr{B}(\mathscr{K}_1), \sigma)$, there exists a finite constant c' with

$$\Lambda_p\left(n^{-1/p} \sum_{i=1}^n \int_{\mathscr{K}_1} f(x) \, dM^i(x)\right) \le c'\left(\int_{\mathscr{K}_1} |f(x)|^p d\sigma(x)\right)^{1/p}, \quad n \in \mathbb{N}. \tag{5.6}$$

(Compare with Lemma 3.6, loc. cit.) Choose compact subsets A_n of $\mathscr{K}_1$ which satisfy $\sigma(\mathscr{K}_1 \backslash A_n) \to 0$. Then, by (5.6),

$$\Lambda_p\left(n^{-1/p} \sum_{i=1}^n \int_{A_n^c} \|x\| \, dM(x)\right) \le c'(\sigma(A_n^c))^{1/p} \to 0. \tag{5.7}$$

So, by the approximation lemma in e.g. Giné and Marcus (1982, Lemma 2.1), it is enough to prove (5.5) under the assumption that σ *has compact support* (in $(\mathscr{K}(\mathbb{B}), \delta)$). Let $S_\sigma := \mathrm{supp}(\sigma)$. Since S_σ is compact, it is totally bounded. Therefore there exists a sequence of compact sets $M_i \subset \mathbb{B}$, $M_1 = \{0\}$, such that

$$\sup_{x \in S_\sigma} \delta(x, F_m^+) := \varepsilon_m \to 0 \quad \text{as} \quad m \to \infty \tag{5.8}$$

where F_m^+ is the "positive" linear span of $M_1, \cdots, M_m$. Since $\delta(\sum \alpha_i x_i, F_m^+) \le \sum \alpha_i \delta(x_i, F_m^+)$ if $x_i \in \mathrm{co}\mathscr{K}(\mathbb{B})$ and $\alpha_i \ge 0$, it follows that

$$\delta\left(\int_{\mathscr{K}_1} x \, dM(x), F_m^+\right) \le \int_{\mathscr{K}_1} \delta(x, F_m^+) \, dM(x). \tag{5.9}$$

Hence, using (5.6), (5.8) and (5.9),

$$\lim_{m\to\infty}\limsup_{t\to\infty} t^p P\left\{\delta\left(\int_{\mathscr{K}_1} x\, dM(x),\, F_m^+\right) > t\right\}$$

$$\le \lim_{m\to\infty}\limsup_{t\to\infty} t^p P\left\{\int_{\mathscr{K}_1} \delta(x, F_m^+)\, dM(x) > t\right\}$$

$$\le \lim_{m\to\infty}(c')^p \varepsilon_m(\sigma(\mathscr{K}_1))^p = 0.$$

Condition (b) in Theorem 1.19 is thus proved (with $a_n = n^{1/p}$).

Now, (5.3) makes it possible to apply Lemma 4.1 in Giné and Marcus (1983) (here in a simpler situation), to obtain that

$$\int_{\mathscr{K}_1} \phi(x)\, dM(x) \in \mathrm{DNA}\left(\int_{\mathscr{K}_1} \phi(x)\, dM_p(x)\right).$$

This is condition (a) in Theorem 1.19. The claim is thus proved. □

5.10 EXAMPLE. Fix $0 < p < 1$ and let $\theta, \theta_1, \theta_2, \cdots$ be i.i.d. standard positive p-stable random variables. Let ξ_i be i.i.d. nonnegative random variables in the DNA of θ. Select $m_i \ge 0$ with $\sum m_i < \infty$ and $K_i \in \mathscr{K}_1(\mathbb{B})$. Then a proof similar to the previous one yields

$$\textstyle\sum_{i=1}^\infty m_i^{1/p}\, \xi_i K_i \in \mathrm{DNA}(\sum_{i=1}^\infty m_i^{1/p}\, \theta_i K_i).$$

(If $\mathscr{L}(\xi_i) = \mathrm{Pois}\,\mu$, μ as in Example 5.2, then this example is a special case of Example 5.2; otherwise it is distinct from Example 5.2.) □

The following example should be compared to the results on domains of attraction for $p > 1$ in Giné, Hahn and Zinn (1983). Here we use the notation $\| K \|_G := \inf\{\lambda \ge 0: K \subseteq \lambda G\}$, where G and K are compact convex sets and G is symmetric.

5.11 EXAMPLE. Let K be a random compact convex set in $\mathbb{B}$ such that:

(a) $\phi(K)$ is in the DNA of a real p-stable law ($p < 1$), for all $\phi \in L^+(\mathscr{K}(\mathbb{B}), \mathbb{R})$;
(b) $\lim_{t\to\infty} t^p P\{\| K \|_G > t\} < \infty$ for some compact convex symmetric set G in $\mathbb{B}$.

Then X is in the DNA of a p-stable compact convex set.

The proof of this example is based on Theorem 1.19 and the following

5.12 LEMMA. *Let $H \in \mathrm{co}\,\mathscr{K}(\mathbb{B})$. Then the set $F = \{K \in \mathrm{co}\,\mathscr{K}(\mathbb{B}): K \subset H\}$ is compact in* $(\mathrm{co}\,\mathscr{K}(\mathbb{B}), \delta)$.

PROOF. It is enough to establish compactness in $C(\mathbb{B}_1^*, \mathrm{w}^*)$ of the corresponding set of support functions, which we also denote by F. Since for $K \subset H$ we have

$$\| K \|_\infty = \| K \| \le \| H \|,$$

F is uniformly bounded. So, by Arzelà-Ascoli's theorem, the result will follow upon showing that F is equicontinuous for some distance d metrizing $(\mathbb{B}_1^*, w^*)$. One such distance is $d(f, g) = \sum_{i=1}^{\infty} 2^{-i}|f(x_i) - g(x_i)|$, $\{x_i\}_{i=1}^{\infty}$ being a countable dense subset of the unit ball of $\mathbb{B}$. Since H is compact, given any $\varepsilon > 0$ there exist $r < \infty$ and integers $i_i < \cdots < i_r$ such that the set of points $\{x_{i_1}, \cdots, x_{i_r}\}$ is $(\varepsilon/6\|H\|)$-dense in $H/\|H\|$. Let $\delta = 2^{-i_r}\varepsilon/3\|H\|$ and let $f, g \in \mathbb{B}_1^*$ satisfy $d(f, g) < \delta$. Then we have

$$|K(f) - K(g)| \le |H(f-g)| \le \|H\| [2\varepsilon/3\|H\| + \max_{j\le r}|f(x_{i_j}) - g(x_{i_j})|] < \varepsilon.$$

Hence, F is equicontinuous for d. □

PROOF OF EXAMPLE 5.11. Find compact convex sets M_i, $M_1 = \{0\}$, such that if F_m^+ is the positive linear span of $M_1, \cdots, M_m$, then $\varepsilon_m := \sup_{K\subset F}\delta(K, F_m^+) \to 0$. Then, (b) in 5.11 implies (b) in 1.19 because

$$\delta(K, F_m^+) = \|K\|_G\delta(K/\|K\|_G, F_m^+) \le \varepsilon_m\|K\|_G. \quad \square$$

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