The Annals of Probability
1985, Vol. 13, No. 2, 379–384

SOME EXAMPLES OF NONLINEAR DIFFUSION PROCESSES HAVING A TIME-PERIODIC LAW

BY MICHAEL SCHEUTZOW

Universität Kaiserslautern

In this paper it is shown that contrary to "linear" homogeneous nondegenerate diffusion processes, "nonlinear" diffusion processes can have a time-periodic law. We study four qualitatively different simple examples: a law-dependent Ornstein-Uhlenbeck process, a deterministic example, a one-dimensional example with a "semilinear" drift and a two-dimensional example with a linear drift.

It is well known that regular (i.e. nonexploding) solutions of equations of the type

(1) $$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t)$$

with $b:\mathbb{R}^d \to \mathbb{R}^d$ locally Lipschitz continuous, $\sigma:\mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d)$ locally Lipschitz-continuous and nondegenerate (i.e. the eigenvalues of $A(x) = (a_{ij}(x))_{1\le i,j\le m}\, a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x)$ are strictly positive on $\mathbb{R}^d$) and $(W(t))_{t\ge 0}$ an m-dimensional Brownian motion cannot have a time-periodic law since they are either positive recurrent, null recurrent or transient (see Hasminskii, 1980). In the case $d = m = 1$ periodic laws cannot even occur if σ is degenerate.

Solutions of certain equations of the type

(2) $$dX(t) = a(X(t), \mathscr{L}(X(t)))dt + b(X(t), \mathscr{L}(X(t)))dW(t),$$

$\mathscr{L}(\cdot)$ denoting the law, however can have a periodic law, as we show in this paper. Solutions of (2) are usually referred to as "nonlinear" diffusion processes since the associated Fokker-Planck-equation is a nonlinear PDE. They have been studied by a number of authors, among them Dawson (1983), McKean (1969), Scheutzow (1983), Sznitman (1983) and Tanaka (1982). Quite frequently they appear as limiting equations of systems of N interacting particles as $N \to \infty$ (Dawson, 1983, Sznitman, 1983). One particular case for which the existence of a periodic law has been shown is the stochastic mean-field "Brusselator," a model for a chemical reaction described by the following pair of equations:

$$dX(t) = (a - (b+1)X(t) + X^2(t)Y(t) + D_1(EX(t) - X(t)))dt + g_1(X(t))dW_1(t)$$

$$dY(t) = (bX(t) - X^2(t)Y(t) + D_2(EY(t) - Y(t)))dt + g_2(Y(t))dW_2(t)$$

where the parameters a, b, D_1 and D_2 and the functions g_1 and g_2 have to satisfy suitable conditions (see Scheutzow, 1983).

Received December 1983; revised May 1984.

AMS 1980 *subject classifications.* Primary 35K55, 60J60; secondary 35B10, 60H10.

Key words and phrases. Nonlinear diffusion process, process with periodic law, Ornstein-Uhlenbeck process, nonlinear PDE.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Probability.* www.jstor.org

Before we discuss examples of nonlinear diffusions with a time-periodic law let us note that one-dimensional equations of the type

$$dX(t) = (f(EX(t))X(t) + g(EX(t)))dt + h(X(t), EX(t))dW(t) \tag{3}$$

with f and g locally Lipschitz-continuous and h Lipschitz-continuous and bounded, cannot have a periodic law. To see this integrate both sides of (3), take expected values and differentiate again. Then we have

$$(d/dt)EX(t) = f(EX(t))EX(t) + g(EX(t)).$$

This is a one-dimensional ODE with locally Lipschitz-continuous right hand side which cannot have a (nondegenerate) periodic solution i.e. if (3) has a periodic law, then $EX(t)$ must be constant. Inserting a constant for $EX(t)$ in (3) results in an equation of type (1) which cannot have a periodic law. (The boundedness of h can of course be relaxed. It was introduced only to avoid complicated arguments showing that the expectation of the stochastic integral is zero.)

To get periodic laws we either have to consider a drift which depends on $\mathscr{L}(X(t))$ not only through $EX(t)$ but which may be linear in X (Example 1) or the drift only depends on X and EX but is nonlinear in X (Examples 2 and 3) or we have to consider the multidimensional linear case with coefficients depending on EX (Example 4).

Let $VX := EX^2 - (EX)^2$ denote the variance of a random variable X.

EXAMPLE 1. Let us consider the following SDE.

$$\begin{aligned} &dX(t) = (f_1(EX(t), VX(t))(X(t) - EX(t)) + f_2(EX(t), VX(t)))dt + dW(t) \\ &\mathscr{L}(X(0)) = \mu. \end{aligned} \tag{4}$$

(4) has a periodic law if and only if the following ODE (5) has a periodic solution $(m(t), v(t))$ satisfying $v(t) > 0$ for any $t \geq 0$.

$$\begin{aligned} (d/dt)m(t) &= f_2(m(t), v(t)) \\ (d/dt)v(t) &= 2f_1(m(t), v(t))v(t) + 1. \end{aligned} \tag{5}$$

Accordingly, if we start from the following ODE (6) which has a periodic solution $(m(t), v(t))$ satisfying $v(t) > 0$ for any $t \geq 0$,

$$\begin{aligned} (d/dt)m(t) &= b_1(m(t), v(t)) \\ (d/dt)v(t) &= b_2(m(t), v(t)), \end{aligned} \tag{6}$$

we obtain a nonlinear diffusion process defined by (4), setting $f_1(m, v) = (b_2(m, v) - 1)/2v$ and $f_2(m, v) = b_1(m, v)$.

PROOF. If (4) has a periodic law μ, it is easy to see that $m(t) = E_\mu X(t)$ and $v(t) = V_\mu X(t)$ is a solution of (5) satisfying $v(t) > 0$ for any $t \geq 0$. Conversely, for any periodic solution of (5) $(m(t), v(t))$ satisfying $v(t) > 0$ for any $t \geq 0$, it is obvious that if $\mathscr{L}(X(0)) = \mathscr{N}(m(0), v(0))$ $\mathscr{L}(X(t)) = \mathscr{N}(m(t), v(t))$ holds for any $t \geq 0$, noting that $\{X(t)\}$ is a Gaussian process.

EXAMPLE 2. Let us first give a very simple example of an equation with a nonlinear (but semilinear i.e. linear on $\mathbb{R}_+$ and on $\mathbb{R}_-$) drift and $\sigma \equiv 0$ i.e. randomness only enters through the initial condition.

We want to have the following solution:

$$X(t) = \begin{cases} \sin t - 2 & \text{with probability } 1/2 \\ \cos t + 2 & \text{with probability } 1/2. \end{cases}$$

The expectation of X is then given by

$$EX(t) = 1/2(\cos t + \sin t).$$

Therefore X, which has a periodic law, is a solution of the equation

$$dX(t) = f(X(t), EX(t))dt$$

$$\mathscr{L}(X(0)) = 1/2(\varepsilon_{-2} + \varepsilon_3) \quad (\varepsilon_x \text{ denoting a unit mass in } x)$$

with

$$f(x, y) = \begin{cases} 2y - x - 2 & \text{if } x < 0 \\ -2y + x - 2 & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Of course one can change f arbitrarily outside the range of X and EX.

EXAMPLE 3. The following example is based on the same idea as the previous one namely to separate the state space $\mathbb{R}$ into two parts, neither of which can be reached by the process if it starts in the other one, but it is less trivial because we allow nonzero diffusion.

We consider the equation

$$dX(t) = f(X(t), EX(t))\,dt + g(X(t))\,dW(t) \tag{7}$$

with g Lipschitz-continuous and bounded, $g(0) = 0$ and

$$f(x, y) = \begin{cases} -x + \alpha y + \beta & \text{if } x > 0 \\ -2x + \gamma y + \delta & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where we want to choose the constants α, β, γ, δ in such a way that for a suitable initial condition, $\mathscr{L}(X(t))$ is periodic. (Obviously we cannot take the same f as in the previous example because for $x > 0$ the drift is linearly increasing in x.)

Let us first look at a solution $X_1(\cdot)$ of (7) where $EX(t)$ is replaced by some deterministic continuous function $e(t)$ and which satisfies $P\{X_1(0) > 0\} = 1$ and $EX_1(0) < \infty$. Define $e_1(t) := EX_1(t)$. Then

$$e_1'(t) = -e_1(t) + \alpha e(t) + \beta$$

provided $\inf_{t\geq 0} \alpha e(t) + \beta \geq 0$ to guarantee that $P\{X_1(t) \geq 0 \text{ for all } t \geq 0\} = 1$. Similarly define X_2 with $P\{X_2(0) < 0\} = 1$ and $EX_2(0) > -\infty$ and $e_2(t) := EX_2(t)$.

Then

$$e_2'(t) = -2e_2(t) + \gamma e(t) + \delta$$

provided $\sup_{t\geq 0}\gamma e(t) + \delta \leq 0$. Now define

$$X(\cdot) = \begin{cases} X_1(\cdot) & \text{with probability } 1/2 \\ X_2(\cdot) & \text{with probability } 1/2. \end{cases}$$

X solves (7) if $EX(t) = 1/2(e_1(t) + e_2(t)) = e(t)$. To show that $EX(t)$ is periodic in t we have to show that

(8) $$\begin{aligned} e_1'(t) &= -e_1(t) + (\alpha/2)(e_1(t) + e_2(t)) + \beta \\ e_2'(t) &= -2e_2(t) + (\gamma/2)(e_1(t) + e_2(t)) + \delta \end{aligned}$$

has a periodic solution with

(9) $$\begin{aligned} &\min_{t\geq 0}(\alpha/2)(e_1(t) + e_2(t)) + \beta \geq 0 \quad \text{and} \\ &\max_{t\geq 0}(\gamma/2)(e_1(t) + e_2(t)) + \delta \leq 0. \end{aligned}$$

It is well known that (8) has a periodic solution if and only if the matrix

$$M = \begin{bmatrix} -1 + (\alpha/2) & (\alpha/2) \\ (\gamma/2) & (\gamma/2) - 2 \end{bmatrix}$$

has a pair of purely imaginary (nonzero) eigenvalues. It is easy to check that this is the case if $\alpha + \gamma = 6$ and $\alpha < -2$. Let us choose $\alpha = -4$ (and $\gamma = 10$). This implies that the period of the solutions of (8) is 2π. The solutions of (8) (after a suitable phase shift) are of the form

$$e_1(t) = -2A \sin t - 3\beta - 2\delta$$

$$e_2(t) = 3A \sin t + A \cos t + 5\beta + 3\delta.$$

Conditions (9) are

$$-2(|A|\sqrt{2} + 2\beta + \delta) + \beta \geq 0$$

and

$$5(|A|\sqrt{2} + 2\beta + \delta) + \delta \leq 0.$$

These are satisfied e.g. for $\delta = 0$, $\beta = -1$ and $A = 1$. So we have shown that (7) with

$$f(x, y) = \begin{cases} -x - 4y - 1 & \text{if } x > 0 \\ -2x + 10y & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a solution with the periodic expectation function

$$EX(t) = 1/2(\sin t + \cos t - 2).$$

To see that there exists an initial condition such that not only $EX(\cdot)$ but also $\mathscr{L}(X(\cdot))$ is periodic, consider the SDE (7) with $EX(t) = e(t) = 1/2(\sin t + \cos t - 2)$ on the half line $(0, +\infty)$. Denoting by $X_1(t)$ the solution of

(7), $\{X_1(2\pi n)\}_{n\in N_0}$ is a homogeneous time Markov chain on $(0, +\infty)$, the transition function $P(\cdot, \cdot, \cdot)$ of which satisfies the Feller property. Assuming $EX_1(0) = 3$, $EX_1(0)^2 < +\infty$ and $EX_1(0)^{-1} < +\infty$ it is easy to see that $EX_1(t) = e_1(t) = -2 \sin t + 3$, $\sup_{t\geq 0} EX_1(t)^2 < +\infty$ and $\sup_{t\geq 0} EX_1(t)^{-1} < +\infty$. Hence

$$\left\{P_n(\cdot) = \frac{1}{n} \sum_{k=1}^{n} \int_0^\infty P(x_0, k, \cdot)\, dF_{X_1(0)}(x_0)\right\}$$

is tight as a family of probability measures on $(0, +\infty)$ and any limit μ_1 of $\{P_n\}$ is an invariant probability measure of $\{X_1(2\pi n)\}_{n\in N_0}$ and $\int_0^\infty x\mu_1(dx) = 3$. Therefore μ_1 is a periodic law on $(0, +\infty)$ of (7) with $EX(t) = e(t)$ and $E_{\mu_1}(X_1(t)) = e_1(t)$. In the same way we obtain a periodic law μ_2 with the same period as μ_1 of SDE (7) on $(-\infty, 0)$ with $EX(t) = e(t)$, and $E_{\mu_2}(X_2(t)) = e_2(t)$. Thus we see that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ is a periodic law of (7) since $e(t) = \frac{1}{2}(e_1(t) + e_2(t))$ holds.

REMARKS. (a) If we choose $A = 0$ in the proof of the example, then $EX(t) = -1$ for all $t \geq 0$ which shows (using similar arguments as before, for details see Hasminskii, 1980) that (7) also has an invariant probability measure.

(b) The proof also shows that (7) has infinitely many periodic laws because A can be any real number which is sufficiently small.

(c) The time evolution of the $e_1(t)$ and $e_2(t)$ do not depend on the diffusion g at all. In particular the case $g \equiv 0$ is also covered by Example 3.

(d) Although, due to the fact that X is a nonlinear diffusion, the law of the solution of (7) with initial condition $\lambda\mu_1 + (1 - \lambda)\mu_2$ ($1 > \lambda > 0$ and μ_1 and μ_2 being probability measures on $\mathbb{R}$) is not the same as the same convex combination of the solutions with initial conditions μ_1 and μ_2, this is true for the expected values because (8) is a linear system.

EXAMPLE 4. Using a slight modification of Example 3 we can construct a two-dimensional nonlinear diffusion process with a periodic law satisfying a pair of equations with a drift which is (contrary to Example 3) linear in X (and which is also linear in EX):

$$\begin{aligned} dX_1(t) &= (-X_1(t) + (\alpha/2)E(X_1(t) + X_2(t)) + \beta)dt \\ &\quad + g_1(X_1(t), X_2(t))dW_1(t) \\ dX_2(t) &= (-2X_2(t) + (\gamma/2)E(X_1(t) + X_2(t)) + \delta)dt \\ &\quad + g_2(X_1(t), X_2(t))dW_2(t) \end{aligned}$$

where α, β, γ and δ are chosen as in Example 3 and g_1 and g_2 are bounded Lipschitz-continuous functions.

The expected values $e_1(t) := EX_1(t)$ and $e_2(t) := EX_2(t)$ satisfy the same pair of ordinary differential equations as before. Therefore $(e_1(t), e_2(t))$ are periodic with period 2π and by the same argument as before a periodic law exists. In this example we do not have to worry about conditions (9). These results show that Example 3 is a "disguised" two-dimensional example.

Acknowledgement. I wish to thank D. A. Dawson for valuable discussions on the subject and the referee for his helpful suggestions.

REFERENCES

ARNOLD, L. (1973). *Stochastische Differentialgleichungen.* Oldenbourg Verlag, München.

DAWSON, D. A. (1983). Critical dynamics and fluctuations for a mean-field model of cooperative behavior. *J. Statist. Phys.* **31** 29–85.

HASMINSKII, R. S. (1980). *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, Alphen aan den Rijn.

LIPTSER, R. S. and SHIRYAYEV, A. N. (1977). *Statistics of Random Processes I.* Springer Verlag, Berlin.

MCKEAN, JR., H. P. (1969). Propagation of chaos for a class of nonlinear parabolic equations, *Lecture Series in Differential Equations*, **2** 177–193. Van Nostrand Reinhold, New York.

SCHEUTZOW, M. (1983). Periodic behavior of a stochastic Brussel's model. Technical Report Series of the Laboratory for Research in Statistics and Probability No. 9, Ottawa.

SZNITMAN, A. S. (1983). An example of nonlinear diffusion process with normal reflecting boundary conditions and some related limit theorems. (To appear).

TANAKA, H. (1982). Limit theorems for certain diffusion processes with interaction. *Proc. of the Taniguchi Intern. Symp. on Stochastic Analysis*, Katata and Kyoto. (To appear).

FACHBEREICH MATHEMATIK
UNIVERSITÄT KAISERSLAUTERN
ERWIN-SCHRÖDINGER-STRASSE
6750 KAISERSLAUTERN
WEST GERMANY