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ON EVALUATING THE DONSKER-VARADHAN I-FUNCTION

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Let $x(t)$ be a Feller process on a complete separable metric space A and consider the occupation measure $L_t(\omega, \cdot) = \int_0^t \chi_{(\cdot)}(x(s))\, ds$. The I-function is defined for $\mu \in \mathscr{P}(A)$, the set of probability measures on A, by $I(\mu) = -\inf_{u \in \mathscr{D}^+} \int_A (Lu/u)\, d\mu$ where $(L, \mathscr{D})$ is the generator of the process and $\mathscr{D}^+ \subset \mathscr{D}$ consists of the strictly positive functions in $\mathscr{D}$. The I-function determines the asymptotic rate of decay of $P((1/t)L_t(\omega, \cdot) \in G)$ for $G \subset \mathscr{P}(A)$. The first difficulty encountered in evaluating $I(\mu)$ is that the domain $\mathscr{D}$ is generally not known explicitly. In this paper, we prove a theorem which allows us to restrict the calculation of the infimum to a nice subdomain. We then apply this general result to diffusion processes with boundaries.

1. Introduction. Let $x(t)$ be a time homogeneous Markov process with state space A, a complete separable metric space. Let T_t be the semigroup induced by the process and denote by $(L, \mathscr{D})$ the infinitesimal generator. We assume that the process is Feller and lives on $D([0, \infty), A)$, the space of right continuous trajectories with left-hand limits at every point. Denote by Ω_x, the collection of paths $x(\cdot) \in D([0, \infty), A)$ with $x(0) = x$. The process above induces a measure P_x on Ω_x.

For $\omega \in \Omega_x$ and a Borel set $B \subset A$, consider

$$L_t(\omega, B) = \int_0^t \chi_{(B)}(x(s))\, ds.$$

So $(1/t)L_t(\omega, B)$ is the proportion of time up to t that a particular path $\omega = x(\cdot)$ spends in the set B. Thus $(1/t)L_t(\omega, \cdot) \in \mathscr{P}(A)$, the set of probability measures on A; $(1/t)L_t(\omega, \cdot)$ is the occupation measure of the process. For $\mu \in \mathscr{P}(A)$, define

$$(1.1) \quad I(\mu) = -\inf_{u \in \mathscr{D}^+} \int_A \frac{Lu}{u}\, d\mu \quad \text{where} \quad \mathscr{D}^+ = \mathscr{D} \cap \{u \colon u \geq c > 0\}.$$

It is easy to see that $I(\mu)$ is lower semicontinuous under weak convergence on $\mathscr{P}(A)$. Under suitable recurrence and transitivity conditions, Donsker and Varadhan ([2], [3]) have proven that, for open sets $G \subset \mathscr{P}(A)$,

$$(1.2) \quad \liminf_{t \to \infty} (1/t) \log P_x((1/t)L_t(\omega, \cdot) \in G) \geq -\inf_{\mu \in G} I(\mu)$$

and for closed sets $C \subset \mathscr{P}(A)$,

$$(1.3) \quad \limsup_{t \to \infty} (1/t) \log P_x((1/t)L_t(\omega, \cdot) \in C) \leq -\inf_{\mu \in C} I(\mu).$$

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We see that for large t, if U_μ is a small neighborhood of μ, then

$$\exp(-t(I(\mu)+\varepsilon)) \le P_x((1/t)L_t(\omega,\cdot)\in U_\mu) \le \exp(-t(I(\mu)-\varepsilon)).$$

We call $I(\mu)$ the I-function for the process. Donsker and Varadhan have done much theoretical work with the I-function (see in particular [1], [2], [3]), however little work has been directed towards evaluating the I-function explicitly. The one main result along these lines is for the selfadjoint case. Suppose there exists a σ-finite reference measure β with respect to which the semigroup is selfadjoint. Then there exists a corresponding negative semidefinite selfadjoint generator $(\hat{L}, \hat{\mathscr{D}})$ on $L_2(A, \beta(dx))$. Donsker and Varadhan [2] have shown that for $\mu \in \mathscr{P}(A)$,

$$I(\mu) = \|(-\hat{L})^{1/2} f^{1/2}\|_2^2 \quad \text{if } \mu \text{ has a density } d\mu/d\beta = f \text{ with } f^{1/2}\in\hat{\mathscr{D}}_{1/2}$$
$$= \infty, \quad \text{otherwise.}$$

Here, $\hat{\mathscr{D}}_{1/2}$ is the domain of the selfadjoint operator $(-\hat{L})^{1/2}$

The first difficulty that arises in computing the I-function is that the domain $\mathscr{D}$ is generally not known explicitly. One would like to be able to restrict the calculation of the infimum to a nice dense (in sup norm) subdomain $\tilde{\mathscr{D}}^+ \subset \mathscr{D}^+$. That is, we would like it to be true that

$$I(\mu) \equiv -\inf_{u\in\mathscr{D}^+}\int_A \frac{Lu}{u}\,d\mu = -\inf_{u\in\tilde{\mathscr{D}}^+}\int_A \frac{Lu}{u}\,d\mu.$$

Of course, if $\tilde{\mathscr{D}}$ is a core for the operator L, then the result is trivial. However, even a core is not generally available explicitly. In this paper we prove that the above result holds as long as $(L, \tilde{\mathscr{D}})$ determines a unique Markov semigroup, or equivalently, there is a unique solution to the martingale problem for $(L, \tilde{\mathscr{D}})$ (see below).

The corresponding problem for invariant measures was considered by Echeverria [4]. It is well known that $\int_A Lu\,d\mu = 0$ for all $u\in\mathscr{D}$ if and only if μ is invariant for the process. Echeverria showed that if $\int_A Lu\,d\mu = 0$ for all $u \in \tilde{\mathscr{D}}$, where $(L, \tilde{\mathscr{D}})$ determines a unique Markov semigroup, then in fact $\int_A Lu\,d\mu = 0$ for all $u\in\mathscr{D}$ and hence μ is invariant for the process.

Consider the case of a Markov diffusion process in a bounded region, $A\subset R^n$; that is the case when the generator is an extension of $(L, \tilde{\mathscr{D}})$ where $L = \frac{1}{2} a\nabla\cdot\nabla + b\cdot\nabla$ and $\tilde{\mathscr{D}} = \{u\in C^2(A): J\cdot\nabla u = 0 \text{ on } \partial A\}$ with $-J\cdot n \ge \gamma > 0$, where n is the outward normal vector. Let T_t denote the semigroup for the process. If the coefficients a, b and J are in $C^n(A)$, A has a C^n boundary and a is strictly elliptic, then T_t leaves $C^n(A)$ invariant. From this, it is not difficult to show that for $n\ge 2$,

$$-\inf_{u\in C^n\cap\mathscr{D}^+}\int_A \frac{Lu}{u}\,d\mu = -\inf_{u\in\mathscr{D}^+}\int_A \frac{Lu}{u}\,d\mu,$$

so that $\tilde{\mathscr{D}}^+ = C^n(A)\cap\mathscr{D}^+$ would be a nice subdomain to restrict to, since it would consist of C^n functions satisfying $u>0$ in A and $J\cdot\nabla u = 0$ on ∂A.

But we would like to consider diffusions with less restrictive conditions on the coefficients. In particular, we would like to consider diffusion processes for which the martingale problem has a unique solution. The martingale problem for any operator $(L, \tilde{\mathscr{D}})$ on a metric space A, where $\tilde{\mathscr{D}}$ is a dense subset of $C(A)$ on which L is defined, is the problem of finding for each $x \in A$, a probability measure P_x on $D([0, \infty), A)$ with

(i) $P_x(x(0) = x) = 1$
(ii) $f(x(t)) - \int_0^t Lf(x(s))\, ds$ is a P_x-martingale for each $f \in \tilde{\mathscr{D}}$.

The generator of the process will then be some extension of $(L, \tilde{\mathscr{D}})$. In Lemma 3.1 at the end of Section 3, we recast the Stroock and Varadhan submartingale uniqueness theorem for diffusions [6] into a martingale uniqueness theorem. This lemma tells us that when

$$L = \tfrac{1}{2} a\nabla \cdot \nabla + b \cdot \nabla, \quad \tilde{\mathscr{D}} = \{f \in C^2(A): J \cdot \nabla f = 0 \text{ on } \partial A\},$$

and A is an open, bounded region in R^n, then the martingale problem has a unique solution in $C([0, \infty), A)$, the space of continuous trajectories on $[0, \infty)$, if the following conditions are met: a is continuous and strictly elliptic, $b \in C(A)$, $J \in C^1(\partial A)$, A has a C^2 boundary and the normal component of J is bounded away from zero. It is for this class of diffusions that we would like to calculate the I-function. (Indeed, uniqueness persists even if b is only bounded and measurable but our methods do not cover this case.)

In Section 2, we prove a theorem which allows us to restrict the calculation of the infimum to a nice dense subdomain. In an upcoming paper, this result will be used to obtain an explicit representation of the I-function for diffusion processes with boundaries.

THEOREM 1.4. *Let A be a compact metric space. Let $\tilde{\mathscr{D}}$ be a dense subset of $C(A)$ (with sup norm) and let $L\colon \tilde{\mathscr{D}} \to C(A)$ be an operator satisfying the maximum principle. Assume that for each $x \in A$, the martingale problem for $(L, \tilde{\mathscr{D}})$ is well posed, that is, has a unique solution. The infinitesimal generator of the process is an extension of $(L, \tilde{\mathscr{D}})$, call it $(L, \mathscr{D})$. If $\tilde{\mathscr{D}}$ satisfies the condition that $g_1, \cdots, g_n \in \tilde{\mathscr{D}}$ implies $\psi(g_1, \cdots, g_n) \in \tilde{\mathscr{D}}$ when $\psi\colon R^n \to R$ is a smooth function, then*

$$-\inf_{u \in \mathscr{D}^+} \int_A \frac{Lu}{u}\, d\mu = -\inf_{u \in \tilde{\mathscr{D}}^+} \int_A \frac{Lu}{u}\, d\mu \quad \textit{for all} \quad \mu \in \mathscr{P}(A).$$

Hence $I(\mu) = -\inf_{u \in \tilde{\mathscr{D}}^+} \int_A (Lu/u)\, d\mu$.

COROLLARY 1.5. *If $\int_A (Lu/u)\, d\mu \geq 0$ for all $u \in \tilde{\mathscr{D}}^+$, then μ is invariant for the process.*

PROOF. This follows directly from the fact that I is a nonnegative functional and that $I(\mu) = 0$ if and only if μ is invariant for the process [2].

Consider Theorem 1.4 applied to diffusion processes with boundaries and with

$(L, \tilde{\mathscr{D}})$ as in Lemma 3.1. The lemma tells us that there is a unique solution to the martingale problem for $(L, \tilde{\mathscr{D}})$. Let $\mathscr{D}$ be the domain of the generator. Since $J \in C^1(\partial A)$, we see that $\tilde{\mathscr{D}} = C^2 \cap \mathscr{D} = \{u \in C^2 \colon J \cdot \nabla u = 0 \text{ on } \partial A\}$ is dense and satisfies the condition of Theorem 1.4.

COROLLARY 1.6. *Let A be an open bounded region in R^n. Let $(L, \mathscr{D})$ be the generator of the unique Markov diffusion process which solves the martingale problem for $(L, \tilde{\mathscr{D}})$ where $L = \frac{1}{2} a \nabla \cdot \nabla + b \cdot \nabla$ and*

$$\tilde{\mathscr{D}} = \{u \in C^2(\bar{A}) \colon J \cdot \nabla u = 0 \text{ on } \partial A\}.$$

Assume a belongs to $C(\bar{A})$ and is strictly elliptic, $b \in C(\bar{A})$, $J = C^1(\partial A)$, A has a C^2 boundary and $|J \cdot n| \geq \gamma > 0$. Then

$$I(\mu) = -\inf_{\{u \in C^2(\bar{A})^+ : J \cdot \nabla u = 0\}} \int_A \frac{Lu}{u}\, d\mu.$$

(Here $C^m(A)^+$ denotes the strictly positive functions in $C^m(A)$.)

The case in which A is a complete separable metric space may be treated similarly. In Section 4, we briefly outline the necessary revisions.

2. Proof of the Theorem.

NOTATION. Given any function space $\mathscr{F}$, we let $\mathscr{F}^+ = \{u \in \mathscr{F} \colon u > 0\}$. Since A is compact, note that functions in $C(A)^+$ are bounded away from zero.

Since

$$-\inf_{u \in \mathscr{D}^+} \int_A \frac{Lu}{u}\, d\mu \geq -\inf_{u \in \tilde{\mathscr{D}}^+} \int_A \frac{Lu}{u}\, d\mu,$$

we need only prove the reverse inequality. Fix an arbitrary measure $\mu \in \mathscr{P}(A)$. We will prove that if

$$\int_A \frac{Lu}{u}\, d\mu \geq -k \quad \text{for all} \quad u \in \tilde{\mathscr{D}}^+,$$

then

$$\int_A \frac{Lu}{u}\, d\mu \geq -k \quad \text{for all} \quad u \in \mathscr{D}^+.$$

Let $\lambda > 0$ and consider $I - \lambda L \colon \tilde{\mathscr{D}} \to C(A)$. Let $M_\lambda = (I - \lambda L)\tilde{\mathscr{D}}$ and let $\Pi_\lambda = (I - \lambda L)^{-1} \colon M_\lambda \to \tilde{\mathscr{D}}$. Note that Π_λ is the restriction to M_λ of the operator $(1/\lambda)R_{1/\lambda} \colon C(A) \to \mathscr{D}$ where $R_\lambda = (\lambda I - L)^{-1}$ is the resolvent.

LEMMA 2.1. (i) *If $f \in M_\lambda^+$, then* $\Pi_\lambda f \in C(A)^+$.

(ii) $1 \in M_\lambda$ *and* $\Pi_\lambda 1 = 1$

(iii) $\int_A \log(\Pi_\lambda g/g)\, d\mu \geq -k\lambda$, *for all* $g \in M_\lambda^+$.

It is clear that (i) and (ii) hold since Π_λ is the restriction of $(1/\lambda)R_{1/\lambda}$ to M_λ

and $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} E_x f(x(t))\, dt$ where $x(t)$ is the process with generator $(L, \mathscr{D})$.

Now consider (iii). Pick any $g \in M_\lambda^+$. Then there exists $u \in \tilde{\mathscr{D}}$ with $(I - \lambda L)u = g$ or $\Pi_\lambda g = u$. Since $g \in M_\lambda^+$, $u \in \tilde{\mathscr{D}}^+$. We have $\lambda Lu = u - g = \Pi_\lambda g - g$. Thus

$$-k \le \int_A \frac{Lu}{u}\, d\mu = \frac{1}{\lambda} \int_A \frac{\Pi_\lambda g - g}{\Pi_\lambda g}\, d\mu = \frac{1}{\lambda} \int_A \left(1 - \frac{g}{\Pi_\lambda g}\right) d\mu.$$

Since $x - 1 \ge \log x$, for all $x > 0$, $(g/\Pi_\lambda g) - 1 \ge \log(g/\Pi_\lambda g)$. Hence

$$-k \le \frac{1}{\lambda} \int_A \left(1 - \frac{g}{\Pi_\lambda g}\right) d\mu \le \frac{1}{\lambda} \int_A \log\left(\frac{\Pi_\lambda g}{g}\right) d\mu$$

or

$$\int_A \log\left(\frac{\Pi_\lambda g}{g}\right) d\mu \ge -k\lambda.$$

We now want to extend Π_λ from the subspace M_λ to all of $C(A)$ as a positive operator $\tilde{\Pi}_\lambda$ satisfying

$$\int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right) d\mu \ge -k\lambda \quad \text{for all} \quad g \in C(A)^+.$$

First we prove

LEMMA 2.2. *If $\tilde{\Pi}_\lambda$ is a strictly positive operator, then the condition*

$$\int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right) d\mu \ge -k\lambda \quad \textit{for all} \quad g \in C(A)^+ \tag{2.3}$$

is equivalent to the condition

$$\log \int_A f\tilde{\Pi}_\lambda g\, d\mu - \int_A \log fg\, d\mu \ge -k\lambda \quad \textit{for all} \quad f, g \in C(A)^+. \tag{2.4}$$

PROOF. Assume (2.3) holds. Then

$$\begin{aligned} -k\lambda \le \int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right) d\mu &= \int_A \log\left(\frac{f\tilde{\Pi}_\lambda g}{fg}\right) d\mu \\ &= \int_A \log f\tilde{\Pi}_\lambda g\, d\mu - \int_A \log fg\, d\mu \\ &\le \log \int_A f\tilde{\Pi}_\lambda g\, d\mu - \int_A \log fg\, d\mu \end{aligned}$$

by Jensen's inequality. To go the other way, given any $g \in C(A)^+$, $\tilde{\Pi}_\lambda g \in C(A)^+$ and hence we may define $f \in C(A)^+$ by $f = 1/\tilde{\Pi}_\lambda g$. Using this f in (2.4) gives us (2.3).

LEMMA 2.5. *Π_λ: $M_\lambda \to C(A)$ can be extended to $\tilde{\Pi}_\lambda$: $C(A) \to B(A)$, the space of bounded measurable functions, with the following conditions holding:*
(i) $\tilde{\Pi}_\lambda f(x) = \Pi_\lambda f(x)$ *for all* $x \in A, f \in M_\lambda$
(ii) $\tilde{\Pi}_\lambda f \geq 0$, if $f \geq 0$
(iii) $\int_A \log(\tilde{\Pi}_\lambda g/g)\, d\mu \geq -k\lambda$ *for all* $g \in C(A)^+$.

PROOF. By Lemma 2.2, it suffices to show (2.4) in place of condition (iii). For any $g \in M_\lambda^+$, we have

$$\int_A \log\left(\frac{\Pi_\lambda g}{g}\right) d\mu \geq -k\lambda$$

and hence

$$\log \int_A f\Pi_\lambda g\, d\mu - \int_A \log fg\, d\mu \geq -k\lambda \quad \text{for all} \quad f \in C(A)^+,$$

again by Lemma 2.2.

This may be written as

$$\int_A f\Pi_\lambda g\, d\mu \geq \exp(-k\lambda)\exp\left(\int_A \log fg\, d\mu\right) \tag{2.6}$$
$$\text{for all} \quad f \in C(A)^+, \quad g \in M_\lambda^+.$$

Consider the subspace

$$V = \{\textstyle\sum_{i=1}^m f_i(x)g_i(y) \colon f_i \in C(A),\, g_i \in M_\lambda\} \quad \text{of} \quad C(A \times A)$$

and define the linear functional Λ_λ on V by

$$\Lambda_\lambda\{\textstyle\sum_{i=1}^m f_i(x)g_i(y)\} = \displaystyle\int_A \textstyle\sum_{i=1}^m f_i(x)\Pi_\lambda g_i(x)\, d\mu(x).$$

To see that Λ_λ is well defined, assume $v \in V$ and $v = \sum_{i=1}^n f_i(x)g_i(y) = \sum_{i=1}^m h_i(x)k_i(y)$ with f_i, $h_i \in C(A)$ and g_i, $k_i \in M_\lambda$. For fixed x, $\sum_{i=1}^n f_i(x)g_i(y) = \sum_{i=1}^m h_i(x)k_i(y)$ is a function in $C(A)$. Thus we have

$$\begin{aligned}\textstyle\sum_{i=1}^n f_i(x)\Pi_\lambda g_i(y) &= \textstyle\sum_{i=1}^n f_i(x)(1/\lambda)R_{1/\lambda}g_i(y) = (1/\lambda)R_{1/\lambda}(\sum_{i=1}^n f_i(x)g_i(y))\\ &= (1/\lambda)R_{1/\lambda}(\textstyle\sum_{i=1}^m h_i(x)k_i(y)) = \sum_{i=1}^m h_i(x)(1/\lambda)R_{1/\lambda}k_i(y)\\ &= \textstyle\sum_{i=1}^m h_i(x)\Pi_\lambda k_i(y).\end{aligned}$$

Let Q_λ: $C(A \times A)^+ \to R$ be a functional defined by

$$Q_\lambda(u) = \sup_{c(x)d(y)\leq u(x,y);\, c,d\in C(A)^+} \exp(-k\lambda)\exp\left(\int_A \log c(x)\, d(x)\, d\mu(x)\right).$$

Hence by (2.6), for $f \in C(A)^+$, $g \in M_\lambda^+$, we have

$$\Lambda_\lambda(f(x)g(y)) \geq Q_\lambda(f(x)g(y)).$$

Note that $Q(\alpha u) = \alpha Q(u)$ if $\alpha > 0$. We now state two lemmas needed to prove

Lemma 2.5. Their proofs will be deferred until the completion of the proof of the lemma.

LEMMA 2.7. *Q_λ is superadditive, that is, for $u, v \in C(A \times A)^+$, $Q_\lambda(u + v) \geq Q_\lambda(u) + Q_\lambda(v)$.*

LEMMA 2.8. *For $u \in V^+$, $\Lambda_\lambda(u) \geq Q_\lambda(u)$.*

We are now in the position to apply the Hahn-Banach theorem. In the standard version, a functional F is defined on a subspace M of a space $\mathscr{G}$ and is bounded from above by a subadditive functional ρ defined on $\mathscr{G}$. The functional F can then be extended to all of $\mathscr{G}$ and still be bounded by ρ. By considering $F' = -F$ and $\rho' = -\rho$, we see that the hypothesis of the theorem might just as well be that the functional F' to be extended is bounded from below by a superadditive functional ρ'. In the present case, Λ_λ is bounded from below by a superadditive functional Q_λ. However Q_λ is only defined on positive functions whereas, to apply the standard Hahn-Banach theorem, it should be defined on the entire space $C(A \times A)$. By chasing through the proof of the Hahn-Banach theorem, we will find that this is no problem. By the Hausdorff Maximal Principle, we extend Λ_λ: $V \to R$ to a maximal operator $\tilde{\Lambda}_\lambda$: $\tilde{V} \to R$ satisfying $\tilde{\Lambda}_\lambda(u) \geq Q_\lambda(u)$ for all $u \in \tilde{V}^+$. Then to show that in fact $\tilde{V} = C(A \times A)$, we prove that if $h(x, y) \notin \tilde{V}$, then $\tilde{\Lambda}_\lambda$ can be extended to the subspace generated by $\tilde{V}$ and h with the lower bound from Q holding for all positive functions in this larger subspace. This contradicts the supposed maximality of $\tilde{V}$. Suppose $h(x, y) \notin \tilde{V}$. Then for any $g_1, g_2 \in \tilde{V}$ satisfying $g_1 - h > 0$, $g_2 + h > 0$, $\tilde{\Lambda}_\lambda(g_1) + \Lambda_\lambda(g_2) = \tilde{\Lambda}_\lambda(g_1 + g_2) \geq Q_\lambda(g_1 + g_2) \geq Q_\lambda(g_1 - h) + Q_\lambda(g_2 + h)$. Thus $\tilde{\Lambda}_\lambda(g_1) - Q_\lambda(g_1 - h) \geq Q_\lambda(g_2 + h) - \tilde{\Lambda}_\lambda(g_2)$ and hence

$$\inf_{g_1 \in \tilde{V}; g_1 - h > 0} \tilde{\Lambda}_\lambda(g_1) - Q_\lambda(g_1 - h) \geq \sup_{g_2 \in \tilde{V}; g_2 + h > 0} Q_\lambda(g_2 + h) - \tilde{\Lambda}_\lambda(g_2).$$

Pick α satisfying

$$\inf_{g_1 \in \tilde{V}; g_1 - h > 0} \tilde{\Lambda}_\lambda(g_1) - Q_\lambda(g_1 - h) \geq \alpha \geq sup_{g_2 \in \tilde{V}; g_2 + h > 0} Q_\lambda(g_2 + h) - \tilde{\Lambda}_\lambda(g_2). \tag{2.9}$$

Define $\tilde{\Lambda}_\lambda(h) = \alpha$. We must show that for any $t \in R$, $g \in \tilde{V}$ with $g + th > 0$, $\tilde{\Lambda}_\lambda(g + th) \geq Q_\lambda(g + th)$. First assume $t > 0$. Then since $g/t + h > 0$, we get from the right-hand inequality of (2.9) that

$$\tilde{\Lambda}_\lambda(g + th) = \tilde{\Lambda}_\lambda(g) + t\alpha \geq \tilde{\Lambda}_\lambda(g) + t[Q_\lambda((g/t) + h) - \tilde{\Lambda}_\lambda(g/t)] = Q_\lambda(g + th).$$

If $t < 0$, we use the left-hand inequality in (2.9). This then proves that the Hahn-Banach theorem is applicable even when Q_λ is defined only on positive functions.

Thus we obtain an extension $\tilde{\Lambda}_\lambda$: $C(A \times A) \to R$ of Λ_λ satisfying $\tilde{\Lambda}_\lambda(u) \geq Q_\lambda(u)$, for all $u \in C(A \times A)^+$. Since $Q_\lambda(u) \geq 0$, $\tilde{\Lambda}_\lambda$ is a positive functional on $C(A \times A)$ and hence, by the Riesz representation theorem, there exists a measure B with $\Lambda_\lambda(u) = \int_{A \times A} u(x, y)\, dB(x, y)$. Since $\tilde{\Lambda}_\lambda 1 = \Lambda_\lambda 1 = \int_A 1\, d\mu$, we see that B is a probability measure on $A \times A$. Let E^B denote integration with respect to B and let $\mathscr{F}_x$ denote the σ-field on $C(A \times A)$ generated by all functions which depend only on x. Then $E^B(\cdot \mid \mathscr{F}_x)$ depends only on x. We write $E^B(\cdot \mid x)$ for the version of $E(\cdot \mid \mathscr{F}_x)$ which is a regular conditional probability distribution. For a proof of the existence of a regular conditional probability distribution, see [7]. For $q \in C(A)$, we have $\int_A q(x)\, d\mu(x) = \Lambda_\lambda(q) = \tilde{\Lambda}_\lambda(q) = \int_{A\times A} q(x)\, dB(x, y)$. In fact,

$$\int_A q(x)\, d\mu(x) = \int_{A\times A} q(x)\, dB(x, y)$$

then holds for all bounded measurable $q(x)$. In particular, applying this when $q(x) = f(x)E^B(g \mid x)$ where $f \in C(A)$, $g \in M_\lambda$, we see that

$$\begin{aligned}\int_A f(x)\Pi_\lambda g(x)\, d\mu &= \Lambda_\lambda(fg) = \tilde{\Lambda}_\lambda(fg) = \int_{A\times A} f(x)g(y)\, dB(x, y)\\ &= \int_{A\times A} f(x)E^B(g \mid x)\, dB(x, y)\\ &= \int_A f(x)E^B(g \mid x)\, d\mu.\end{aligned}$$

Since this holds for all $f \in C(A)$, we have $\Pi_\lambda g(x) = E^B(g \mid x)$ for a.e. $[\mu]$ $x \in A$. Define $\bar{\Pi}_\lambda$: $C(A) \to B(A)$ by $\bar{\Pi}_\lambda g(x) = E^B(g \mid x)$. Now pick a countable set $\{g_m\}$ which is dense in M_λ. For each g_m pick a μ-null set N_m with $\bar{\Pi}_\lambda g_m(x) = \Pi_\lambda g_m(x)$ for all $x \in A - N_m$. Let $= \bigcup_{n=1}^{\infty} N_n$. Then for any $g \in M_\lambda$, pick a subsequence g_{n_k} converging to g in the sup norm. For $x \in A - N$ we have $\bar{\Pi}_\lambda g_{n_k}(x) = E^B(g_{n_k} \mid x)$ and the latter is a regular conditional probability distribution. Furthermore Π_λ is a restriction of $(1/\lambda)R_{1/\lambda}$, which is a bounded operator in sup norm. Thus, letting $n_k \to \infty$, we may conclude that $\bar{\Pi}_\lambda g(x) = \Pi_\lambda g(x)$ for all $x \in A - N$. Also, since $E^B(\cdot \mid x)$ is a regular conditional probability distribution, $E^B(g \mid x) \geq 0$ for all x if $g \geq 0$. Hence $\bar{\Pi}_\lambda$ is a nonnegative operator. Let's define $\tilde{\Pi}_\lambda$: $C(A) \to B(A)$ by

$$\begin{aligned}\tilde{\Pi}_\lambda g(x) &= \bar{\Pi}_\lambda g(x) &&\text{for}\quad x \in A - N\\ \tilde{\Pi}_\lambda g(x) &= (1/\lambda)R_{1/\lambda} g(x) &&\text{for}\quad x \in N.\end{aligned}$$

Then since Π_λ is the restriction of $(1/\lambda)R_{1/\lambda}$ to M_λ, we have

$$\begin{aligned}&\tilde{\Pi}_\lambda g(x) = \Pi_\lambda g(x) &&\text{for all}\quad x \in A,\quad g \in M_\lambda\\ &\tilde{\Pi}_\lambda g(x) \geq 0 &&\text{for all}\quad x \in A\quad \text{if}\quad g \in C(A)\quad \text{and}\quad g \geq 0.\end{aligned}$$

Also, for $f, g \in C(A)^+$, $\tilde{\Lambda}_\lambda(f(x)g(y)) \geq Q_\lambda(f(x)g(y))$ or

$$\int_A f(x)\bar{\Pi}_\lambda g(x)\, d\mu(x) \geq \exp(-k\lambda)\exp \int_A \log f(x)g(x)\, d\mu(x).$$

Since $\tilde{\Pi}_\lambda g = \bar{\Pi}_\lambda g$, a.e. $[\mu]$,

$$\int_A f(x)\tilde{\Pi}_\lambda g(x)\, d\mu(x) \geq \exp(-k\lambda)\exp \int_A \log f(x)g(x)\, d\mu(x),$$

and thus by Lemma 2.2,

$$\int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right) d\mu \geq -k\lambda \quad \text{for all} \quad g \in C(A)^+.$$

This proves Lemma 2.5.

PROOF OF LEMMA 2.7. We must show that given $u, v \in C(A \times A)^+$ and $f, g, h, k \in C(A)^+$ with $u(x, y) \geq f(x)g(y)$, $v(x, y) \geq h(x)k(y)$, we have

$$\sup_{c(x)d(y) \leq u(x,y)+v(x,y); c,d \in C^+(A)} \exp \int_A \log c(x)\, d(x)\, d\mu(x)$$
$$\geq \exp \int_A \log f(x)g(x)\, d\mu(x) + \exp \int_A \log h(x)k(x)\, d\mu(x).$$

Hence it is sufficient to show

$$\exp \int_A \log fg\, d\mu + \exp \int_A \log hk\, d\mu$$
$$\leq \sup_{c(x)d(y) \leq f(x)g(y)+h(x)k(y); c,d \in C(A)^+} \exp \int_A \log cd\, d\mu. \tag{2.10}$$

For $0 < p < 1$, we have

$$\begin{aligned} f(x)g(y) + h(x)k(y) &= p\,\frac{f(x)g(y)}{p} + (1-p)\,\frac{h(x)k(y)}{1-p} \\ &\geq \left(\frac{f(x)g(y)}{p}\right)^p \left(\frac{h(x)k(y)}{1-p}\right)^{1-p} = \frac{f^p(x)h^{1-p}(x)}{p^p}\,\frac{g^p(y)k^{1-p}(y)}{(1-p)^{1-p}}. \end{aligned}$$

Let $c_p(x) = f^p(x)h^{1-p}(x)/p^p$ and $d_p(y) = g^p(y)k^{1-p}(y)/(1-p)^{1-p}$. So $c_p(x)d_p(x) \leq f(x)g(y) + h(x)k(y)$. Thus to show (2.10), it suffices to show that

$$\exp \int_A \log fg\, d\mu + \exp \int_A \log hk\, d\mu \leq \sup_{0<p<1} \exp \int_A \log c_p d_p\, d\mu. \tag{2.11}$$

Maximizing the exponent on the right hand side, we obtain the equation

$$\int_A \log\left(\frac{fg}{hk}\right) d\mu - \log p + \log(1-p) = 0.$$

Letting p_0 be the value of p which solves the equation, we have

$$\int_A \log\left(\frac{fg}{hk}\right) d\mu = \log\left(\frac{p_0}{1-p_0}\right). \tag{2.12}$$

Using (2.12), we obtain

$$\int_A \log c_{p_0}\, d_{p_0}\, d\mu$$

$$= \int_A \log\left[\frac{(fg)^{p_0}}{p_0{}^{p_0}}\right]\left[\frac{(hk)^{1-p_0}}{(1-p_0)^{1-p_0}}\right] d\mu$$

$$= p_0 \int_A \log\left(\frac{fg}{hk}\right) d\mu + \int_A \log hk\, d\mu - p_0 \log p_0 - (1-p_0)\log(1-p_0)$$

$$= p_0 \log\left(\frac{p_0}{1-p_0}\right) + \int_A \log hk\, d\mu - p_0\log p_0 - (1-p_0)\log(1-p_0)$$

$$= \int_A \log hk\, d\mu - \log(1-p_0).$$

Thus

$$\exp\left(\int_A \log c_{p_0}\, d_{p_0}\, d\mu\right) = \frac{1}{1-p_0} \exp\left(\int_A \log hk\, d\mu\right)$$

is what we have on the right hand side of (2.11). On the left hand side of (2.11), we have, using (2.12),

$$\exp\left(\int_A \log fg\, d\mu\right) + \exp\left(\int_A \log hk\, d\mu\right)$$

$$= \exp\left(\int_A \log hk\, d\mu\right)\left[1 + \exp\left(\int_A \log \frac{fg}{hk}\, d\mu\right)\right]$$

$$= \exp\left(\int_A \log hk\, d\mu\right)\left[1 + \frac{p_0}{1-p_0}\right] = \frac{1}{1-p_0} \exp\left(\int_A \log hk\, d\mu\right).$$

Hence (2.11) does indeed hold, proving Lemma 2.7.

To prove Lemma 2.8, we need one other lemma.

LEMMA 2.13. *Let $\Phi(x_1, x_2, \cdots, x_n)$: $R^n \to R$ be a concave function and let $g_1, g_2, \cdots, g_n \in M_\lambda$ with $\Phi(g_1, \cdots, g_n) > 0$ and bounded from above. (By concavity, the same is true of $\Phi(\Pi_\lambda g_1, \cdots, \Pi_\lambda g_n)$.) Then*

$$\int_A \log\left(\frac{\Phi(\Pi_\lambda g_1, \cdots, \Pi_\lambda g_n)}{\Phi(g_1, \cdots, g_n)}\right) d\mu \geq -k\lambda.$$

PROOF. Without loss of generality, assume Φ is smooth since we may convolve any concave Φ with smooth test functions which approach a δ-function and obtain smooth concave approximations of Φ which converge to Φ in the sup norm. To simplify notation, let $u = (u_1, u_2, \cdots, u_n)$ and $Lu = (Lu_1, \cdots, Lu_n)$.

Let

$$H(\lambda) \equiv \int_A \log \Phi(u)\, d\mu - \int_A \log \Phi((I - \lambda L)u)\, d\mu + k\lambda$$

for $u_i \in \tilde{\mathscr{D}}$ defined by $u_i = \Pi_\lambda g_i$, $i = 1, 2, \cdots, n$. Then

$$H(0) = 0 \tag{2.14}$$

$$H'(\lambda) = \int_A \sum_i \frac{\Phi_{x_i}((I - \lambda L)u)\, Lu_i}{\Phi((I - \lambda L)u)}\, d\mu + k \tag{2.15}$$

$$\begin{aligned} H''(\lambda) &= \int_A - \sum_{i,j} \frac{\Phi_{x_i x_j}((I - \lambda L)u) Lu_i Lu_j}{\Phi((I - \lambda L)u)}\, d\mu \\ &\quad + \int_A \left[\sum_i \frac{\Phi_{x_i}((I - \lambda L)u) Lu_i}{\Phi((I - \lambda L)u)}\right]^2 d\mu \ge 0, \end{aligned} \tag{2.16}$$

since by the concavity of Φ, $\{-\Phi_{x_i x_j}\}$ is a nonnegative semi-definite matrix. Also, for concave Φ and $u_i \in \tilde{\mathscr{D}}$, $i = 1, 2, \cdots, n$, we have the inequality

$$\sum_{i=1}^n \Phi_{x_i}(u) Lu_i \ge L\Phi(u). \tag{2.17}$$

To see this, we have

$$\begin{aligned} L\Phi(u(x)) &= \lim_{t\to 0} \frac{E_x\Phi(u(x(t))) - \Phi(u(x))}{t} \\ &\le \lim_{t\to 0} \frac{\Phi(E_x u(x(t))) - \Phi(u(x))}{t} \end{aligned}$$

by Jensen's inequality. And

$$\begin{aligned} \lim_{t\to 0} \frac{\Phi(E_x u(x(t))) - \Phi(u(x))}{t} &= \frac{d}{dt} \Phi(E_x u(x(t)))\big|_{t=0} \\ &= \sum_{j=1}^n \frac{d\Phi}{dx_j}(u) \frac{d}{dt} E_x(u_j(x(t)))\big|_{t=0} \\ &= \sum_{j=1}^n \frac{d\Phi}{dx_j}(u(x)) Lu_j(x). \end{aligned}$$

Using (2.17), we see from (2.15) that

$$H'(0) = \int_A \sum_{j=1}^n \frac{\Phi_{x_j}(u) Lu_j}{\Phi(u)}\, d\mu + k \ge \int_A \frac{L\Phi(u)}{\Phi(u)}\, d\mu + k. \tag{2.18}$$

But the hypothesis on $\tilde{\mathscr{D}}$ was that $u_1, \cdots, u_n \in \tilde{\mathscr{D}}$ implies $\psi(u_1, \cdots, u_n) \in \tilde{\mathscr{D}}$ for ψ smooth. Thus $\Phi(u_1, \cdots, u_n) \in \tilde{\mathscr{D}}$. Since $\Phi(u(x)) = \Phi(\Pi_\lambda g_1(x), \cdots, \Pi_\lambda g_n(x)) > 0$, we have $\Phi(u) \in \tilde{\mathscr{D}}^+$. However, we have assumed that

$\int_A (Lv/v)\, d\mu \geq -k$ for all $v \in \tilde{\mathscr{D}}^+$. Hence

$$\int_A \frac{L\Phi(u)}{\Phi(u)}\, d\mu \geq -k$$

and from (2.18), $H'(0) \geq 0$. But by (2.16), $H(\lambda)$ is convex. Thus, since $H(0) = 0$, we have $H(\lambda) \geq 0$ for all $\lambda \geq 0$. That is

$$\int_A \log \Phi(u)\, d\mu - \int_A \log \Phi((I - \lambda L)u)\, d\mu + k\lambda \geq 0.$$

Or, since $u_j = \Pi_\lambda g_j$, and $(I - \lambda L)u_j = g_j$,

$$\int_A \log \frac{\Phi(\Pi_\lambda g_1, \cdots, \Pi_\lambda g_n)}{\Phi(g_1, \cdots, g_n)}\, d\mu \geq -k\lambda,$$

proving the lemma.

Now we give the

PROOF OF LEMMA 2.8. We need to show that if $f_i \in C(A)$, $g_i \in M_\lambda$, $i = 1, 2, \cdots, m$, and $f, g \in C(A)^+$ with

$$\textstyle\sum_{i=1}^m f_i(x) g_i(y) \geq f(x) g(y),$$

then

$$\int_A \textstyle\sum_{i=1}^m f_i(x) \Pi_\lambda g_i(x)\, d\mu(x) \geq \exp(-k\lambda) \exp\left(\int_A \log fg\, d\mu\right). \tag{2.19}$$

Since A is compact, $f \geq c > 0$ and hence we may define $h_i \in C(A)^+$ by $h_i = f_i/f$. Substituting this into (2.19) and taking logarithms, we must show that

$$\log \int_A \{\textstyle\sum_{i=1}^m h_i(x) \Pi_\lambda g_i(x)\} f(x)\, d\mu(x) + k\lambda - \int_A \log fg\, d\mu \geq 0 \tag{2.20}$$

for $g_i \in M_\lambda$, $h_i \in C(A)$, $f, g \in C(A)^+$ and $\sum_{i=1}^m h_i(x) g_i(y) \geq g(y)$. We minimize the left hand side of (2.20) with respect to $f(x)$. The variational equation for the minimal $f(x)$ is seen to be

$$\frac{\int_A \sum_{i=1}^m h_i(x) \Pi_\lambda g_i(x) q(x)\, d\mu(x)}{\int_A \sum_{i=1}^m h_i(x) \Pi_\lambda g_i(x) f(x)\, d\mu(x)} = \int_A \frac{q}{f}\, d\mu$$

for all $q \in C(A)^+$. One sees that the solution to this is

$$f_0(x) = [\textstyle\sum_{i=1}^m h_i(x) \Pi_\lambda g_i(x)]^{-1}.$$

Plugging this minimal $f_0(x)$ into (2.20), we see that for all $f \in C(A)^+$,

$$\log \int_A (\textstyle\sum_{i=1}^m h_i(x) \Pi_\lambda g_i(x)) f(x)\, d\mu(x) + k\lambda - \int_A \log fg\, d\mu$$

$$\geq \int_A \log\left(\frac{\sum_{i=1}^m h_i(x) \Pi_\lambda g_i(x)}{g(x)}\right) d\mu + k\lambda.$$

Thus it suffices to show that

$$\int_A \log\left(\frac{\sum_{i=1}^m h_i(x)\Pi_\lambda g_i(x)}{g(x)}\right) d\mu \geq -k\lambda.$$

Since $\sum_{i=1}^m h_i(x)g_i(y) \geq g(y)$, we have $\inf_x \sum_{i=1}^m h_i(x)g_i(y) \geq g(y)$. Let $\Phi(z) = \Phi(z_1, \cdots, z_n) = \inf_x \sum_{i=1}^m h_i(x)z_i$. Φ is concave. We have $\Phi(g_1(y), \cdots, g_n(y)) \geq g(y) > 0$, and by the concavity of Φ, $\Phi(\Pi_\lambda g_1, \cdots, \Pi_\lambda g_n) \geq 0$. Also, $\Phi(\Pi_\lambda g_1(x), \cdots, \Pi_\lambda g_n(x)) \leq \sum_{i=1}^m h_i(x)\Pi_\lambda g_i(x)$. Hence

$$\int_A \log\left(\frac{\sum_{i=1}^m h_i\Pi_\lambda g_i}{g}\right) d\mu \geq \int_A \frac{\log \Phi(\Pi_\lambda g_1, \cdots, \Pi_\lambda g_n)}{\Phi(g_1, \cdots, g_n)} d\mu.$$

By Lemma 2.13, $\int_A \log \Phi(\Pi_\lambda g_1, \cdots, \Pi_\lambda g_n)/\Phi(g_1, \cdots, g_n)\, d\mu \geq -k\lambda$. Thus

$$\int_A \log\left(\frac{\sum_{i=1}^m h_i\Pi_\lambda g_i}{g}\right) d\mu \geq -k\lambda,$$

proving the lemma.

Now consider the operator $\tilde{\Pi}_\lambda$: $C(A) \to B(A)$. We will show that

$$\lim_{\lambda\to 0; n\to\infty; \lambda n\downarrow t;} (\tilde{\Pi}_\lambda)^n = T_t$$

where $T_t f(x) = E_x f(x(t))$ is the semigroup corresponding to the unique solution of the martingale problem. With this and the following lemma, taken from [2, Lemma 3.1], we may prove our theorem.

LEMMA 2.21. *Let T_t be the semigroup of a Markov process on A with generator $(L, \mathscr{D})$. Then*

$$\frac{1}{t}\left[-\inf_{f\in C(A)^+} \int_A \log \frac{T_t f}{f}\, d\mu\right] \leq -\inf_{u\in\mathscr{D}^+} \int_A \frac{Lu}{u}\, d\mu$$

and

$$\lim_{t\to 0} \frac{1}{t}\left[-\inf_{f\in C(A)^+} \int_A \log \frac{T_t f}{f}\, d\mu\right] = -\inf_{u\in\mathscr{D}^+} \int_A \frac{Lu}{u}\, d\mu \quad \textit{for all} \quad \mu \in \mathscr{P}(A).$$

REMARK. The corresponding I-function for a Markov chain with transition function $\Pi(x, y)$ is $I_\Pi(\mu) = -\inf_{f\in C(A)^+} \int \log (\Pi f/f)\, d\mu$ where $\Pi f(x) = \int f(y)\Pi(x, dy)$. Thus the left hand side in this lemma is just $(1/t)I_t(\mu)$ where I_t is the I-function for the Markov chain $(x(0), x(t), x(2t), \cdots)$ induced by T_t. Hence the lemma states that $(1/t)I_t(\mu) \leq I(\mu)$ and $\lim_{t\to 0}(1/t)I_t(\mu) = I(\mu)$.

We now prove Theorem 1.4 using $\lim_{\lambda\to 0; n\to\infty; \lambda n\downarrow t}(\tilde{\Pi}_\lambda)^n = T_t$. Then we will return to prove this. We have

$$\inf_{g\in C(A)^+}\int_A \log\frac{\tilde{\Pi}_\lambda^n g}{g}\,d\mu$$

$$= \inf_{g\in C(A)^+}\int_A \log\frac{\tilde{\Pi}_\lambda^{n-1}\tilde{\Pi}_\lambda g}{\tilde{\Pi}_\lambda g}\,\frac{\tilde{\Pi}_\lambda g}{g}\,d\mu$$

$$\geq \inf_{g\in C(A)^+}\int_A \log\left(\frac{\tilde{\Pi}_\lambda^{n-1}\tilde{\Pi}_\lambda g}{\tilde{\Pi}_\lambda g}\right)d\mu + \inf_{g\in C(A)^+}\int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right)d\mu$$

$$\geq \inf_{g\in C(A)^+}\int_A \log\frac{\tilde{\Pi}_\lambda^{n-1} g}{g}\,d\mu + \inf_{g\in C(A)^+}\int_A \log\frac{\tilde{\Pi}_\lambda g}{g}\,d\mu.$$

Hence

$$\inf_{g\in C(A)^+}\int \log\left(\frac{\tilde{\Pi}_\lambda^n g}{g}\right)d\mu \geq n\,\inf_{g\in C(A)^+}\int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right)d\mu. \tag{2.22}$$

But from Lemma 2.5,

$$\inf_{g\in C(A)^+}\int_A \log\left(\frac{\tilde{\Pi}_\lambda g}{g}\right)d\mu \geq -k\lambda.$$

Thus, from (2.22),

$$\inf_{g\in C(A)^+}\int_A \log\left(\frac{\tilde{\Pi}_\lambda^n g}{g}\right)d\mu \geq -nk\lambda. \tag{2.23}$$

But since $\tilde{\Pi}_\lambda^n \to T_t$ as $n \to \infty$, $\lambda \to 0$, and $n\lambda \downarrow t$, we see from (2.23), that for any fixed $g \in C(A)^+$,

$$\int_A \log\left(\frac{T_t g}{g}\right)d\mu \geq -kt.$$

Hence

$$\inf_{g\in C(A)^+}\int_A \log\left(\frac{T_t g}{g}\right)d\mu \geq -kt.$$

Using this with Lemma 2.21 gives

$$-\inf_{u\in\mathscr{D}^+}\int_A \frac{Lu}{u}\,d\mu = \lim_{t\to 0}\frac{1}{t}\left[-\inf_{f\in C(A)^+}\int_A \log\left(\frac{T_t f}{f}\right)d\mu\right] \leq k.$$

This proves Theorem 1.4.

We are left with showing that $\tilde{\Pi}_\lambda^n \to T_t$ as $\lambda \to 0$, $n \to \infty$, and $n\lambda \downarrow t$. We construct a Markov chain x_0^λ, x_1^λ, $\cdots$ with $\tilde{\Pi}_\lambda$ as the Markov transition function. (By Lemma 2.5, $\tilde{\Pi}_\lambda$ is a Markov transition function.) We need the following lemma.

LEMMA 2.24. *Let A be a locally compact separable metric space and $x_0, x_1, \cdots$ a time homogeneous Markov chain with transition function $q(x, dy)$ and state space A. Let $G(x) \equiv (\Pi_q - I)f(x) = \int_A (f(y) - f(x))q(x, dy)$. Then $z_n \equiv f(x_n) - \sum_{j=0}^{n-1} G(x_j)$ is an $\mathscr{F}_n$-martingale where $\mathscr{F}_n$ is the σ-algebra generated by $x_0, x_1, \cdots, x_n$.*

PROOF. $E(f(x_{n+1} \mid \mathscr{F}_n) = \int_A f(y)q(x_n, dy)$ and

$$G(x_n) = \int_A (f(y) - f(x_n))q(x_n, dy).$$

Thus

$$E(z_{n+1} \mid \mathscr{F}_n) = E(f(x_{n+1}) - \textstyle\sum_{j=0}^{n} G(x_j) \mid \mathscr{F}_n) = f(x_n) - \sum_{j=0}^{n-1} G(x_j) = z_n.$$

Apply this lemma to the process $x_0^\lambda, x_1^\lambda, \cdots$. For $f \in \tilde{\mathscr{D}}$, let $g_\lambda = f - \lambda Lf \in M_\lambda$. So $(\tilde{\Pi}_\lambda - I)g_\lambda = \lambda Lf$.

Hence, by the lemma

$$z_n = g_\lambda(x_n^\lambda) - \lambda \textstyle\sum_{j=0}^{n-1} Lf(x_j^\lambda) \tag{2.25}$$

is a martingale. Construct the process $Y_t^\lambda \in D([0, \infty), A)$ by $Y_t^\lambda \equiv x_{[t/\lambda]}^\lambda$. Let Q_x^λ be the probability measure on $A \times A \times \cdots$ induced by the $\{x_n^\lambda\}$ process starting at $x_0^\lambda = x$ and let P_x^λ be the measure on $D([0, \infty), A)$ induced by the Y_t^λ process. Endow $D([0, \infty), A)$ with the Skorohod metric. We now show that P_x^λ is weakly relatively compact as $\lambda \to 0$. Let $\psi(\lambda, x, \delta, t) = P_x^\lambda(\tau_\delta \le t)$ where $\delta > 0$ and τ_δ is the first exit time from the ball $B(x, \delta)$. Then to prove compactness of $\{P_x^\lambda\}$ as $\lambda \to 0$, it suffices to prove that

$$\lim_{t\to 0}\limsup_{\lambda\to 0}\sup_{x\in A}\psi(\lambda, x, \delta, t) = 0, \quad \text{for all} \quad \delta > 0 \quad [5, 8].$$

By compactness of A, in fact, it suffices to show that for arbitrary $x_0 \in A$,

$$\lim_{t\to 0}\limsup_{\lambda\to 0}\sup_{x\in B(x_0,\delta/2)}\psi(\lambda, x, \delta, t) = \quad \text{for all} \quad \delta > 0.$$

Pick $p \in (0, \frac{1}{2})$. Let $\theta \in C(A)$ be such that $\theta \equiv 2p$ on $B(x_0, \delta/2)$ and $\theta \equiv 1$ on $B(x_0, \delta)^c$ and $2p \le \theta \le 1$. Since $\tilde{\mathscr{D}}$ is dense in $C(A)$, we can find $p \le \theta_p \in \tilde{\mathscr{D}}$ satisfying $\sup_{x\in A} | \theta_p(x) - \theta(x) | \le p$. Hence

$$\sup\{\theta_p(x) : d(x, x_0) < \delta/2\} \le 3p \tag{2.26}$$

and

$$\inf\{\theta_p(x) : d(x, x_0) > \delta\} \ge 1 - p \tag{2.27}$$

where $d(x, y)$ is the metric on A. Let $\varphi_{p,\lambda} = (I - \lambda L)\theta_p$. Thus $\Pi_\lambda \varphi_{p,\lambda} = \theta_p$. Since $\theta_p \in \tilde{\mathscr{D}}$, there exists a constant A_p with $| L\theta_p | \le A_p$, and thus by (2.26),

$$| \varphi_{p,\lambda}(x) | = | \theta_p(x) - \lambda L\theta_p(x) | \le 3p + \lambda A_p \quad \text{for all} \quad x \in B(x_0, \delta/2). \tag{2.28}$$

Also, for all $x \in A$,

$$\varphi_{p,\lambda}(x) \ge p - \lambda A_p \ge 0 \quad \text{if} \quad \lambda \le p/A_p. \tag{2.29}$$

Consider now only such λ. By (2.27),

$$|\varphi_{p,\lambda}(x)| \geq |\theta_p(x)| - \lambda |L\theta_p(x)| \geq 1 - p - \lambda A_p \quad \text{for all} \quad x \in B(x_0, \delta)^c. \tag{2.30}$$

Using (2.25) with $f = \theta_p$ and $g_\lambda = \varphi_{p,\lambda}$ and using the process $Y_t^\lambda \equiv x_{[t/\lambda]}^\lambda$, we see that

$$\varphi_{p,\lambda}(Y_{n\lambda}^\lambda) - \lambda \textstyle\sum_{j=0}^{n-1} L\theta_p(Y_{j\lambda}^\lambda)$$

is a martingale and thus $\varphi_{p,\lambda}(Y_{n\lambda}^\lambda) - n\lambda A_p$ is a supermartingale since $|L\theta_p| \leq A_p$. With brackets denoting the greatest integer function, we have

$$\begin{aligned}\varphi_{p,\lambda}(Y_{t\wedge\tau_\delta}^\lambda) - A_p(t\wedge\tau_\delta) &= \varphi_{p,\lambda}(Y_{[t\wedge\tau_\delta/\lambda]\lambda}^\lambda) - A_p(t\wedge\tau_\delta)\\ &\leq \varphi_{p,\lambda}(Y_{[t\wedge\tau_\delta/\lambda]\lambda}^\lambda) - A_p\left[\frac{t\wedge\tau_\delta}{\lambda}\right]\lambda.\end{aligned}$$

Since $\{\varphi_{p,\lambda}(Y_{(t\wedge\tau_\delta)}^\lambda) - A_p(t\wedge\tau_\delta)\}$ is uniformly bounded, Doob's stopping time theorem gives

$$\begin{aligned}E^{P_x^\lambda}(\varphi_{p,\lambda}(Y_{t\wedge\tau_\delta}^\lambda) - A_p(t\wedge\tau_\delta)) &\leq E^{P_x^\lambda}\left(\varphi_{p,\lambda}(Y_{[t\wedge\tau_\delta/\lambda]\lambda}^\lambda) - A_p\left[\frac{t\wedge\tau_\delta}{\lambda}\right]\lambda\right)\\ &\leq E^{P_x^\lambda}\left(\varphi_{p,\lambda}(Y_{[0\wedge\tau_\delta/\lambda]\lambda}^\lambda) - A_p\left[\frac{0\wedge\tau_\delta}{\lambda}\right]\lambda\right),\end{aligned}$$

Hence,

$$E^{P_x^\lambda}(\varphi_{p,\lambda}(Y_{t\wedge\tau_\delta}^\lambda)) - E^{P_x^\lambda}(A_p(t\wedge\tau_\delta)) \leq \varphi_{p,\lambda}(x) \leq 3p + \lambda A_p \tag{2.31}$$

for $x \in B(x_0, \delta/2)$. Also by (2.29) and (2.30) for $x \in B(x_0, \delta/2)$,

$$E^{P_x^\lambda}(\varphi_{p,\lambda}(Y_{t\wedge\tau_\delta}^\lambda)) \geq (1 - p - \lambda A_p)P_x[\tau_\delta \leq t]. \tag{2.32}$$

Since $E^{P_x^\lambda}(A_p(t\wedge\tau_\delta)) \leq tA_p$, we have from (2.31) and (2.32) that

$$\begin{aligned}(1 - p - \lambda A_p)P_x^\lambda[\tau_\delta \leq t] - tA_p &\leq E^{P_x^\lambda}(\varphi_{p,\lambda}(Y_{t\wedge\tau_\delta}^\lambda)) - E^{P_x^\lambda}(A_p(t\wedge\tau_\delta))\\ &\leq 3p + \lambda A_p \quad \text{for} \quad x \in B(x_0, \delta/2).\end{aligned}$$

Since $\lambda < p/A_p$ and $p < 1/2$,

$$P_x^\lambda[\tau_\delta \leq t] \leq \frac{3p + \lambda A_p + tA_p}{1 - p - \lambda A_p} \leq \frac{4p + tA_p}{1 - 2p} \quad \text{for all} \quad x \in B(x_0, \delta/2).$$

So

$$\lim_{t\to 0}\limsup_{\lambda\to 0}\sup_{x\in B(x_0,\delta/2)} P_x^\lambda[\tau_\delta \leq t] \leq \frac{4p}{1-2p}.$$

Since we have already let $\lambda \to 0$, there is no longer any restriction on p. Hence, letting $p \to 0$ gives

$$\lim_{t\to 0}\lim_{\lambda\to 0}\sup_{x\in B(x_0,\delta/2)} P_x^\lambda[\tau_\delta \leq t] = 0.$$

Thus $\{P_x^\lambda\}$ is relatively compact as $\lambda \to 0$ and there exists a sequence λ_j with

$P_j^{\lambda_j} \to Q_x$ as $\lambda_j \to 0$, where Q_x is a probability measure on $D([0, \infty), A)$. We now identify Q_x. From (2.25) and the fact that $Y_t^\lambda \equiv x_{[t/\lambda]}^\lambda$, we have that

$$\zeta_{n_j} \equiv g_{\lambda_j}(Y_{n_j\lambda_j}^{\lambda_j}) - \frac{\lambda_j n_j}{n_j} \sum_{k=0}^{n_j-1} Lf(Y_{k\lambda_j}^{\lambda_j})$$

is a $P_x^{\lambda_j}$-martingale for $f \in \tilde{\mathscr{D}}$ and $g_{\lambda_j} = f - \lambda_j Lf$. We may consider $\zeta_{n_j} = \zeta_{n_j}(X(\cdot))$ as a function from $D([0, \infty), A) \to R$ defined by

$$\zeta_{n_j}(X(\cdot)) = g_{\lambda_j}(X(n_j\lambda_j)) - \frac{\lambda_j n_j}{n_j} \sum_{k=0}^{n_j-1} Lf(X(k\lambda_j)).$$

Then $\zeta_{n_j}(X(\cdot))$ tends to $\zeta(X(\cdot)) \equiv f(X(t)) - \int_0^t Lf(X(s))\, ds$ as $\lambda_j \to 0$, $n_j \to \infty$, $\lambda_j n_j \downarrow t$. Now the $\{\zeta_{n_j}\}$ are continuous functions on $D([0, \infty), A)$ with the Skorohod topology and, in fact, the convergence to ζ is uniform for compact subsets $C \subset D([0, \infty), A)$. This allows us to conclude that $\int_C \zeta_{n_j} dP_x^j \to \int_C \zeta\, dQ_x$ for compact $C \subset D([0, \infty), A)$. But ζ_{n_j} is a $P_x^{\lambda_j}$ martingale. Also, since $D([0, \infty), A)$ is a complete separable metric space, there exists for each $\varepsilon > 0$ a compact C_ε with $P_x^{\lambda_j}(C_\varepsilon) \geq 1 - \varepsilon$, for all j. Hence we may conclude that $\zeta = f(X(t)) - \int_0^t Lf(X(s))\, ds$ is a Q_x martingale. Yet we have assumed that there is a unique solution, call it P_x, to the martingale problem. Thus $Q_x = P_x$. Now $(\tilde{\Pi}_\lambda)^n g(x) = E^{P_x^\lambda} g(X(\lambda n))$, so since $P_x^{\lambda_j} \Rightarrow P_x$, $\lambda_n j_n \downarrow t$, and the measures are supported on right continuous functions,

$$\lim_{j\to\infty} (\tilde{\Pi}_{\lambda_j})^{n_j} g(x) = \lim_{j\to\infty} E^{P_x^{\lambda_j}} g(X(\lambda_j n_j)) = E^{P_x} g(X(t)) = T_t g(x),$$

where T_t is the semigroup corresponding to the unique solution to the martingale problem.

3. Martingale uniqueness for diffusions. We now prove a lemma which, as discussed in the Introduction, allows us to obtain Corollary 1.6. Stroock and Varadhan have proven the following submartingale uniqueness theorem [6, Theorem 5.8]. (Actually, their result is more general—we are taking only what we need.) Let $A \subset R^n$ be a bounded region described by $\theta(x_1, \cdots, x_n) \geq 0$ with $\theta \in C^2(R^n)$ and $|\nabla\theta| \neq 0$ on ∂A. If a is positive definite with entries in $C(A)$, $b \in B(A)$, $J = (J_1, \cdots, J_n)$ satisfies $J \cdot \nabla\theta \geq \beta > 0$ on ∂A and $J_i \in C^1(\partial A)$, $i = 1, 2, \cdots, n$, then starting from any point x, there exists a unique homogeneous Markov diffusion process $x(t)$ corresponding to a measure P_x for which

(i) $P_x(x(0) = x) = 1$

(ii) $P_x(x(t) \in A) = 1$

(iii) $f(x(t)) - \int_0^t Lf(x(s))\, ds$ is a P_x-submartingale for $f \in C^2(A)$ with $J \cdot \nabla f \geq 0$ on ∂A.

By considering the functions f and $-f$, it is clear that if $f \in \tilde{\mathscr{D}} \equiv \{f \in C^2(A): J \cdot \nabla f = 0 \text{ on } \partial A\}$, then $f(x(t)) - \int_0^t Lf(x(s))ds$ is a martingale.

LEMMA 3.1. *With the conditions stipulated above on a, b, J, and A, there is a unique Markov diffusion process $x(t)$ for which $f(x(t)) - \int_0^t Lf(x(s))\,ds$ is a martingale when $f \in \tilde{\mathscr{D}}$.*

PROOF. By Stroock and Varadhan, it suffices to show that if $f(x(t)) - \int_0^t Lf(x(s))\,ds$ is a martingale when $f \in C^2(A)$ and $J \cdot \nabla f = 0$ on ∂A, then $f(x(t)) - \int_0^t Lf(x(s))\,ds$ is a submartingale when $f \in C^2(A)$ and $J \cdot \nabla f \geq 0$ on ∂A. By changing coordinates locally, it suffices to consider the case in which $\theta(x_1, \cdots, x_n) = x_1$ so that ∂A is the $x_1 = 0$ hyperplane. Assume $J \cdot \nabla f \geq 0$ on $x_1 = 0$. Consider $h \equiv f(x) + \varphi(x)$. We want to pick $\varphi \in C^2$ so that $J \cdot \nabla h = 0$ on $x_1 = 0$ and so that $h = f$ on $x_1 = 0$. We need $\varphi(0, x_2, \cdots, x_n) = 0$. This gives us $\varphi_{x_i}(0, x_2, \cdots, x_n) = 0$ for $i = 2, 3, \cdots, n$. Thus, in order that $J \cdot \nabla h = 0$, we need $\varphi_{x_1}(0, x_2, \cdots, x_n) = (-J \cdot \nabla f/J_1)(0, x_2, \cdots, x_n) \equiv \gamma(x_2, \cdots, x_n)$. Note that $\gamma \leq 0$ and $\gamma \in C^1$. We will also need $\varphi < 0$ for $x_1 > 0$. We now prove the lemma assuming such a φ exists and then we will come back and exhibit such a φ.

Let $\tilde{h} = f + (e^{n\varphi}/n)$. Note that $\nabla \tilde{h} \cdot J = 0$ on $x_1 = 0$. Hence by assumption, $\tilde{h}(x(t)) - \int_0^t L\tilde{h}(x(s))\,ds$ is a martingale, that is,

$$E\left(\tilde{h}(x(t)) - \int_0^t L\tilde{h}(x(u))\,du \mid \mathscr{F}_s\right) = \tilde{h}(x(s)).$$

Thus,

$$\begin{aligned}
&E\left(f(x(t)) - \int_s^t Lf(x(u))\,ds \mid \mathscr{F}_s\right) \\
&\quad = -E\left(\frac{\exp(n\varphi(x(t)))}{n} \,\Big|\, \mathscr{F}_s\right) + E\left(\int_s^t \exp(n\varphi(x(u)))L\varphi(x(u))\,du \,\Big|\, \mathscr{F}_s\right) \\
&\qquad + E\left(\int_s^t n\exp(n\varphi(x(u)))(\nabla\varphi a \nabla\varphi)(x(u))\,du \,\Big|\, \mathscr{F}_s\right) \\
&\qquad + f(x(s)) + \frac{\exp(n\varphi(x(s)))}{n} \\
&\quad \geq -E\left(\frac{\exp(n\varphi(x(t)))}{n} \,\Big|\, \mathscr{F}_s\right) + E\left(\int_s^t \exp(n\varphi(x(u)))L\varphi(x(u))\,du \,\Big|\, \mathscr{F}_s\right) \\
&\qquad + f(x(s)) + \frac{\exp(n\varphi(x(s)))}{n}.
\end{aligned}$$

Since almost all paths spend zero time on $x_1 = 0$, [6], and since $\varphi < 0$ for $x_1 > 0$, letting $n \to \infty$ in (3.2) gives

$$E\left(f(x(t)) - \int_s^t Lf(x(u))\,du \,\Big|\, \mathscr{F}_s\right) \geq f(x(s))$$

for almost all paths. This proves the lemma.

Now we go back and exhibit such a φ. Let $\psi(x) \equiv \psi(x_2, \cdots, x_n) > 0$ be a Schwartz class function on R^{n-1} satisfying $\int_{R^{n-1}} \psi(x)\, dx = 1$. Let $y = (x_2, \cdots, x_n)$ and $x = (x_1, x_2, \cdots, x_n)$. Define for $x_1 > 0$,

$$\varphi(x) = \varphi(x_1, y) = x_1^{-n+2} \int_{R^{n-1}} \psi\left(\frac{y-z}{x_1}\right) \gamma(z)\, dz.$$

We may assume that γ is not identically zero since otherwise there is nothing to prove. Thus $\varphi < 0$ for $x_1 > 0$. Changing variables, one sees readily that $\varphi(0, y) \equiv \lim_{x \to 0} \varphi(x, y) = 0$. Then

$$\begin{aligned}\varphi_{x_1}(0, x_2, \cdots, x_n) &= \lim_{x_1 \to 0} \frac{\varphi(x_1, x_2, \cdots, x_n)}{x_1} \\ &= \lim_{x_1 \to 0} x_1^{-n+1} \int_{R^{n-1}} \psi\left(\frac{y-z}{x_1}\right) \gamma(z)\, dz \\ &= \lim_{x_1 \to 0} \int_{R^{n-1}} \psi(u)\gamma(y - x_1 u)\, du = \gamma(y) = \gamma(x_2, \cdots, x_n).\end{aligned}$$

Finally we must show that $\varphi \in C^2$. We write

$$\varphi(x) = x_1 \int_{R^{n-1}} \psi(u)\gamma(y - x_1 u)\, du.$$

Thus,

$$\varphi_{x_1}(x) = \int_{R^{n-1}} \psi(u)\gamma(y - x_1 u)\, du - x_1 \int_{R^{n-1}} \psi(u) \frac{\partial \gamma}{\partial y} (y - x_1 u) \cdot u\, du.$$

Changing variables in the second term on the right hand side above, we obtain

$$\begin{aligned}\varphi_{x_1}(x) &= \int_{R^{n-1}} \psi(u)\gamma(y - x_1 u)\, du \\ &\quad - x^{-n+1} \int_{R^{n-1}} \psi\left(\frac{y-v}{x_1}\right) \frac{\partial \gamma}{\partial y} (v) \cdot (y - v)\, dv.\end{aligned}$$

Thus

$$\begin{aligned}\varphi_{x_1, x_1}(x) &= - \int_{R^{n-1}} \psi(u) \left(\frac{\partial \gamma}{\partial y} (y - x_1 u) \cdot u\right) du \\ &\quad + (n-1)x_1^{-n} \int_{R^{n-1}} \psi\left(\frac{y-v}{x_1}\right) \frac{\partial \gamma}{\partial y} (v) \cdot (y - v)\, dv \\ &\quad + x^{-n+1} \int_{R^{n-1}} \left(\frac{\partial \psi}{\partial y} \left(\frac{y-v}{x_1}\right) \cdot \left(\frac{y-v}{x_1^2}\right)\right)\left(\frac{\partial \gamma}{\partial y} (v) \cdot (y - v)\right) dv\end{aligned}$$

$$= - \int_{R^{n-1}} \psi(u) \left(\frac{\partial \gamma}{\partial y}(y - x_1 u) \cdot u\right) du$$

$$+ (n-1) \int_{R^{n-1}} \psi(w) \left(\frac{\partial \gamma}{\partial y}(y - x_1 w) \cdot w\right) dw$$

$$+ \int_{R^{n-1}} \left(\frac{\partial \psi}{\partial y}(w) \cdot w\right)\left(\frac{\partial \gamma}{\partial y}(y - w x_1) \cdot w\right) dw.$$

Since $\gamma \in C^1$, it is clear that $\varphi_{x_1 x_1}(x)$ is bounded and continuous for all $x \geq 0$ and one can show similarly that the same is true of the other mixed partial derivatives. Hence $\varphi \in C^2$. This completes the proof of Lemma 3.1.

As discussed in the Introduction, Corollary 1.6 follows from this lemma and Theorem 1.4.

4. The noncompact case. We now indicate how to prove Theorem 1.4 in the noncompact case. We assume A is a complete separable locally compact metric space, and we consider $C_{b,\infty}(A)$, the space of bounded continuous functions on A with a limit at ∞. Let $\tilde{\mathscr{D}}$ be a dense subset of $C_{b,\infty}(A)$ and assume $L: \tilde{\mathscr{D}} \to C_{b,\infty}(A)$. This is certainly no restriction for diffusion processes, since the set of continuous functions which are constant off a compact set is dense and L maps this set into $C_{b,\infty}(A)$. With this setup, we can prove Theorem 1.4 for A as above. We assume

$$\int_A \frac{Lu}{u}\, d\mu \geq -k \quad \text{for all} \quad u \in \tilde{\mathscr{D}}$$

and must show that

$$\int_A \frac{Lu}{u}\, d\mu \geq -k \quad \text{for all} \quad u \in \mathscr{D}.$$

Define $(I - \lambda L): \tilde{\mathscr{D}} \to M_\lambda \subset C_{b,\infty}$ and $\Pi_\lambda = (I - \lambda L)^{-1}: M_\lambda \to \tilde{\mathscr{D}}$. As before, $\Pi_\lambda 1 = 1$, $\Pi_\lambda f \geq 0$ if $f \geq 0$ and

$$\int_A \log \frac{\Pi_\lambda g}{g}\, d\mu \geq -k\lambda \quad \text{for all} \quad g \in M_\lambda^+.$$

We want to extend Π_λ to a positive operator $\tilde{\Pi}_\lambda$ so that

$$\int_A \log \frac{\tilde{\Pi}_\lambda g}{g}\, d\mu \geq -k\lambda, \quad \text{for all} \quad g \in C_{b,\infty}^+(A).$$

To do this, consider the subspace $V \subset C_{b,\infty}(A \times A)$ defined by

$$V = \{\textstyle\sum_{i=1}^m f_i(x) g_i(y): f_i \in C_{b,\infty}(A),\ g_i \in M_\lambda^+\}.$$

We define a linear functional

$$\Lambda(\textstyle\sum_{i=1}^m f_i(x) g_i(y)) = \sum_{i=1}^m f_i(x) \Pi g_i(y).$$

As before, with a few minor changes, we can extend Λ to $\tilde{\Lambda}$ defined on $C_{b,\infty}(A \times A)$ and bounded from below by Q_λ. But $C_{b,\infty}(A \times A)$ is isomorphic to $C(\widehat{A \times A})$, where $\widehat{A \times A}$ is the one point compactification of $A \times A$. Hence we may consider $\tilde{\Lambda}$ to live on $C(\widehat{A \times A})$, and, as $\widehat{A \times A}$ is compact, we may use the Riesz theorem to conclude that $\tilde{\Lambda}$ corresponds to a measure β on $\widehat{A \times A}$. It is not hard to show that in fact β is supported on $A \times A$. We obtain $\tilde{\Pi}$ from $\tilde{\Lambda}$ as before. Then we must show that $\tilde{\Pi}_\lambda^n \to T_t$ as $\lambda \to 0$, $n \to \infty$, $\lambda n \downarrow t$. This comes down to showing that $\{P_n^\lambda\}$ is relatively compact as $\lambda \to 0$. This may be done using [7, Lemma 11.1.1]. See [4, Lemma 5, and proof of Theorem 2] for details. Hence we obtain

$$\inf_{g \in C_{b,\infty}^+(A)} \int \log \frac{T_t g}{g} \, d\mu \geq -kt.$$

Since T_t is a bounded operator, we have in fact

$$\inf_{g \in C_b^+(A)} \int \log \frac{T_t g}{g} \, d\mu \geq -kt,$$

where $C_b(A)$ is the space of bounded continuous functions on A. In the terminology of Lemma 2.21, $(1/t)I_t(\mu) \leq k$. To conclude the proof, we need only apply Lemma 2.21, whose proof in the noncompact case requires only a slight modification of the proof in [2].

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