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ON AN INEQUALITY OF CHERNOFF[1]

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An inequality due to Chernoff is generalized and a related Cramér-Rao type of inequality is studied.

1. Introduction. Let X be a standard normal random variable and let $G: \mathbb{R} \to \mathbb{R}$ be a function which is absolutely continuous with respect to Lebesgue measure with Radon-Nikodym derivative g. In Chernoff (1980, 1981) the elegant inequality

$$\text{var } G(X) \le Eg^2(X) \tag{1.1}$$

has been presented and proved by a method involving Hermite polynomials. As has been shown in Chen (1982) this result can also be proved by the Cauchy-Schwarz inequality and Fubini's theorem, as follows:

$$\begin{aligned} \text{var } G(X) &\le \int_{-\infty}^{\infty} \left(\int_0^x g(y)\, dy \right)^2 \phi(x)\, dx \le \int_{-\infty}^{\infty} x \int_0^x g^2(y)\, dy\, \phi(x)\, dx \\ &= -\int_{-\infty}^{0} \int_{-\infty}^{y} g^2(y) x\phi(x)\, dx\, dy + \int_0^{\infty} \int_y^{\infty} g^2(y) x \phi(x)\, dx\, dy \\ &= \int_{-\infty}^{\infty} g^2(y)\phi(y)\, dy = Eg^2(X), \end{aligned} \tag{1.2}$$

where ϕ is the standard normal density.

Applying this device of Cauchy-Schwarz inequality and Fubini's theorem one can obtain upper bounds for var $G(X)$ in terms of g, also when X is not normal. This has been done in Cacoullos (1982) for absolutely continuous random variables X and for nonnegative integer valued random variables X. In Section 2 we present a generalization of Chernoff's inequality (1.1) which is valid for arbitrary random variables X. The key ideas leading to this generalization are:

- define the relation between G and g properly (cf. (2.5))
- choose a nonnegative function h and rewrite g in (1.2) as $(gh^{-1/2})h^{1/2}$ before applying the Cauchy-Schwarz inequality (cf. (2.9)).

The resulting inequality (2.8) looks rather complicated, but in many cases it is sharper and simpler than the corresponding one in Cacoullos (1982).

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It has been pointed out by Cacoullos (1982) that a suitable application of the Cramér-Rao inequality and a partial integration yields lower bounds for var $G(X)$, again in terms of g. For a standard normal random variable X the resulting inequality is

$$\text{var } G(X) \geq \{Eg(X)\}^2, \tag{1.3}$$

which resembles (1.1) very much. Indeed,

$$\begin{aligned}
&\text{var } G(X) \\
&\quad\geq \{E((G(X) - EG(X))X)\}^2 = \left\{\int_{-\infty}^{\infty}\int_0^x g(y)x\phi(x)\,dy\,dx\right\}^2 \\
&\quad= \left\{-\int_{-\infty}^{0}\int_{-\infty}^{y} g(y)x\phi(x)\,dx\,dy + \int_0^{\infty}\int_y^{\infty} g(y)x\phi(x)\,dx\,dy\right\}^2 \qquad (1.4)\\
&\quad= \left\{\int_{-\infty}^{\infty} g(y)\phi(y)\,dy\right\}^2 = \{Eg(X)\}^2
\end{aligned}$$

and we see that, just like (1.1), inequality (1.3) can be proved by the Cauchy-Schwarz inequality and Fubini's theorem, which are of course closely related to the Cramér-Rao inequality and partial integration respectively. In Section 3 we generalize (1.4) to the case of arbitrary random variables X. The resulting inequality provides the same lower bounds as obtained in Cacoullos (1982) as well as other ones.

We conclude this section by noting that, as pointed out above, the device of Cauchy-Schwarz inequality and Fubini's theorem yields both lower and upper bounds for var $G(X)$ in terms of g and that these bounds may have some value in cases where they are easier to compute than var $G(X)$ itself or where more is known about the behavior of g than about that of G.

2. Chernoff's inequality. In the situation of the introduction, where G has derivative g with respect to Lebesgue measure, there exist b and c such that

$$G(x) = \int_b^x g(y)\,dy + c. \tag{2.1}$$

With the kernel $\chi\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\chi(x, y) = 1_{(b,x]}(y) - 1_{(x,b]}(y) \tag{2.2}$$

or

$$\chi(x, y) = 1_{[b,x)}(y) - 1_{[x,b)}(y), \tag{2.3}$$

we may write this also as

$$G(x) = \int \chi(x, y)g(y)\,dy + c. \tag{2.4}$$

When μ is some σ-finite measure, we may generalize (2.4) to

$$G(x) = \int \chi(x, y)g(y)\, d\mu(y) + c. \tag{2.5}$$

If in (2.5) μ is counting measure on the integers and χ is defined by (2.2), then $g(x) = G(x) - G(x-1)$ and if χ is as in (2.3) then $g(x) = G(x+1) - G(x)$.

In view of the above, (2.5) seems to be a useful generalization of (2.1). Indeed we obtain quite general inequalities with it, by very easy proofs.

In what follows, we use the convention that the variance of a random variable is infinite iff the second moment of that random variable is infinite.

THEOREM 2.1. *Let μ be a σ-finite measure on $(\mathbb{R}, \mathscr{B})$ and X a random variable with density f with respect to μ. Let χ: $\mathbb{R}^2 \to \mathbb{R}$ be a measurable function such that for μ-almost all $x \in \mathbb{R}$ the function $\chi(x, \cdot)$: $\mathbb{R} \to \mathbb{R}$ does not change sign. Furthermore, g: $\mathbb{R} \to \mathbb{R}$ is a measurable function such that G: $\mathbb{R} \to \mathbb{R}$ is well-defined by* (2.5) *for some $c \in \mathbb{R}$. Finally, h: $\mathbb{R} \to \mathbb{R}$ is a nonnegative measurable function such that H: $\mathbb{R} \to \mathbb{R}$ is well-defined by*

$$H(x) = \int \chi(x, y)h(y)\, d\mu(y). \tag{2.6}$$

If

$$\mu(\{x \in \mathbb{R} \mid g(x) \neq 0, f(x)h(x) = 0\}) = 0, \tag{2.7}$$

then the inequality

$$\operatorname{var}_f G(X) \leq E_f\left\{\frac{g^2(X)}{f(X)h(X)} \int \chi(z, X)H(z)f(z)\, d\mu(z)\right\} \tag{2.8}$$

holds.

PROOF. Analogously to (1.2), i.e., by the Cauchy-Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned}
\operatorname{var}_f G(X) &\leq E_f\left\{\int |\chi(X, y)|^{1/2} \frac{g(y)}{h^{1/2}(y)} |\chi(X, y)|^{1/2} h^{1/2}(y)\, d\mu(y)\right\}^2 \\
&\leq E_f\left\{\int \chi(X, y) \frac{g^2(y)}{h(y)}\, d\mu(y) \int \chi(X, y)h(y)\, d\mu(y)\right\} \\
&= \int \int \chi(x, y) \frac{g^2(y)}{h(y)} H(x)f(x)\, d\mu(y)\, d\mu(x) \\
&= \int \frac{g^2(y)}{f(y)h(y)} \int \chi(x, y)H(x)f(x)\, d\mu(x) f(y)\, d\mu(y).
\end{aligned} \tag{2.9}$$

Note that Fubini's theorem may be applied here since $\chi(x, y)H(x) = \chi(x, y) \int \chi(x, z)h(z)\, dz$ is nonnegative for μ-almost all x and y. □

From this proof it is clear that equality in (2.8) holds iff the variance is infinite or $E_f G(X) = c$ holds and there exists a measurable function $C: \mathbb{R} \to \mathbb{R}$ such that $\chi(x, y)\{g(y) - C(x)h(y)\} = 0$ for $(f \times \mu)$-almost all $(x, y) \in \mathbb{R}^2$.

Let us consider the case in which μ is Lebesgue measure and χ is as in (2.2) or (2.3) with b chosen in such a way that H satisfies $E_f H(X) = 0$. Then $\int \chi(z, x)H(z)f(z)\,dz = \int_x^\infty H(z)f(z)\,dz$ holds and g and h are derivatives of G and H respectively. In Table 2.1 the bound from (2.8) for this situation is given for some choices of f and H.

Note that (1.1) is implied by Examples 1 and 6 of Table 2.1 and that Example 5 of Table 2.1 with $\alpha = 1$, $\sigma = \theta^{-1}$ improves Propositions 4.1 and 4.2 of Cacoullos (1982). When we try to choose H such that the bound in (2.8) becomes $dE_f g^2(X)$, for some $d > 0$, then we have to solve the integral equation $\int_x^\infty Hf = df(x)h(x)$. Differentiation of this equation leads to the differential equation $H'' + f'f^{-1}H' + d^{-1}H = 0$, which by the transformation $H' = f^{-1}H\psi$ is equivalent to the Riccati-equation $\psi' + f^{-1}\psi^2 + d^{-1}f = 0$, for which a general solution seems to be unknown. A solution for some particular cases is given in Examples 1 through 4 of Table 2.1.

Here the question poses itself for which densities f there exists a finite constant d such that all measurable functions g satisfy

$$\text{(2.10)} \qquad \operatorname{var}_f G(X) \le dE_f g^2(X).$$

Existence of a solution for the above mentioned Riccati-equation is not necessary, of course.

The next theorem provides a necessary and almost sufficient condition on f for (2.10) to hold.

THEOREM 2.2. *Let μ be Lebesgue measure and F a distribution function with density f. If* (2.10) *holds for some finite d then we have*

$$\text{(2.11)} \qquad \limsup_{\varepsilon \downarrow 0} \varepsilon \int \{F \wedge (1 - F)\}^{1+\varepsilon}/f < \infty.$$

On the other hand if $\lim_{u \downarrow 0} u/f(F^{-1}(u))$ and $\lim_{u \uparrow 1}(1 - u)/f(F^{-1}(u))$ exist, then (2.11) *implies the existence of a finite d satisfying* (2.10) *for all g.*

TABLE 2.1
The value of the upper bound of (2.8) for some choices of f and H with μ Lebesgue measure and χ as in (2.2)

	name	f	H	(2.8)
1.	normal	$(2\pi\sigma^2)^{-1/2}e^{-x^2\sigma^{-2}/2}$	x	$\sigma^2 E_f g^2(X)$
2.	exponential	$\lambda e^{-\lambda x}, \quad x > 0$	$(x - 2/\lambda)e^{1/2\lambda x}$	$4\lambda^{-2}E_f g^2(X)$
3.	Laplace	$\frac{1}{2}\lambda e^{-\lambda\|x\|}$	$xe^{1/2\lambda\|x\|}$	$4\lambda^{-2}E_f g^2(X)$
4.	logistic	$2\lambda(e^{\lambda x} + e^{-\lambda x})^{-2}$	$e^{\lambda x} - e^{-\lambda x}$	$\lambda^{-2}E_f g^2(X)$
5.	gamma	$\{\sigma^\alpha\Gamma(\alpha)\}^{-1}x^{\alpha-1}e^{-x/\sigma}, \quad x > 0$	$x - \alpha\sigma$	$\sigma E_f X g^2(X)$
6.	—	$c(\alpha, \sigma)\|x\|^{\alpha-1}e^{-1/2\|x/\sigma\|^{\alpha+1}}$	x	$\frac{2\sigma^{\alpha+1}}{\alpha+1}E_f\|X\|^{1-\alpha}g^2(X)$

PROOF. For all measurable functions ψ and g: $\mathbb{R} \to [0, \infty)$ we have

$$\begin{aligned}
\operatorname{var}_f G(X) &= \inf_{a\in\mathbb{R}} \left\{ \int_{-\infty}^{a} \left(\int_{x}^{a} g(y)\, dy \right)^2 f(x)\, dx \right. \\
&\qquad \left. + \int_{a}^{\infty} \left(\int_{a}^{x} g(y)\, dy \right)^2 f(x)\, dx \right\} \\
&\geq \inf_{a\in\mathbb{R}} \left\{ \frac{(\int_{-\infty}^{a} \int_{x}^{a} g(y)\, dy \psi(x) f(x)\, dx)^2}{\int_{-\infty}^{a} \psi^2(x) f(x)\, dx} \right. \\
&\qquad \left. + \frac{(\int_{a}^{\infty} \int_{a}^{x} g(y)\, dy \psi(x) f(x)\, dx)^2}{\int_{a}^{\infty} \psi^2(x) f(x)\, dx} \right\} \qquad (2.12) \\
&\geq \inf_{a\in\mathbb{R}} \left\{ \frac{(\int_{-\infty}^{a} \{\int_{-\infty}^{y} \psi f\} \wedge \{\int_{y}^{\infty} \psi f\} g(y)\, dy)^2}{\int_{-\infty}^{a} \psi^2 f} \right. \\
&\qquad \left. + \frac{(\int_{a}^{\infty} \{\int_{-\infty}^{y} \psi f\} \wedge \{\int_{y}^{\infty} \psi f\} g(y)\, dy)^2}{\int_{a}^{\infty} \psi^2 f} \right\}.
\end{aligned}$$

Since for all $C, D \in \mathbb{R}$

$$\inf_{0\leq\alpha\leq 1, x\in\mathbb{R}} \frac{x^2}{\alpha D^2} + \frac{(C - x)^2}{(1 - \alpha)D^2} = \frac{C^2}{D^2} \qquad (2.13)$$

holds, inequality (2.12) yields (for positive g and ψ)

$$\frac{\operatorname{var}_f G(X)}{E_f g^2(X)} \geq \frac{(\int_{-\infty}^{\infty} \{\int_{-\infty}^{y} \psi f\} \wedge \{\int_{y}^{\infty} \psi f\} g(y)\, dy)^2}{\int \psi^2 f \int g^2 f}. \qquad (2.14)$$

With $\psi(x) = \{F(x) \wedge (1 - F(x))\}^{\varepsilon/2 - 1/2}$ and

$$g_K(x) = (\{F(x) \wedge (1 - F(x))\}^{\varepsilon/2 + 1/2}/f(x)) \wedge K$$

for $\varepsilon > 0$ and $K > 0$ this implies

$$\begin{aligned}
\sup_g \frac{\operatorname{var}_f G(X)}{E_f g^2(X)} &\geq 2^{1+\varepsilon} \varepsilon (1 + \varepsilon)^{-2} \lim_{K\to\infty} \int \{F \wedge (1 - F)\}^{\varepsilon/2 + 1/2} g_K \\
&= 2^{1+\varepsilon} \varepsilon (1 + \varepsilon)^{-2} \int \{F \wedge (1 - F)\}^{1+\varepsilon}/f.
\end{aligned} \qquad (2.15)$$

Taking the limit for $\varepsilon \downarrow 0$ we see that, if (2.11) is not valid then (2.10) can not hold.

We assume now that (2.11) holds and that $L_0 = \lim_{u\downarrow 0} u/f(F^{-1}(u))$ and $L_1 = \lim_{u\uparrow 1} (1 - u)/f(F^{-1}(u))$ exist. If $L_0 = \infty$, then for every $K > 0$ there exists a $u_K > 0$ such that $u/f(F^{-1}(u)) \geq K$ on $(0, u_K]$ and the inequality

$$\varepsilon \int_0^{1/2} u^{1+\varepsilon}/f^2(F^{-1}(u))\, du = \varepsilon \int_0^{1/2} \left\{ \frac{u}{f(F^{-1}(u))} \right\}^2 u^{\varepsilon - 1}\, du \geq K^2 u_K^{\varepsilon} \qquad (2.16)$$

shows that (2.11) can not hold. We conclude that L_0 and L_1 have to be finite.

Consequently there exist $a, e \in \mathbb{R}$ with $0 < F(a) < \frac{1}{2} < F(e) < 1$ such that $\{F \wedge (1 - F)\}/f$ is bounded outside (a, e). Furthermore (2.11) shows that $\int_a^e f^{-1}$ is finite and we see that the support of f has to be an interval. Choosing now $h(x) = (f(x))^{-1}$ on (a, e) and $h(x) = \{F(x) \wedge (1 - F(x))\}^{-3/2} f(x)$ outside (a, e) and choosing $\chi(x, y)$ as in (2.2) for some $b \in (a, e)$ and H as in (2.6), we see that

$$\int \chi(z, x) H(z) f(z)\, dz \{f(x) h(x)\}^{-1}$$

is uniformly bounded a.e. on the support of f. Together with (2.8) this yields (2.10). Note that (2.7) is no restriction here, since the support of f is an interval. □

If f is such that there exists an interval over which $1/f$ is integrable and outside which $(F \wedge (1 - F))/f$ is bounded, then we will say that Condition 2.1 holds. From Theorem 2.2 and its proof we see that the following string of implications holds

$$(2.10) \Rightarrow (2.11) \Rightarrow \text{Condition 2.1} \Rightarrow (2.10),$$

where we have used the regularity conditions of Theorem 2.2 in order to prove the second implication.

LEMMA 2.1. *If the support of f is an interval on which* $\log f$ *is concave, i.e., if f is strongly unimodal, then Condition* 2.1 *and hence* (2.10) *hold.*

PROOF. In view of the unimodality of f it suffices to prove the boundedness of $F(x)/f(x)$ for x "close" to $F^{-1}(0+) = a$. First assume that $a > -\infty$. If $f(a) = 0$ then f is nondecreasing and positive on (a, b) for some $b > a$ and we have

$$F(x)/f(x) = \int_a^x f(y)\, dy/f(x) \le \int_a^x dy \le b - a, \quad a < x < b.$$

If $f(a) > 0$ the boundedness of $F(x)/f(x)$ on some interval (a, b) is trivial.

Finally, assume that $a = -\infty$. Because $\log f(x)$ is concave with $\log f(-\infty) = -\infty$, there exist constants $b \in \mathbb{R}$ and $c > 0$ such that

$$\log f(x) - \log f(y) \ge c(x - y), \quad y \le x \le b.$$

Consequently we have

$$F(x)/f(x) = \int_{-\infty}^x f(y)\, dy/f(x) \le \int_{-\infty}^x e^{c(y-x)}\, dy = 1/c, \quad x \le b. \quad \square$$

For μ counting measure on the integers we consider only the case $h(x) = 1$ and $H(x) = x - \nu$ with $\nu = E_f X$. This may be obtained by the choice

$$\chi(x, y) = 1_{[[\nu], x)}(y) - 1_{[x, [\nu])}(y) - (\nu - [\nu]) 1_{\{[\nu]\}}(y), \tag{2.17}$$

where $[\nu]$ is the integer part of ν. For this choice of χ the function g satisfies

TABLE 2.2
The value of the upper bound of (2.8) *for some choices of f and H with μ counting measure on the integers and χ as in* (2.17)

name	f	H	(2.8)
1. Poisson	$e^{-\lambda}\lambda^x(x!)^{-1}$	$x-\lambda$	$\lambda E_f g^2(X)$
2. binomial	$\binom{n}{x} p^x(1-p)^{n-x}$	$x-np$	$pE_f(n-X)g^2(X)$
3. negative binomial	$\binom{x-1}{k-1} p^k(1-p)^{x-k}$	$x-k/p$	$(1-p)p^{-1}E_f Xg^2(X)$

$g(x) = G(x+1) - G(x)$. A few examples of (2.8) for this situation are given in Table 2.2.

REMARK 1. It is easy to verify that Examples 1 and 2 of Table 2.2 yield only the first summands in the right-hand sides of inequalities (5.1) and (6.1) respectively of Cacoullos (1982), thus providing better bounds.

3. Cramér-Rao inequality. Here we present a Cramér-Rao type of inequality which generalizes (1.3) and (1.4) and which is related to the inequalities in Lemmas 2.2.3 and 2.2.4 of Klaassen (1981).

THEOREM 3.1. *Let μ, X, f, χ, g and G be as in Theorem* 2.1 *and let k: $\mathbb{R} \to \mathbb{R}$ be a measurable function for which $E_f k^2(X)$ is positive and finite and for which $E_f k(X)$ vanishes. If K: $\mathbb{R} \to \mathbb{R}$ can be defined in such a way that it satisfies*

$$K(y)f(y) = \int \chi(x, y)k(x)f(x)\, d\mu(x), \quad y \in \mathbb{R}, \tag{3.1}$$

then the inequality

$$\operatorname{var}_f G(X) \geq (E_f K(X)g(X))^2(\operatorname{var}_f k(X))^{-1} \tag{3.2}$$

holds, with equality iff G is linear in k, f-almost everywhere.

PROOF. Analogously to (1.4) and in view of (3.1) we have

$$\begin{aligned}\operatorname{var}_f G(X)\operatorname{var}_f k(X) &\geq \{E_f G(X)k(X)\}^2 \\ &= \left\{\int \int \chi(x, y)g(y)\, d\mu(y)k(x)f(x)\, d\mu(x)\right\}^2 \\ &= \left\{\int K(y)g(y)f(y)\, d\mu(y)\right\}^2 = (E_f K(X)g(X))^2. \quad \square\end{aligned}$$

The Cramér-Rao inequality as quoted in Lemma 2.2 of Cacoullos (1982) has been used in that paper to derive lower bounds for $\operatorname{var}_f G(X)$. Its relation to our Theorem 3.1 becomes clear when we substitute $n = 1$, $f(x, \theta) = f(x)$ and $(\partial/\partial\theta)\log f(x, \theta) = k(x)$.

Let us consider the case in which μ is Lebesgue measure and χ is as in (2.2) or (2.3). Then $\int \chi(x, y)k(x)f(x)\,d\mu(x) = \int_y^\infty k(x)f(x)\,dx$ holds. If f is absolutely continuous with derivative f' and finite Fisher information $I(f)$ with respect to a location parameter, i.e., $I(f) = \int (f'/f)^2 f < \infty$, then we may choose k to be a multiple of f'/f and (3.2) reduces to

$$\text{var}_f G(X) \geq (E_f g(X))^2/I(f). \tag{3.3}$$

For a characterization of those distribution functions F for which there exists a positive constant d satisfying

$$\text{var}_F G(X) \geq d(E_F g(X))^2 \tag{3.4}$$

for all absolutely continuous functions G with derivative g, the reader is referred to Definition 4.1 and Theorem 4.2 of Huber (1981) concerning the Fisher information.

Some examples of (3.3) and, more generally, (3.2) are given in Table 3.1.

REMARK 2. Note that Examples 1, 6 and 7 of Table 3.1 yield the same lower bounds as Propositions 3.2, 3.3 and 4.4 respectively of Cacoullos (1982). Choosing in the normal case $k(x) = x^2 - \sigma^2 + 2\sigma^2 x E_f g(X)(E_f X g(X))^{-1}$ the lower bound

TABLE 3.1
The value of the lower bound of (3.2) for some choices of f and k with μ Lebesgue measure and χ as in (2.2)

name	f	k	(3.2)
1. normal	$(2\pi\sigma^2)^{-1/2}e^{-x^2\sigma^{-2}/2}$	x	$\sigma^2(E_f g(X))^2$
2. Cauchy	$\{\pi\sigma(1 + x^2\sigma^{-2})\}^{-1}$	$x(1 + x^2\sigma^{-2})^{-1}$	$2\sigma^2(E_f g(X))^2$
3. Laplace	$\frac{1}{2}\lambda e^{-\lambda\lvert x\rvert}$	$\text{sgn } x$	$\lambda^{-2}(E_f g(X))^2$
4. logistic	$2\lambda(e^{\lambda x} + e^{-\lambda x})^{-2}$	$(e^{\lambda x} - e^{-\lambda x})(e^{\lambda x} + e^{-\lambda x})^{-1}$	$\frac{3}{4}\lambda^{-2}(E_f g(X))^2$
5. gamma	$\{\sigma^\alpha\Gamma(\alpha)\}^{-1}x^{\alpha-1}e^{-x/\sigma}$	$1 - (\alpha - 1)\sigma x^{-1}$	$\sigma^2(\alpha - 2)(E_f g(X))^2$, $\alpha \geq 2$
6. normal	$(2\pi\sigma^2)^{-1/2}e^{-x^2\sigma^{-2}/2}$	$x^2 - \sigma^2$	$\frac{1}{2}(E_f X g(X))^2$
7. gamma	$\{\sigma^\alpha\Gamma(\alpha)\}^{-1}x^{\alpha-1}e^{-x/\sigma}$	$\alpha\sigma - x$	$\alpha^{-1}(E_f X g(X))^2$

TABLE 3.2
The value of the lower bound of (3.2) for some choices of f and k with μ counting measure on the integers and g the forward difference of G

name	f	k	(3.2)
1. Poisson	$e^{-\lambda}\lambda^x(x!)^{-1}$	$x - \lambda$	$\lambda(E_f g(X))^2$
2. binomial	$\binom{n}{x}p^x(1-p)^{n-x}$	$x - np$	$\frac{p}{n(1-p)}(E_f(n - X)g(X))^2$
3. negative binomial	$\binom{x-1}{k-1}p^k(1-p)^{x-k}$	$x - k/p$	$\frac{1-p}{k}(E_f X g(X))^2$

from (3.2) becomes $\sigma^2(E_f g(X))^2 + \frac{1}{2}(E_f Xg(X))^2$ which is the same one as in Proposition 3.4 of Cacoullos (1982).

For μ counting measure on the integers we choose χ such that for all g and G satisfying (2.5) we have $g(x) = G(x+1) - G(x)$. It is not difficult to verify that $\chi(x, y) = \chi(y, y) + 1_{\{x>y\}}$ and hence $\int \chi(x, y)k(x)f(x)\,d\mu(x) = \sum_{x=y+1}^{\infty} k(x)f(x)$ hold in this case. Table 3.2 consists of a few examples for this situation.

REMARK 3. Note that in Examples 1 and 2 of Table 3.2 the same lower bounds are obtained as in Propositions 5.1 and 6.1 of Cacoullos (1982).

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