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ON RANDOMIZED TACTICS AND OPTIMAL STOPPING IN THE PLANE

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Given a two-parameter filtration $(\mathscr{F}_z)$ satisfying the conditional independence assumption (F4), we prove the existence of an optimal stopping point for adapted processes (X_z) indexed by $\mathbb{N}^2$ or $\mathbb{R}_+^2$ which are of class (D), and have regularity properties which generalize the usual one-parameter ones, and are expressed in terms of sequences of 1- and 2-stopping points.

Optimal stopping of processes indexed by directed sets was first studied by Haggstrom [12]. Haggstrom defined control variables, and proved the existence of optimal stopping points for very special processes. Then Cairoli and Gabriel [5] introduced the notion of increasing path, and Krengel and Sucheston [13] refined this notion to that of tactic. Krengel and Sucheston had a new approach: they used the linear embedding of tactics to reduce two-dimensional stopping problems to classical one-dimensional ones for sums of independent identically distributed random variables. Mandelbaum and Vanderbei [16] introduced a similar notion of strategy for countable directed sets, and Walsh [24] extended increasing paths and tactics to the continuous two-parameter situation. The importance of these notions lies mainly in the fact that under the conditional independence assumption (F4), every stopping point is given by a tactic [13], [16], [24]. Intuitively speaking, given a stopping point T, there exists an increasing path passing through T such that if the path leads to $z < T$, then the information contained in $\mathscr{F}_z$ tells one not to stop and also where to go. Here we generalize the notions defined by Cairoli and Gabriel for discrete parameter processes, and Walsh for continuous parameter ones.

In [13] Krengel and Sucheston proved the existence of optimal stopping points for functionals (e.g., averages) of independent identically distributed random variables $(X_z, z \in \mathbb{N}^2)$; given a tactic T, the linear embedding defines i.i.d. random variables $(Y_n, n \in \mathbb{N})$, and a stopping time σ for the filtration of (Y_n) such that X_T and Y_σ have the same distribution. Mandelbaum and Vanderbei [16], Cairoli [4], and Mazziotto and Szpirglas [17] worked on the two-parameter Snell envelope. In [16] Mandelbaum and Vanderbei used Snell's envelope to study the optimal stopping problem for independent Markov chains, and its relation with the theory of multiharmonic functions. In [17] Mazziotto and Szpirglas proved the existence of an optimal stopping point for a process (X_z) indexed by $\mathbb{N}^2 \cup \{\infty\}$ with no independence assumption, but such that $X_\infty \geq \limsup X_z$. Using a decomposition of Snell's envelope, they first proved the existence of an

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optimal optional increasing path, and then solved a usual one-parameter stopping problem. At each step z the successor was chosen to keep Snell's envelope as large as possible, conditionally in $\mathscr{F}_z$. However their method only gave partial results in the continuous case, because of the lack of a "good" decomposition of supermartingales.

In this paper our approach to the optimal stopping problem is different, but well-known for one-parameter processes; see, e.g., Bismut [3], El Karoui [10], and Edgar, Millet and Sucheston [9]. The method consists in introducing a set containing the set of stopping points, showing the existence of an optimal element in the larger set, and then proving that the optimal element can be chosen in the original set of stopping points. This technique is used for processes indexed by $\mathbb{N}^2$ or $\mathbb{R}_+^2$. More precisely, we generalize both the notion of randomized stopping time, first introduced by Baxter and Chacon [1] in the one-dimensional situation, and the notions of optimal increasing path and tactic in the formulation of Walsh [24]. Thus we define the set Θ of randomized tactics, which is a compact subset of a product space with convex sections. Note that this method heavily depends on the fact that every stopping point can be reached by a tactic, since it splits the problem into two successive one-dimensional steps. Krengel and Sucheston [13] and Mandelbaum and Vanderbei [16] have given examples of stopping points in $\mathbb{N}^3$ which cannot be obtained via tactics. Hence in dimension strictly larger than two the technique used in this paper, or that in [13], [16], and [17], would allow us to solve the optimal stopping problem when the supremum is taken over the set of tactics, and not over the set of stopping points. Randomized tactics can be seen as special families $(\nu_a,\ a \in \mathbb{R}_+)$ of probabilities each on $\mathbb{R}_+ \times \{(s, t): s + t = a\} \times \Omega$, and whose projection on Ω is P. As in [1] the conditions on the ν_a will ensure the proper measurability of the increasing processes obtained by desintegrating the probabilities ν_a according to P; see also [18]. Note that a stopping point lying on a given stopping line corresponds to a probability on $\mathbb{R}_+ \times \Omega$ as in the one-dimensional case, and that it is possible to have simultaneously the correct measurability properties and a "good" integral representation. But in the general case we work with families of probabilities (ν_a) rather than with the fairly natural notion of probability on $\mathbb{R}_+^2 \times \Omega$ as in Ghoussoub [11]. Indeed, this last notion does not seem to possess the required measurability properties together with a useful integral representation, while we prove the representation of a randomized tactic as an integral over the set of tactics for some probability measure. This representation is not quite as simple as the one obtained in [9] for one-parameter stopping times, and uses Choquet's theorem. Note that Dynkin's notion of sufficient σ-algebra [8], which avoids the use of topological considerations, cannot be used in our setting since Ghoussoub [11] proved that the convex sets under study are not Choquet simplexes, even in the one-parameter situation. Thus we define EX_θ for $\theta \in \Theta$, and show the existence of an optimal randomized tactic θ^* under boundedness assumptions, and regularity conditions on sequences of 1- and 2-stopping points which extend the usual one-parameter ones, and ensure the continuity of the map $\theta \longrightarrow EX_\theta$. The integral representation of θ^* implies the existence of an optimal stopping point T^*.

In the first section we state the definitions, study the main properties of the set Θ, and characterize the extreme points of the sections of Θ. The second section establishes the integral representation of randomized tactics, and defines EX_θ, $\theta \in \Theta$. Finally the third section studies the continuity of the map $\theta \to EX_\theta$, and gives sufficient conditions for the existence of an optimal stopping point. The discrete and continuous parameter cases are treated simultaneously, and in most cases the proof is written only for continuous parameter processes; the proofs in the discrete case are similar and somewhat easier.

1. Definitions and notations. Let D denote the set of dyadic numbers. Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $\bar{\mathbb{N}}^2 = \mathbb{N}^2 \cup \{(\infty, \infty)\}$ denote the Alexandrov compactifications of $\mathbb{N}$ and $\mathbb{N}^2$ for the discrete topology. Set $\bar{D} = D \cup \{\infty\}$, $\bar{D}^2 = D^2 \cup \{(\infty, \infty)\}$; let $\bar{\mathbb{R}}_+ = [0, \infty]$, and $\bar{\mathbb{R}}_+^2 = \mathbb{R}_+^2 \cup \{(\infty, \infty)\}$ denote the Alexandrov compactifications of $\mathbb{R}_+$ and $\mathbb{R}_+^2$ for the usual topology. Set I to be $\bar{\mathbb{N}}^2$ or $\bar{\mathbb{R}}_+^2$, and $J = \bar{\mathbb{N}}$ or $\bar{\mathbb{R}}_+$, and define on I the usual order $z = (s, t) \le (s', t') = z'$ if $s \le s'$ and $t \le t'$; set $z \ll z'$ if $s < s'$ and $t < t'$. For every $z = (s, t) \in I$ set $|z| = s + t$, $R(z) = [0, s] \times [0, t]$. For every $a \in J$, set $\Delta_a = \{z \in I: |z| = a\}$. Denote by $\mathscr{C}(I)$ $[\mathscr{C}(J)]$ the set of continuous functions on I $[J]$ with the norm $\|f\| = \sup |f(x)|$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space such that $\mathscr{F}$ is countably generated, let $(\mathscr{F}_t^1, t \in J)$ and $(\mathscr{F}_t^2, t \in J)$ be two increasing one-parameter filtrations. For every $z = (s, t) \in I$, set $\mathscr{F}_z = \mathscr{F}_s^1 \cap \mathscr{F}_t^2$. We assume that $\mathscr{F}_{0,0}$ contains all the null sets of $\mathscr{F}$, and that the filtration $(\mathscr{F}_z)$ satisfies the usual assumption (F4), i.e., that given any $z = (s, t) \in I$, $\mathscr{F}_s^1$ and $\mathscr{F}_t^2$ are conditionally independent given $\mathscr{F}_z$ (cf., for example, [6], [19]). Observe that our results remain true if the condition (F4) is weakened to the following one of *qualitative conditional independence* of $\mathscr{F}_s^1$ and $\mathscr{F}_t^2$ given $\mathscr{F}_{s,t}$ [13]: for any sets $A \in \mathscr{F}_s^1$ and $B \in \mathscr{F}_t^2$,

$$\{P(A \mid \mathscr{F}_{s,t}) > 0\} \cap \{P(B \mid \mathscr{F}_{s,t}) > 0\} \subset \{P(A \cap B \mid \mathscr{F}_{s,t}) > 0\}.$$

In the continuous parameter case, i.e., if $I = \bar{\mathbb{R}}_+^2$, we also assume that $(\mathscr{F}_z)$ is right-continuous, and hence satisfies the usual conditions [19]. Set $\mathscr{F}_z^1 = \mathscr{F}_s^1$ and $\mathscr{F}_z^2 = \mathscr{F}_t^2$.

A *stopping point* is a random variable $\tau: \Omega \to I$ such that $\{\tau \le z\} \in \mathscr{F}_z$, $\forall z \in I$; let then $\mathscr{F}_\tau = \{A \in \mathscr{F}: A \cap \{\tau \le z\} \in \mathscr{F}_z, \forall z \in I\}$. A process $(X_z, z \in I)$ is of *class* (D) if the set of random variables X_τ is uniformly integrable when τ belongs to the set of stopping points. The notion of tactic is due to Krengel and Sucheston [13] in the discrete parameter case. Using the formulation of Walsh [24] in the continuous parameter case, we have a slightly weaker definition, similar to that set by Cairoli and Gabriel [5]. An *optional increasing path* (o.i.p.) is an increasing family $(\tau_a, a \in J)$ of stopping points such that $|\tau_a| = a$, $\forall a$. A *tactic* is a pair $T = ((\tau_a, a \in J), \sigma)$, where (τ_a) is an optional increasing path, and σ is a stopping time for the one-parameter filtration $(\mathscr{F}_{\tau_a}, a \in J)$. Given a tactic T we denote by $\tau(T)$ the stopping point τ_σ; given a process (X_z) we denote by X_T the process stopped at $\tau(T)$. Since $(\mathscr{F}_z)$ satisfies the condition (F4), given any stopping point τ, there exists a tactic T such that $\tau = \tau(T)$ ([13], [16], [24]).

The following notions extend that of randomized stopping time due to Baxter and Chacon [1]; see also Meyer [18] and Ghoussoub [11].

DEFINITION 1.1. A *randomized path* is a family $\gamma = (\mu_a, a \in J)$ of probabilities on $(I \times \Omega, \mathscr{B}(I) \otimes \mathscr{F})$ whose projections on Ω are equal to P, and such that the support of μ_a is included in $\Delta_a \times \Omega$ for $a \neq \infty$, and in $\{(\infty, \infty)\} \times \Omega$ for $a = \infty$. Given $a \neq \infty$ let $(A_a(s), 0 \le s \le a, s \in J)$ be the one-parameter, right-continuous, measurable increasing process such that

$$(*) \qquad \mu_a(X) = E\left[\int_{[0,a]\cap J} X_{s,a-s}\, dA_a(s)\right]$$

for every positive, bounded, measurable process X indexed by $\Delta_a = \{(s, a - s): 0 \le s \le a, s \in J\}$. Then $A_a(a) = 1$ a.s.

A *randomized optional increasing path* is a randomized increasing path γ such that the increasing processes $A_a(\cdot)$ satisfy the conditions (i) and (ii):

(i) $\forall a \in J, \forall s \in J, 0 \le s \le a < \infty, A_a(s)$ is $\mathscr{F}_{s,a-s}$-measurable.
(ii) $\forall (a, b) \in J^2, \forall s \in J, s \le a \le b < \infty, A_a(s) \ge A_b(s)$ a.s., and $A_a(a - s) \le A_b(b - s)$ a.s.

Since $A_a(s)$ is increasing in s and decreasing in a, we may and do assume that $A_a(s)$ is right-continuous in s and left-continuous in a; hence the inequalities stated in (ii) hold except on a null set for every a, b, s in J. Note that the right-continuity of the filtration $(\mathscr{F}_z)$ allows us to check the measurability condition (i) when $a \in J \cap \bar{D}$, and $s \in J \cap \bar{D} \cap [0, a]$. Let Γ denote the set of optional increasing paths. The random variables $1 - A_a((a - t)^-) = \bar{A}_a(t), 0 \le t \le a < \infty$, form an increasing process such that $\bar{A}_a(t)$ is adapted to $\mathscr{F}_{a-t,t}$ and

$$\mu_a(X) = E\left[\int_{[0,a]\cap J} X_{a-t,t}\, d\bar{A}_a(t)\right]$$

for any positive measurable process X indexed by Δ_a.

The following lemma is similar to a result proved in [9].

LEMMA 1.2. *There is a bijection between the set of optional increasing paths and the set of randomized optional increasing paths such that the increasing processes* $A_a(s)$ *defined in* 1.1 *take on almost surely the values* 0 *and* 1.

PROOF. Let $(\tau_a, a \in \bar{\mathbb{R}}_+)$ be an optional increasing path. Then the family of increasing processes

$$A_a(s) = 1_{\{\tau_a \le (s,a)\}}, \quad 0 \le s \le a < \infty,$$

satisfies the conditions (i) and (ii) of Definition 1.1. Indeed,

$$\{\tau_a \le (s, a)\} = \{\tau_a < (a, a - s)\}^c, \quad 0 \le s \le a < \infty,$$

so that $\{\tau_a \le (s, a)\} \in \mathscr{F}^1_s \cap \mathscr{F}^2_{a-s} = \mathscr{F}_{s,a-s}$, and $A_a(s)$ is $\mathscr{F}_{s,a-s}$-measurable. Also $a \le b$ implies $\tau_a \le \tau_b$, and for every $s \le a$,

$$\{\tau_b \le (b - s, b)\} = \{\tau_b < (b, s)\}^c \supset \{\tau_a < (a, s)\}^c = \{\tau_a \le (a - s, a)\}.$$

Hence $A_b(s) \le A_a(s)$ and $A_b(b - s) \ge A_a(a - s)$. Hence the measures μ_a defined by $\mu_a(X) = E[X_{\tau_a}]$ define a randomized optional increasing path. Conversely, let $(\mu_a, a \in \overline{\mathbb{R}}_+)$ be a randomized optional increasing path such that for every $a \in \overline{\mathbb{R}}_+$ and every $s \in \overline{\mathbb{R}}_+$, with $0 \le s \le a$, the random variable $A_a(s)$ is a.s. equal to 0 or 1. Fix $a \in \bar{D}$; since the increasing process $A_a(\cdot)$ is right-continuous, there exists a map $\tau_a: \Omega \to \overline{\mathbb{R}}_+^2$ such that

$$\{\tau_a \le (s, a)\} = \{A_a(s) = 1\} \quad \text{a.s. for every } s \text{ with } \quad 0 \le s \le a,$$

and

$$\{\tau_a < (a, t)\} = \{A_a(a - t) = 0\} \quad \text{a.s. for every } t \text{ with } \quad 0 < t \le a.$$

The properties (i) and (ii) on $A_a(s)$ ensure that τ_a is a stopping point taking on values in Δ_a. Indeed,

$$\{\tau_a \le (s, t)\} = \{\tau_a \le (s, a)\} \cap \{\tau_a \le (a, t)\} = \{A_a(s) = 1\} \cap \{A_a((a - t)^-) = 0\},$$

and this intersection is empty if $s + t < a$, and belongs to $\mathscr{F}_{s,a-s} \vee \mathscr{F}_{a-t,t}$ if $s + t \ge a$. In both cases the set $\{\tau_a \le (s, t)\} \in \mathscr{F}_{s,t}$, for $a \in \bar{D}$. Furthermore for any $a < b$, $(a, b) \in \bar{D}^2$, and any $z = (s, t) \le (a, a)$, one has that $\{A_a(s) = 1\} \supset \{A_b(s) = 1\}$, and $\{A_a((a - t)^-) = 1\} \subset \{A_b((b - t)^-) = 1\}$. Hence $\{\tau_a \le (s, t)\} \supset \{\tau_b \le (s, t)\}$ a.s., which implies that $\tau_a \le \tau_b$ a.s. We redefine the countably many stopping points $(\tau_a, a \in \bar{D})$ so that the relations $|\tau_a| = a$ and $\tau_a \le \tau_b$ hold for every a and b in $\bar{D}$. Then given any $\omega \in \Omega$ we extend the map $\bar{D} \ni a \to \tau_a(\omega)$ by continuity to $\overline{\mathbb{R}}_+$. The right-continuity of $(\mathscr{F}_z)$ implies that the maps $(\tau_a, a \in \overline{\mathbb{R}}_+)$ are stopping points. Let $a \in \mathbb{R}_+ \backslash D$ and $s \in [0, a] \cap J$; then the property (ii) and the right-continuity of $A_a(\cdot)$ imply that we have almost surely

$$\{A_a(s) = 1\} \subset \cap_{a_n \in D \cap [0,a[} \{A_{a_n}(s) = 1\} = \cap_{a_n \in D \cap [0,a[} \{\tau_{a_n} \le (s, a)\} = \{\tau_a \le (s, a)\},$$

and

$$\{\tau_a \le (s, a)\} = \cap_{a_n \in D \cap [0,a[} \{\tau_{a_n} \le (s, a)\} = \cap_{a_n \in D \cap [0,a[} \{A_{a_n}(a_n - (a_n - s)) = 1\}$$
$$\subset \cap_{a_n \in D \cap [0,a[} \{A_a(a - (a_n - s)) = 1\} \subset \{A_a(s) = 1\}.$$

Hence for every $a \in \overline{\mathbb{R}}_+$ the increasing processes $A_a(\cdot)$ and $1_{\{\tau_a \le (\cdot, a)\}}$ are almost surely equal, and one has that $\mu_a(X) = E(X_{\tau_a})$ for any positive, bounded measurable process X indexed by I.

The proof in the discrete case, being similar and easier, is omitted. □

We define a set Θ containing the set of tactics.

DEFINITION 1.3. Let $\theta = (\nu_a, a \in J)$ be a family of probabilities on $(J \times I \times \Omega, \mathscr{B}(J) \otimes \mathscr{B}(I) \otimes \mathscr{F})$ such that the family of projections μ_a of ν_a on $I \times \Omega$ form a randomized optional increasing path, and such that the projection ν of ν_a on $J \times \Omega$ is independent of a. Let then $(B_b, b \in J)$ be the right-continuous increasing process such that for any positive, bounded, measurable process X on $J \times \Omega$, one has that $\nu(X) = E(\int_J X_b \, dB_b)$.

A family $\theta = (\nu_a, a \in J)$ as above is a *randomized tactic* if for every $(a, b) \in J^2$, $z = (s, t) \in I$ with $b \le a$, $s \le a$ and $|z| \ge a$, the random variable $B_b[A_a(s) - A_a(a - t)]$ is $\mathscr{F}_z$-measurable.

REMARK. If $\theta = (\nu_a)$ is a randomized tactic, then for every $a \in J$, $z = (s, t)$ with $s + t \geq a$, and every $F \in \mathscr{F}$, one has that

$$\nu_a([0, b] \times R_z \times F) = E[1_F B_b\{A_a(s) - A_a((a - t)^-)\}].$$

Note that the continuity properties of the filtration $(\mathscr{F}_z)$, and of the maps $A_a(\cdot)$, $A_.(s)$ and $B_.$ allow us to check the $\mathscr{F}_z$-measurability of the products $B_b[A_a(s) - A_a(a - t)]$ only when $a \in J \cap \bar{D}$, $(s, t, b) \in (J \cap \bar{D} \cap [0, a])^3$.

Let Θ denote the set of randomized tactics. As in the case of randomized optional increasing paths, we obtain a characterization of tactics in Θ.

LEMMA 1.4. *There is a bijection between the set $\mathscr{T}$ of tactics, and the set of randomized tactics Θ such that the random variables $A_a(s)$ and B_b, defined for $a \in J$, $a \geq s \in J$, and $b \in J$, take on almost surely the values* 0 *and* 1.

PROOF. We briefly indicate the correspondence between both sets; the argument, similar to that in Lemma 1.2, is omitted. Given a tactic $T = ((\tau_a), \sigma)$, define a randomized tactic θ by $A_a(s) = 1_{\{\tau_a \leq (s,a)\}}$, and $B_b = 1_{\{\sigma \leq b\}}$. Hence the probabilities ν_a defined on $J \times I \times \Omega$ are given by

$$\nu_a([0, b] \times R_{(s,t)} \times F) = P[F \cap \{\tau_a \leq (s, t)\} \cap \{\sigma \leq b\}],$$

for every $b \in J$, $z = (s, t) \in I$, and $F \in \mathscr{F}$, and for $a \in J$. □

Note that Ghoussoub [11] proved that already in the one-parameter case the set of randomized stopping times is not a Choquet simplex. Hence to obtain an integral representation of randomized tactics we will use a topological argument; see also [3], [9], [10], and LeCam [14] for a general compactness argument. Following Baxter and Chacon [1], we identify the set Γ of randomized optional increasing paths with the subset of $\prod_{a \in J} \prod_{s \in J \cap [0,a]} L^\infty(P)$ such that the elements $\gamma = (A_a(s), a \in J, s \in J \cap [0, a])$ satisfy the conditions (i) and (ii) of Definition 1.1, $0 \leq A_a(s) \leq 1$, and for every $a \in J$ the map $s \mapsto A_a(s)$ is increasing such that $A_a(a) = 1$. Similarly, we identify the set Θ of randomized tactics with the subset of elements θ of the product $(\prod_{a \in J} \prod_{s \in J \cap [0,a]} L^\infty(P)) \times \prod_{b \in J} L^\infty(P)$ such that the projection of θ on $\prod_{a \in J} \prod_{s \in J \cap [0,a]} L^\infty(P)$ belongs to Γ, and such that the projection of θ on $\prod_{b \in J} L^\infty(P)$, namely $(B_b, b \in J)$ satisfies the conditions $0 \leq B_b \leq 1$, the map $b \rightarrow B_b$ is increasing, $B_\infty = 1$, and $B_b[A_a(s) - A_a(a - t)]$ is $\mathscr{F}_{s,t}$-measurable for every $a \in J$, $b \in J \cap [0, a]$, $(s, t) \in I$ with $s \leq a$ and $s + t \geq a$.

The *Baxter-Chacon topology on* Γ is the coarsest topology such that for every $a \in J$, $s \in J \cap [0, a]$, and $Y \in L^1(\Omega, \mathscr{F}, P)$, the map

$$\Gamma \rightarrow \mathbb{R}, \quad \gamma \mapsto E(YA_a(s))$$

is continuous. *The Baxter-Chacon topology on* Θ is the coarsest topology such that for every $a \in J$, $s \in J \cap [0, a]$, $b \in J$, $Y \in L^1(\Omega, \mathscr{F}, P)$ the maps

$$\Theta \rightarrow \mathbb{R}, \quad \theta \mapsto E(YA_a(s)) \quad \text{and} \quad \Theta \rightarrow \mathbb{R}, \quad \theta \mapsto E(YB_b)$$

are continuous.

As in the one-parameter situation ([1], [3], [10], [18]), we have the following:

THEOREM 1.5. *The sets* Γ *of randomized optional increasing paths, and* Θ *of randomized tactics are compact in the Baxter-Chacon topology.*

PROOF. Recall that $(\Omega, \mathscr{F}, P)$ is separable, and hence $L^1(P)$ is separable, and that the filtrations are right-continuous. Hence the Baxter-Chacon topologies on Γ and Θ are metrizable. The argument is similar to that of Theorem 1.5 in [1]. The only difference lies in the introduction of the map

$$y: L^\infty(P) \times L^\infty(P) \to \mathbb{R}, \quad (A, B) \mapsto E(YAB)$$

for fixed $Y \in L^1(P)$. This map is clearly bilinear and separately continuous in both arguments from $(L^\infty(P), \sigma(L^\infty(P), L^1(P)))$ to $\mathbb{R}$. Since the weak-star topology on L^∞ is locally compact, this map is continuous from the product space $L^\infty \times L^\infty$ with the product of the weak-star topologies (see, e.g., Schaefer [23], page 88). Hence the conditions

$$E(YB_b[A_a(s) - A_a(a - t)]) = E(E(Y \mid \mathscr{F}_{s,t})B_b[A_a(s) - A_a(a - t)])$$

for $a \in J$, $(s, t, b) \in (J \cap [0, a])^3$ with $s + t \geq a$, and $Y \in L^1(P)$ define a closed subset of the product space containing Θ. □

Given $\theta = (\gamma, \beta) \in \Theta$ where γ belongs to Γ identified with a subset of $\prod_{a\in J} \prod_{s\in J\cap[0, a]} L^\infty$, and $\beta \in \prod_{b\in J} L^\infty$, set

$$\Gamma(\theta) = \{\gamma' \in \Gamma: (\gamma', \beta) \in \Theta\} \quad \text{and} \quad B(\theta) = \{\beta' \in \textstyle\prod_{b\in J} L^\infty: (\gamma, \beta') \in \Theta\}.$$

The sets Γ, $\Gamma(\theta)$ and $B(\theta)$ are clearly convex for any $\theta \in \Theta$. The following theorem characterizes their extreme points. It is similar to results of [7], [9], and [11].

THEOREM 1.6. (i) *There is a bijection between the set of optional increasing paths and the set of extreme points of the convex set* Γ *of randomized optional increasing paths.*

(ii) *Fix* $\theta = (\gamma, \beta) \in \Theta$ *and suppose that* $\beta = (B_b, b \in J)$ *is such that each* B_b *takes on almost surely the values* 0 *and* 1. *Then there is a bijection between the set of optional increasing paths of* $\Gamma(\theta)$ *and the set of extreme points of* $\Gamma(\theta)$, *and hence between the set of tactics of* $\Gamma(\theta) \times \{\beta\}$ *and the set of extreme points of* $\Gamma(\theta) \times \{\beta\}$.

(iii) *Fix* $\theta \in \Theta$; *there is a bijection between the set of extreme points of* $B(\theta)$ *and the set of elements* $\beta = (B_b, b \in J)$ *of* $B(\theta)$ *such that each* B_b *takes almost surely the values* 0 *and* 1.

PROOF. We suppose that $I = \overline{\mathbb{R}}_+^2$ and $J = \overline{\mathbb{R}}_+$; the proof in the discrete case is similar and somewhat easier, and will be omitted. Using Lemmas 1.2 and 1.4, we identify the set of o.i.p. [tactics] and the corresponding subset of Γ [Θ]. Then every o.i.p. [o.i.p. of $\Gamma(\theta)$] is clearly an extreme point of Γ [$\Gamma(\theta)$], and any $(B_b, b \in J)$ in $B(\theta)$ such that the B_b take almost surely the values 0 and 1 is clearly an extreme point of $B(\theta)$. We prove the converse set inclusions by contradiction.

(i) Let γ be a randomized optional increasing path such that $A_a(s)$ takes on values in $]0, 1[$ on a nonnull set for some $a \in J$ and $s \in [0, a]$. Using the right-continuity of the map $A_a(\cdot)$ and the equalities $A_\infty(s) = 0$ if $s < \infty$ and $A_\infty(\infty) = 1$, we can choose $a_0 \in D$ and $s_0 \in D \cap [0, a_0]$ such that $A_{a_0}(s_0)$ takes on values in $]0, 1[$ on a nonnull set F. Fix $\lambda \in]0, 1[$, and for every $a \in I$ and $s \in [0, a]$ set

$$A'_a(s) = (A_a(s)/\lambda) \wedge 1 \quad \text{and} \quad A''_a(s) = (A_a(s) - \lambda)/(1 - \lambda) \vee 0.$$

Then $A_a(s) = \lambda A'_a(s) + (1 - \lambda)A''_a(s)$, and $A'_a(\cdot)$ and $A''_a(\cdot)$ are increasing, right-continuous processes; for every $s \le a < \infty$ the random variables $A'_a(s)$ and $A''_a(s)$ are $\mathscr{F}_{s,a-s}$-measurable. Since the functions $x \mapsto (x/\lambda) \wedge 1$ and $x \mapsto (x - \lambda)(1 - \lambda)^{-1} \vee 0$ are increasing, the condition (ii) of 1.1 is clearly satisfied by the processes $A'_a(s)$ and $A''_a(s)$. Furthermore, if $\omega \in F$, then $A'_{a_0}(s_0)(\omega) > A_{a_0}(s_0)(\omega)$. Hence γ is a strictly convex combination of the randomized o.i.p. γ' and γ'' defined by the families of increasing processes $A'_a(\cdot)$ and $A''_a(\cdot)$, which completes the proof of (i).

(ii) Let $\theta = (\gamma, \beta) \in \Theta$ be such that every B_b takes almost surely the values 0 and 1, and that $A_a(s)$ takes on values in $]0, 1[$ on a nonnull set for some $a \in J$ and $s \in [0, a]$. We proceed as in the proof of (i), and given $\lambda \in]0, 1[$ we set

$$A'_a(s) = (A_a(s)/\lambda) \wedge 1 \quad \text{and} \quad A''_a(s) = (A_a(s) - \lambda)/(1 - \lambda) \vee 0.$$

It suffices to prove that the elements γ' and γ'' defined by the families $A'_a(s)$ and $A''_a(s)$ belong to $\Gamma(\theta)$. Fix $a \in J$, $b \in J \cap [0, a]$, $z = (s, t) \in I$ such that $s \le a$ and $s + t \ge a$. Then

$$\begin{aligned} B_b[A'_a(s) - A'_a(a-t)] &= (1/\lambda)1_{\{A_a(s)\le\lambda\}}B_b[A_a(s) - A_a(a-t)] \\ &\quad + (1/\lambda)1_{\{A_a(a-t)\le\lambda<A_a(s)\}}1_{\{B_b>0\}}[\lambda - A_a(a-t)] \\ &= (1/\lambda)1_{\{A_a(s)\le\lambda\}}B_b[A_a(s) - A_a(a-t)] \\ &\quad + ([\lambda - A_a(a-t)]/\lambda)1_{\{A_a(a-t)\le\lambda<A_a(s)\}\cap\{B_b[A_a(s)-A_a(a-t)]>0\}}. \end{aligned}$$

This proves the $\mathscr{F}_z$-measurability of this product, and a similar computation gives the $\mathscr{F}_z$-measurability of the product $B_b[A''_a(s) - A''_a(a - t)]$.

(iii) Let $\theta = (\gamma, \beta) \in \Theta$ be such that B_b takes on values in $]0, 1[$ on a nonnull set. The right-continuity of the increasing process $B.$ allows us to choose $b_0 \in D$ and $\lambda \in]0, 2^{-1}[$ such that if $F = \{B_{b_0} \in]\lambda, 1 - \lambda[\}$, we have that $P(F) > 0$. For every $b \in \overline{\mathbb{R}}_+$, set

$$B'_b = (B_b/\lambda) \wedge 1 \quad \text{and} \quad B''_b = (B_b - \lambda)/(1 - \lambda) \vee 0.$$

Clearly $B. = \lambda B'_. + (1 - \lambda)B''_.$ and $B'_{b_0} \neq B_{b_0}$ on the set F. It suffices to check that the families $\beta' = (B'_b)$ and $\beta'' = (B''_b)$ belong to $B(\theta)$. Fix $a \in J$, $b \in J \cap [0, a]$, $z = (s, t) \in I$ such that $s \le a$ and $s + t \ge a$. Then

$$B'_b[A_a(s) - A_a(a-t)] = (1/\lambda)(B_b[A_a(s) - A_a(a-t)]) \wedge \lambda[A_a(s) - A_a(a-t)];$$

this proves the $\mathscr{F}_z$-measurability of this product, and a similar computation gives the $\mathscr{F}_z$-measurability of the product $B_b''[A_a(s) - A_a(a-t)]$. The other conditions on $B_\cdot'$ and $B_\cdot''$ are trivially satisfied. □

2. Stopping by a randomized tactic, and the Choquet theorem. In this section we define EX_θ for a randomized tactic θ, and a properly bounded, measurable process $(X_z, z \in I)$. We also show how the Choquet theorem reduces the original problem of the existence of an optimal stopping point to an optimal stopping problem with randomized tactics. Given a randomized tactic θ, $a \in J$, $b \in J$, and $z = (s, t) \in I$, set

$$C(b, a, z) = [A_a(s) - A_a((a-t)^-)] \cdot B_b \quad \text{if} \quad |z| = s + t \geq a,$$

$$C(b, a, z) = 0 \quad \text{if} \quad |z| < a.$$

Let T be a tactic identified with the corresponding randomized tactic still denoted by $T = ((\tau_a(T), a \in J), \sigma(T))$; then

$$C_T(b, a, z) = 1_{\{\tau_a(T) \leq z\} \cap \{\sigma(T) \leq b\}}.$$

Let $\theta = (\gamma, \beta)$ be a randomized tactic, let $eB(\theta)$ denote the set of extreme points of $\{\gamma\} \times B(\theta)$, and let $e\Gamma(\theta)$ denote the set of extreme points of $\Gamma(\theta) \times \{\beta\}$. Since $\{\gamma\} \times B(\theta)$ is convex, compact, and metrizable, the Choquet theorem asserts the existence of a Borel probability μ_θ on $\{\gamma\} \times B(\theta)$, supported by $eB(\theta)$, such that $\theta = \int_{eB(\theta)} (\gamma, \beta')\, d\mu_\theta(\gamma, \beta')$ (cf., e.g., [15] or [22]). Since the set $\mathscr{T} \cap (\Gamma(\theta) \times \{\beta'\})$ is identified with the set of extreme points of the convex, compact, metrizable set $\Gamma(\theta) \times \{\beta'\}$ for $(\gamma, \beta') \in eB(\theta)$ by Theorem 1.6, the Choquet theorem yields the existence of a Borel probability $\pi_{(\gamma,\beta')}$ on $\Gamma(\theta) \times \{\beta'\}$ supported by $e\Gamma(\gamma, \beta') = \mathscr{T} \cap (\Gamma(\theta) \times \{\beta'\})$ such that

$$(\gamma, \beta') = \int_{e\Gamma(\gamma,\beta')} (\gamma', \beta')\, d\pi_{(\gamma,\beta')}(\gamma', \beta').$$

Hence given a bilinear map $f\colon (\prod_{a\in J} \prod_{s\in J} L^\infty) \times (\prod_{b\in J} L^\infty) \to \mathbb{R}$, continuous for the product of the weak topologies $\sigma(L^\infty, L^1)$, one has that

$$f(\theta) = \int_{eB(\theta)} \left(\int_{\mathscr{T} \cap e\Gamma(\gamma,\beta')} f(T)\, d\pi_{(\gamma,\beta')}(T) \right) d\mu_\theta(\gamma, \beta').$$

To lighten the notation, we will write the above integral representation

$$f(\theta) = \int_{\mathscr{T}} f(T)\, d(\mu\pi)_\theta(T),$$

and will say that μ_θ and $\pi_{(\gamma,\beta')}$ are the corresponding probabilities (on $eB(\theta)$ and $\mathscr{T} \cap e\Gamma(\gamma, \beta')$).

For any $\theta \in \Theta$, $n \in \mathbb{N}$, and $z \in \mathbb{N}^2$ with $|z| = n$, set

$$D(n, z) = C(n, n, z) - C(n-1, n, z).$$

For $z \in \mathbb{N}^2$ set $D(\infty, z) = 0$, and set

$$D(\infty, (\infty, \infty)) = 1 - \lim_n B_n = 1 - \sum_{n \in \mathbb{N}} \sum_{|z|=n} D(n, z).$$

We at first define EX_θ in the discrete parameter case.

DEFINITION 2.1. Let θ be a randomized tactic, and let $(X_z, z \in \mathbb{N}^2)$ be a process such that either (X_z) is positive, or $\sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E[|X_z| D(n, z)] < \infty$. Then set $EX_\theta = \sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E[D(n, z) X_z]$.

In the following proposition, we express EX_θ in terms of EX_T, for (ordinary) tactics T (cf. [9]).

PROPOSITION 2.2. *Let $\theta = (\gamma, \beta)$ be a randomized tactic, and let μ_θ and $\pi_{(\gamma,\beta')}$ be the corresponding probabilities on $eB(\theta)$ and $\mathscr{T} \cap e\Gamma(\gamma, \beta')$. Let $(X_z, z \in \bar{\mathbb{N}}^2)$ be an integrable process which is positive, or satisfies $\sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E[|X_z| D(n, z)] < \infty$. Then*

$$E(X_\theta) = \int_{eB(\theta)} \left[\int_{\mathscr{T} \cap e\Gamma(\gamma,\beta')} E(X_T) \, d\pi_{(\gamma,\beta')}(T) \right] d\mu_\theta(\gamma, \beta').$$

PROOF. The map $(\gamma, \beta) \to E(X^+_{(\gamma,\beta)})$ is bilinear and is the pointwise limit of the sequence of continuous bilinear maps

$$(\gamma, \beta) \to \sum_{0 \le n \le N} \sum_{|z|=n} E(X_z^+ D(n, z)) + E(X^+_{(\infty,\infty)}(1 - B_N)).$$

Fix $n \in \mathbb{N}$, and $z \in \mathbb{N}^2$ with $|z| = n$, and a random variable X_z; then the map $(\gamma, \beta) \to E(X_z D(n, z))$ is bilinear continuous, and has the following integral representation:

$$E[X_z^+ D(n, z)] = \int_{eB(\theta)} \left[\int_{\mathscr{T} \cap e\Gamma(\gamma,\beta')} E(X_z^+ 1_{\{\sigma(T)=n, \tau(T)_n = z\}}) \, d\pi_{(\gamma,\beta')}(T) \right] d\mu_\theta(\gamma, \beta').$$

Hence

$$\sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E(X_z^+ D(n, z)) = \int_{eB(\theta)} \left[\int_{\mathscr{T} \cap e\Gamma(\gamma,\beta')} E(X_T^+) \, d\pi_{(\gamma,\beta')}(T) \right] d\mu_\theta(\gamma, \beta')$$

$$= \int_{\mathscr{T}} E(X_T^+) \, d(\mu\pi)_\theta(T),$$

and if $\sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E(X_z^+ D(n, z)) < \infty$, then $\int_{\mathscr{T} \cap e\Gamma(\gamma,\beta')} E(X_T^+) \, d\pi_{(\gamma,\beta')}(T)$ is finite for μ_θ almost every (γ, β') and $E(X_T^+)$ is finite for $\pi_{(\gamma,\beta')}$ almost every T. Hence under either assumption on (X_z),

$$EX_\theta = \int_{\mathscr{T}} E(X_T) \, d(\mu\pi)_\theta(T). \quad \square$$

REMARK. The proposition above holds if the integrability condition on (X_z) is replaced by the following semi-integrability property:

$$\inf\{\sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E[X_z^+ D(n, z)], \sum_{n \in \bar{\mathbb{N}}} \sum_{|z|=n} E[X_z^- D(n, z)]\} < \infty.$$

The definition of EX_θ in the continuous parameter case is slightly more complicated because the stopped process X_{τ_σ} cannot be written directly in terms of the $C(b, a, z)$. We proceed by discrete approximation. For any integers $n \geq 1$, k, i, such that $2 \leq i \leq k \leq n \cdot 2^n$, and any randomized tactic θ, set

$$\delta(n, k, i) = [B_{k\cdot 2^{-n}} - B_{(k-1)\cdot 2^{-n}}] \cdot [A_{k\cdot 2^{-n}}(i \cdot 2^{-n}) - A_{k\cdot 2^{-n}}((i-1) \cdot 2^{-n})]$$

and set

$$\delta(n, k, 1) = [B_{k\cdot 2^{-n}} - B_{(k-1)\cdot 2^{-n}}]A_{k\cdot 2^{-n}}(2^{-n}).$$

Also set $\delta(n, 1, 1) = B_{2^{-n}}$, and $\delta(n, \infty) = 1 - B_n$.

Given a tactic $T = ((\tau_a), \sigma)$, set for $2 \leq i \leq k \leq n \cdot 2^n$

$$A(n, k, i) = \{(k-1) \cdot 2^{-n} < \sigma \leq k \cdot 2^{-n}\} \cap \{(\tau_{k\cdot 2^{-n}})_1 \in](i-1) \cdot 2^{-n}, i \cdot 2^{-n}]\},$$

and

$$A(n, k, 1) = \{(k-1) \cdot 2^{-n} < \sigma \leq k \cdot 2^{-n}\} \cap \{(\tau_{k\cdot 2^{-n}})_1 \leq 2^{-n}\};$$

also set $A(n, 1, 1) = \{\sigma \leq 2^{-n}\}$ and $A(n, \infty) = \{\sigma > n\}$.

For any fixed n, the sets $\{A(n, k, i): 1 \leq i \leq k \leq n \cdot 2^n\}$, and $A(n, \infty)$ give a partition of Ω such that $\delta_T(n, k, i) = 1_{A(n,k,i)}$ and $\delta_T(n, \infty) = 1_{A(n,\infty)}$. Define the random variables $T[n]: \Omega \to \bar{D}$ as follows:

$$T[n] = \begin{cases} (i \cdot 2^{-n}, (k+1-i) \cdot 2^{-n}) & \text{on } A(n, k, i) \text{ for } 1 \leq i \leq k \leq n \cdot 2^n \\ (\infty, \infty) & \text{on } A(n, \infty) = \{\sigma > n\}. \end{cases}$$

Then $T[n]$ is a stopping point. Furthermore, if $\omega \in A(n, k, i)$ with $2 \leq i \leq k \leq n2^n$, then $\tau_\sigma(\omega)$ belongs to the domain bounded by the lines of equations $x = i \cdot 2^{-n}$, $x + y = k \cdot 2^{-n}$, $y = (k+1-i) \cdot 2^{-n}$, and $x + y = (k-1) \cdot 2^{-n}$, including the first two ones and excluding the last two ones. Similarly, if $\omega \in A(n, k, 1)$, then $\tau_\sigma(\omega)$ belongs to the domain bounded by the lines $x = 0$, $x = 2^{-n}$, $x + y = k \cdot 2^{-n}$, and $x + y = (k-1) \cdot 2^{-n}$, including the first three lines and excluding the last one. It is easy to see that $T[n] \gg \tau_\sigma$, and that $\lim_n T[n] = \tau_\sigma$.

We now adapt the definition of $T[n]$ to the situation of a randomized tactic θ. Let $(X_z, z \in \bar{\mathbb{R}}^2_+)$ be an adapted, right-continuous process of class (D). Given a randomized tactic θ, and $n \geq 1$, set

$$X_{\theta[n]} = \textstyle\sum_{1\leq k\leq n\cdot 2^n} \sum_{1\leq i\leq k} X_{(i\cdot 2^{-n},(k+1-i)\cdot 2^{-n})}\delta(n, k, i) + X_{(\infty,\infty)}\delta(n,\infty).$$

An argument similar to that of the proof of Proposition 2.2 establishes that if μ_θ and $\pi_{(\gamma,\beta')}$ denote the probabilities on $eB(\theta)$ and $\mathscr{T} \cap e\Gamma(\gamma, \beta')$ corresponding to $\theta = (\gamma, \beta)$ by Choquet's theorem, then for $n \geq 1$ one has that

$$E(X_{\theta[n]}) = \textstyle\sum_{1\leq k\leq n\cdot 2^n} \sum_{1\leq i\leq k} \displaystyle\int_{\mathscr{T}} E[X_{(i\cdot 2^{-n},(k+1-i)\cdot 2^{-n})}\delta_T(n, k, i)]\, d(\mu\pi)_\theta(T)$$
$$+ \int_{\mathscr{T}} E[X_{(\infty,\infty)}\delta_T(n, \infty)]\, d(\mu\pi)_\theta(T),$$

where δ_T is the function associated with T considered as a randomized tactic. Fubini's theorem, and the definition of $T[n]$ imply the following:

LEMMA 2.3. *Under the assumptions above on* (X_z) *and* θ, *we have that* $E(X_{\theta[n]}) = \int_{\mathscr{T}} E(X_{T[n]})\, d(\mu\pi)_\theta(T)$, *where* $X_{T[n]}$ *denotes the process* (X_z) *stopped at* $T[n]$ *in the usual meaning.*

Recall that given a tactic $T = ((\tau_a), \sigma)$, we set EX_T for EX_{τ_σ}. We have the following:

PROPOSITION 2.4. *Let* $(X_z, z \in \overline{\mathbb{R}}^2_+)$ *be a process of class* (D) *with almost surely right-continuous trajectories. Let* θ *be a randomized tactic, and let* μ_θ and $\pi_{(\gamma,\beta')}$ *be probabilities on* $eB(\theta)$ *and* $\mathscr{T} \cap e\Gamma(\gamma, \beta')$ *corresponding to* $\theta = (\gamma, \beta)$ *by Choquet's theorem. Then the sequence* $EX_{\theta[n]}$ *converges to* $\int_{\mathscr{T}} EX_T\, d(\mu\pi)_\theta(T)$, *which will be denoted by* EX_θ.

PROOF. Since (X_z) is a.s. right continuous, one has that $X_T = \lim_n X_{T[n]}$ for any tactic T. Since (X_z) is of class (D), $EX_{T[n]}$ converges to EX_T, and $\sup\{E\,|\,X_T\,|: T \in \mathscr{T}\}$ is finite. Hence $EX_{\theta[n]}$ converges to $\int_{\mathscr{T}} EX_T\, d(\mu\pi)_\theta(T)$. □

REMARK. Proposition 2.4 is valid under the following weaker assumptions: $\sup\{\,|\,EX_T\,|: T \in \mathscr{T}\} < \infty$, and $EX_{\tau(n)} \to EX_\tau$ for every sequence $\tau(n)$ of stopping points such that $\tau(n) \to \tau$ with $\tau(n) \gg \tau$.

The following theorem shows how to reduce the usual problem of optimal stopping to that of stopping with randomized tactics. It allows one to "derandomize" optimal randomized tactics (cf [9]).

THEOREM 2.5. *Under either assumptions* (i) *or* (ii):

(i) $(X_z, z \in \overline{\mathbb{N}}^2)$ *is such that* $\sup\{E\,|\,X_T\,|: T \in \mathscr{T}\} < \infty$,
(ii) $(X_z, z \in \overline{\mathbb{R}}^2_+)$ *is a.s. right-continuous, and of class* (D),

one has

$$V = \sup\{EX_T: T \in \mathscr{T}\} = \sup\{EX_\theta: \theta \in \Theta\}.$$

If one of the suprema is achieved, so is the other. Furthermore, if there exists a randomized tactic θ, *such that* $EX_\theta = V$, *and* $\lim B_b = 1$ *almost surely as* $b \to +\infty$, *then there exists a stopping point* τ, *finite a.s., with* $EX_\tau = V$.

PROOF. Clearly $\sup\{EX_T: T \in \mathscr{T}\} \le \sup\{EX_\theta: \theta \in \Theta\}$. Conversely, the assumption (ii) implies that $EX_\theta = \int_{\mathscr{T}} EX_T\, d(\mu\pi)_\theta(T)$ for any randomized tactic θ by Proposition 2.4. Indeed, under the assumption (i), fix a randomized tactic θ with corresponding probabilities μ_θ on $eB(\theta)$ and $\pi_{(\gamma,\beta')}$ on $\mathscr{T} \cap e\Gamma(\gamma, \beta')$, and fix an integer K. Then

$$\sum_{n\le K}\sum_{|z|=n} E[\,|X_z|\,D(n,z)] = \sum_{n\le K}\sum_{|z|=n}\int_{\mathscr{T}} E[\,|X_z|\,1_{\{\sigma(T)=n,\tau_n(T)=z\}}]\,d(\mu\pi)_\theta(T)$$

$$= \int_{\mathscr{T}} E[\,|X_T|\,1_{\{\sigma(T)\le K\}}]\,d(\mu\pi)_\theta(T)$$

$$\le \sup\{E\,|X_T|: T\in\mathscr{T}\}.$$

Since $X_{(\infty,\infty)}\in L^1$, the assumptions of Proposition 2.2 are satisfied, and $EX_\theta = \int_{\mathscr{T}} EX_T\,d(\mu\pi)_\theta(T)$. Hence under either assumption (i) or (ii), one has that for any randomized tactic $\theta=(\gamma,\beta)$ with corresponding probabilities μ_θ on $eB(\theta)$ and $\pi_{(\gamma,\beta')}$ on $\mathscr{T}\cap e\Gamma(\gamma,\beta')$,

$$EX_\theta = \int_{\mathscr{T}} EX_T\,d(\mu\pi)_\theta(T) \le \sup\{EX_T: T\in\mathscr{T}\}.$$

Suppose that there exists an optimal randomized tactic $\theta=(\gamma,\beta)$, and let μ and π be the corresponding probabilities on $eB(\theta)$ and $\mathscr{T}\cap e\Gamma(\gamma,\beta')$. Then $\int_{\mathscr{T}\cap e\Gamma(\gamma,\beta')} EX_T\,d\pi_{(\gamma,\beta')}(T)$ is equal to V for μ almost every $(\gamma,\beta')\in eB(\theta)$. Hence there exists $\beta'\in eB(\theta)$ such that $\int_{\mathscr{T}\cap e\Gamma(\gamma,\beta')} EX_T\,d\pi_{(\gamma,\beta')}(T)=V$. Then $EX_T=V$ for $\pi_{(\gamma,\beta')}$ almost every T in $\mathscr{T}\cap e\Gamma(\gamma,\beta')$. Hence there exists a tactic T in $e\Gamma(\gamma,\beta')$ such that $EX_T=V$. Furthermore, if $\lim B_b=1$ a.s. when $b\to+\infty$, then a similar argument shows that β' and then T can be chosen such that $EX_T=V$, and $P(\sigma(T)<\infty)=1$, i.e., $\tau(T)_{\sigma(T)}\in\mathbb{R}^2$ a.s. □

3. On the existence of optimal stopping points. In this section we give sufficient conditions for the existence of an optimal randomized tactic. In the discrete and continuous cases, our conditions generalize the classical one-parameter ones (cf., for example, [3], [10], [18], [25]); they ensure the continuity of the map $\theta\to EX_\theta$ in the Baxter-Chacon topology.

We at first prove the existence of an optimal stopping point for "general" discrete parameter processes, i.e., with no independence or distribution requirement. The trivial case of constant processes shows that an assumption has to be made on the terminal payoff X_∞. The following theorem is also proved in [17] via a decomposition of Snell's envelope. Krengel and Sucheston [13] and Mandelbaum and Vanderbei [16] have proved some more precise results, with no condition on the terminal payoff, for functionals of i.i.d. random variables and independent Markov chains. The argument below is similar to those in [3], [9], [10].

THEOREM 3.1. *Let $(X_z, z\in\overline{\mathbb{N}}^2)$ be an adapted integrable process of class (D) such that $X_{(\infty,\infty)}\ge\limsup\{X_z: z\in\mathbb{N}^2\}$. Then there exists an optimal stopping point τ^*, i.e., a stopping point such that*

$$EX_{\tau^*} = V = \sup\{EX_\tau: \tau \text{ stopping point}\}$$

$$= \sup\{EX_T: T\in\mathscr{T}\}.$$

PROOF. Since $X^+_{(\infty,\infty)} \geq (\limsup X_z)^+$, and $(\limsup X_z)^- = [\liminf(-X_z)]^+ \leq [s \liminf(-X_z)]^+$, where $s \liminf$ stands for the stochastic lower limit, Fatou's lemma shows that $(\limsup X_z)^- \in L^1$, and $L = \limsup X_z$ is integrable. Fix $\varepsilon > 0$, and choose $\alpha > 0$ such that $P(A) \leq \alpha$ implies that

$$\sup\{E[1_A \mid X_T \mid 1_{\{\tau_\sigma < \infty\}}] \colon T = ((\tau_a), \sigma) \in \mathscr{T}\} \leq \varepsilon,$$

and $E[1_A \mid X_{(\infty,\infty)} \mid] \leq \varepsilon$.

For any ω let $k(\omega) < \infty$ be an integer such that

$$\sup\{X_z(\omega) \colon k(\omega) \leq |z| < \infty\} \leq L(\omega) + \varepsilon.$$

Choose an integer $k_1 < \infty$ such that

$$P[\sup\{X_z(\omega) \colon k_1 \leq |z| < \infty\} \leq L(\omega) + \varepsilon] \geq 1 - \alpha.$$

Let $(\theta_n, n \geq 1)$ be a sequence of randomized tactics such that EX_{θ_n} converges to V. Since Θ is compact, there exists a subsequence $(\theta_{n_i}, i \geq 0)$ which converges to some randomized tactic θ in the Baxter-Chacon topology. Denote again this subsequence by (θ_n); we prove that $EX_\theta = V$. Let $D_n(a, z)$ and $D(a, z)$ denote the random variables associated to θ_n and θ at the beginning of Section 2, and let μ_n, π_n, μ and π denote the probabilities corresponding to θ_n and θ. Since $\sum_{n\in\mathbb{N}} \sum_{|z|=n} D(a, z) \leq 1$, and since $E[\sum_{n\in\mathbb{N}} \sum_{|z|=n} D(a, z) \mid X_z \mid] < \infty$, there exists an integer $k_2 < \infty$ such that

$$E[\mid X_\infty \mid \textstyle\sum_{k_2 \leq a < \infty} \sum_{|z|=a} D(a, z)] < \varepsilon,$$

and

$$E[\textstyle\sum_{k_2 \leq a < \infty} \sum_{|z|=a} D(a, z) \mid X_z \mid] < \varepsilon.$$

Set $k = k_1 \vee k_2$; then

$$\begin{aligned} EX_\theta &= E[\textstyle\sum_{a<k} \sum_{|z|=a} D(a, z)X_z] + E[\sum_{k \leq a < \infty} \sum_{|z|=a} D(a, z)X_z] \\ &\quad + E[D(\infty, (\infty, \infty))X_{(\infty,\infty)}] \\ &\geq E[\textstyle\sum_{a<k} \sum_{|z|=a} D(a, z)X_z] + E[D(\infty, (\infty, \infty))X_{(\infty,\infty)}] - \varepsilon \\ &\geq E[\textstyle\sum_{a<k} \sum_{|z|=a} D(a, z)X_z] + E[\sum_{k \leq a \leq \infty} \sum_{|z|=a} D(a, z)X_{(\infty,\infty)}] - 2\varepsilon. \end{aligned}$$

Set $A_k = \{\omega \colon \{\sup X_z(\omega) \colon |z| = a \geq k\} \leq X_{(\infty,\infty)}(\omega) + \varepsilon\}$; then $P(A_k) \geq 1 - \alpha$. The definition of the Baxter-Chacon topology yields the existence of n_0 such that $n \geq n_0$ implies

$$\mid E[\textstyle\sum_{a<k} \sum_{|z|=a} (D(a, z) - D_n(a, z))X_z] \mid < \varepsilon,$$

and

$$\mid E[\textstyle\sum_{a \geq k} \sum_{|z|=a} (D(a, z) - D_n(a, z))X_{(\infty,\infty)}] \mid < \varepsilon.$$

Hence for every $n \geq n_0$,

$$\begin{aligned} EX_\theta &\geq E[\textstyle\sum_{a<k} \sum_{|z|=a} D_n(a, z)X_z] + E[\sum_{a\geq k} \sum_{|z|=a} D_n(a, z)X_{(\infty,\infty)}] - 4\varepsilon \\ &\geq E[\textstyle\sum_{a<k} \sum_{|z|=a} D_n(a, z)X_z] - E[1_{A_k^c} \sum_{k\leq a<\infty} \sum_{|z|=a} D_n(a, z)\,|\,X_{(\infty,\infty)}\,|\,] \\ &\quad + E[1_{A_k} \textstyle\sum_{k\leq a<\infty} \sum_{|z|=a} D_n(a, z)(X_z - \varepsilon)] \\ &\quad + E[D_n(\infty, (\infty, \infty))X_{(\infty,\infty)}] - 4\varepsilon \\ &\geq E[\textstyle\sum_{a<k} \sum_{|z|=a} D_n(a, z)X_z] + E[D_n(\infty, (\infty, \infty))X_{(\infty,\infty)}] - 6\varepsilon \\ &\quad + \int_{\mathscr{T}} E[1_{A_k} \textstyle\sum_{k\leq a<\infty} \sum_{|z|=a} X_z 1_{\{\sigma(T)=a, \tau_a(T)=z\}}]\, d(\mu\pi)_{\theta_n}(T). \end{aligned}$$

The last term in the preceding inequality can be written

$$\int_{\mathscr{T}} E[1_{A_k \cap \{k\leq|\tau(T)_{\sigma(T)}|<\infty\}} X_T]\, d(\mu\pi)_{\theta_n}(T);$$

hence

$$\begin{aligned} EX_\theta &\geq E[\textstyle\sum_{a<k} \sum_{|z|=a} D_n(a, z)X_z] + E[D_n(\infty, (\infty, \infty))X_{(\infty,\infty)}] \\ &\quad + \int_{\mathscr{T}} E[1_{\{k\leq|\tau(T)_{\sigma(T)}|<\infty\}} X_T]\, d(\mu\pi)_{\theta_n}(T) - 7\varepsilon \\ &\geq EX_{\theta_n} - 7\varepsilon. \end{aligned}$$

Since the inequality $EX_\theta \geq EX_{\theta_n} - 7\varepsilon$ holds for every $n \geq n_0$, we have $EX_\theta \geq V - 7\varepsilon$. Hence θ is optimal, and Theorem 2.5 completes the proof. □

The separability assumption of $\mathscr{F}$ (to ensure the sequential compactness of Θ) is satisfied when $\mathscr{F}$ is generated by the process $(X_z, z \in \overline{\mathbb{N}}^2)$.

We now study the existence of an optimal randomized tactic for some classes of real-valued continuous-parameter processes. We at first prove the existence of an optimal randomized tactic if $EX_{T[n]}$ converges to EX_T uniformly in $T \in \mathscr{T}$. This uniformity is obviously satisfied if (X_z) has almost surely continuous trajectories.

THEOREM 3.2. *Let $(X_z, z \in \overline{\mathbb{R}}_+^2)$ be a process of class (D) with almost surely right-continuous trajectories, and such that*

$$\lim_n \sup\{\,|\,E(X_T - X_{T[n]})\,|: T \in \mathscr{T}\} = 0.$$

Then there exists an optimal stopping point τ^* such that

$$EX_{\tau^*} = V = \sup\{EX_\tau: \tau \text{ stopping point}\}.$$

PROOF. Let (θ_j) be a sequence of randomized tactics such that (EX_{θ_j}) converges to V. Since Θ is compact, there exists a subsequence of (θ_j) that converges to some randomized tactic θ in the Baxter-Chacon topology; let (θ_j) denote again this subsequence. Fix $n \geq 1$, and let $(A_a^{(j)}(\cdot))$, $(B_\cdot^{(j)})$, and $\delta^{(j)}$ denote the processes and the function associated with θ_j as in Sections 1 and 2. Then

$$E[X_{\theta_j[n]}] = \textstyle\sum_{1\leq k\leq n\cdot 2^n} \sum_{1\leq i\leq k} E[X_{(i\cdot 2^{-n},(k+1-i)\cdot 2^{-n})}\delta^{(j)}(n, k, i)] + E[X_{(\infty,\infty)}\delta^{(j)}(n, \infty)].$$

For fixed n the definition of the Baxter-Chacon topology on Θ implies that $E[X_{(\infty,\infty)}\delta^{(j)}(n, \infty)] = E[X_{(\infty,\infty)}(1 - B_n^{(j)})]$ converges to $E[X_{(\infty,\infty)}(1 - B_n)] = E[X_{(\infty,\infty)}\delta(n, \infty)]$ as $j \to +\infty$. Fix n, k, and i with $2 \leq i \leq k \leq n \cdot 2^n$. Then since $\delta^{(j)}(n, k, i)$ is equal to

$$[B_{k\cdot 2^{-n}}^{(j)} - B_{(k-1)\cdot 2^{-n}}^{(j)}][A_{k\cdot 2^{-n}}^{(j)}(i \cdot 2^{-n}) - A_{k\cdot 2^{-n}}^{(j)}((i - 1) \cdot 2^{-n})],$$

and since the definition of the Baxter-Chacon topology yields the convergence of $E[XB_b^{(j)} \cdot A_a^{(j)}(s)]$ to $E[XB_bA_a(s)]$ as $j \to +\infty$ for $X = X_{(i\cdot 2^{-n},(k+1-i)\cdot 2^{-n})}$, $b = k \cdot 2^{-n}$ or $(k - 1) \cdot 2^{-n}$, $a = k \cdot 2^{-n}$, and $s = i \cdot 2^{-n}$ or $(i - 1) \cdot 2^{-n}$, the sequence $E[X_{(i\cdot 2^{-n},(k+1-i)\cdot 2^{-n})}\delta^{(j)}(n, k, i)]$ converges to $E[X_{(i\cdot 2^{-n},(k+1-i)\cdot 2^{-n})}\delta(n, k, i)]$ as $j \to +\infty$. A similar argument proves the convergence of the analogous expression when $i = 1$ or $k = 1$. This concludes the proof of the convergence of $E[X_{\theta_j[n]}]$ to $E[X_{\theta[n]}]$ for fixed n as $j \to +\infty$. Let $\theta' = (\gamma', \beta')$ be a fixed randomized tactic with corresponding probabilities $\mu_{\theta'}$ and $\pi_{(\gamma',\beta)}$. Then

$$\begin{aligned} |EX_{\theta'} - EX_{\theta'[n]}| &\leq \int_{\mathscr{T}} |EX_T - EX_{T[n]}|\, d(\mu\pi)_{\theta'}(T) \\ &\leq \sup\{|EX_T - EX_{T[n]}|: T \in \mathscr{T}\} \end{aligned}$$

by Proposition 2.4. Hence EX_{θ_j} converges to $EX_\theta = V$, and Theorem 2.5 yields the existence of an optimal stopping point. □

We now give sufficient conditions on the process (X_z) for the uniform convergence of $EX_{T[n]}$ to EX_T. Again we suppose that (X_z) has a.s. right-continuous trajectories, and is of class (D). Our approach is similar to that of Theorem 1.10 [1], and of Theorem 4.7 [9]. It depends on several lemmas. Since $(X_z, z \in \overline{\mathbb{R}}_+^2)$ is of class (D), it suffices to prove that given any $\varepsilon > 0$,

$$\lim_n \sup\{P(|X_T - X_{T[n]}| > \varepsilon): T \in \mathscr{T}\} = 0.$$

Using the continuous time-change $z' = (\text{arc tan } s, \text{arc tan } t)$, we may and do assume that (X_z) is indexed by $[0, 1]^2$. Since (X_z) has right-continuous trajectories, we can replace T by a stopping point $\gamma[n]$ in the computation of $P(|X_T - X_{T[n]}| > \varepsilon)$, where $\gamma[n]$ takes on values in D^2, and satisfies $\tau(T) \ll \gamma[n] \ll T[n]$, and $\sup\{E(|X_T - X_{\gamma[n]}|): T \in \mathscr{T}\} \to 0$ as $n \to \infty$.

Let f: $\mathbb{R} \times \mathbb{R} \to [0, 1]$ be defined by $f(x, y) = |x - y|/(1 + |x - y|)$. Given $(a, b) \in D^2$ with $a < b$, set

$$Y(a, b) = \sup_{t\in D}\sup\{f(X_{s,t}, X_{a,t}): a \leq s \leq b, s \in D\}.$$

For fixed $m \geq 1$, $t \in D$ and $\varepsilon > 0$, set

$$H(t, m) = \{E[Y(t, t + 2^{-m}) \mid \mathscr{F}_t^1] > \varepsilon\},$$

and for fixed $m \geq 0$ and $n \geq 0$ set

$$R(n, m)(\omega) = \inf\{k2^{-n}: \omega \in H(k2^{-n}, m)\},$$

with the usual convention $\inf \varnothing = +\infty$. These definitions are similar to that set by Baxter-Chacon [1]. For fixed m, the sequence $(R(n, m), n \geq 0)$ of $(\mathscr{F}_t^1)$ stopping times decreases to a $(\mathscr{F}_t^1)$ stopping time, say $R(m)$. The sequence $R(m)$ increases. The two lemmas below are similar to the results proved in [1].

LEMMA 3.3. *Let $(X_z, z \in [0, 1]^2)$ be a process such that for any $(\mathscr{F}_t^1)$ stopping time σ*

$$\lim_{a \to \sigma(\omega), a \in D} \sup_{b \in D} | X_{a,b}(\omega) - X_{\sigma(\omega),b}(\omega) | = 0 \quad a.s.$$

Then $\lim P[R(m) \leq 1] = 0$.

PROOF. For any integer $m \geq 1$, the right-continuity of $X_{(.,t)}$ being "uniform in t" by assumption, one has that

$$a(m, n) = sup\{f(X_{s,t}, X_{R(m)+2^{-m},t}): R(m) + 2^{-m} \leq s \leq R(m, n) + 2^{-m}, (s, t) \in D^2\}$$

converges to zero as n converges to infinity. Hence the triangle inequality applied to f shows that

$$Y[R(m, n), R(m, n) + 2^{-m}]$$
$$\leq Y[R(m), R(m) + 2^{-m}] \vee a(m, n) + \sup\{f(X_{(R(m,n),t)}, X_{(R(m),t)}): t \in D\}.$$

Hence

$$\limsup_n Y[R(m, n), R(m, n) + 2^{-m}] \leq Y[R(m), R(m) + 2^{-m}].$$

Conversely,

$$Y[R(m), R(m) + 2^{-m}]$$
$$\leq Y[R(m), R(m, n)] \vee (Y[R(m, n), R(m, n) + 2^{-m}] + Y[R(m), R(m, n)]),$$

so that

$$Y[R(m), R(m) + 2^{-m}] \leq \liminf_n Y[R(m, n), R(m, n) + 2^{-m}].$$

Hence

$$\lim_n Y[R(m, n), R(m, n) + 2^{-m}] = Y[R(m), R(m) + 2^{-m}].$$

The definition of $R(m, n)$ and $R(m)$ implies that

$$E[1_{\{R(m)<\infty\}} Y[R(m), R(m) + 2^{-m}]] \geq \varepsilon P[R(m) < \infty].$$

Since $R(m)$ increases to an $(\mathscr{F}_t^1)$ stopping time R, the assumption made on the left limits at $(\mathscr{F}_t^1)$ stopping points yields that $P[R(m) \leq 1] \to 0$. □

Exchanging the roles of the coordinates in the plane, we obtain the following similar results: For any $(a, b) \in D^2$ let

$$Z(a, b) = \sup\{f(X_{s,t}, X_{s,a}): a \leq t \leq b, (s, t) \in D^2\}.$$

For any $s \in D$, $m \geq 0$, and $n \geq 0$, set

$$K(s, m) = \{E[Z(s, s + 2^{-m}) \mid \mathscr{F}^2_s] > \varepsilon\},$$

$$S(n, m)(\omega) = \inf\{k \cdot 2^{-n}: \omega \in K(k \cdot 2^{-n}, m)\}.$$

The sequence $(S(n, m), n \geq 0)$ of $(\mathscr{F}^2_s)$ stopping times decreases to $S(m)$. Under the assumptions on (X_z) obtained by exchanging the first and the second planar coordinates in Lemma 3.3, one has that $\lim P[S(m) \leq 1] = 0$. The following lemma relates tactics to the one-parameter stopping points $R(m)$ and $S(m)$.

LEMMA 3.4. *Let $(X_z, z \in [0, 1]^2)$ be a right-continuous process such that for any $(\mathscr{F}^1_t)$ stopping time σ, and any $(\mathscr{F}^2_t)$ stopping time τ,*

$$\lim_{a \to \sigma, a \in D}(\sup_{b \in D} | X_{a,b} - X_{\sigma,b} |) = 0 \quad a.s.$$

and

$$\lim_{b \to \tau, b \in D}(\sup_{a \in D} | X_{a,b} - X_{a,\tau} |) = 0 \quad a.s.$$

then

$$\sup\{E[f(X_T, X_{T[m]})]: T \in \mathscr{T}\} \leq 4\varepsilon + 2P[R(m-1) \leq 1] + 2P[S(m-1) \leq 1].$$

PROOF. Fix $\alpha > 0$, $m \geq 1$, and a tactic $T \in \mathscr{T}$. Let γ be a stopping point taking on values in $D \times D$, such that $E[f(X_T, X_\gamma)] \leq \alpha$, and $\tau(T) \ll \gamma \ll T[m]$. Set $\sigma[m] = (T[m]_1, \gamma_2)$ and $\tau[m] = (\gamma_1, T[m]_2)$; remark that $\tau(T)_1 \leq T[m]_1 \leq \tau(T)_1 + 2^{1-m}$, and $\tau(T)_2 \leq T[m]_2 \leq \tau(T)_2 + 2^{1-m}$. Then $E[f(X_\gamma, X_{\sigma[m]})] \leq E[Y(\gamma_1, \gamma_1 + 2^{1-m})]$. Let σ be a one-dimensional $(\mathscr{F}^1_t)$ stopping time taking on dyadic values; we prove that for every $k \geq 0$ one has that $E[Y(\sigma, \sigma + 2^{-k})] \leq \varepsilon + P[R(k) \leq \sigma]$. Fix k; for any $j \geq 0$, let σ_j be the $(\mathscr{F}^1_t)$ dyadic stopping time defined by $\sigma_j = n2^{-j}$ on $\{\sigma \in [(n-1) \cdot 2^{-j}, n \cdot 2^{-j}[\}$ and set $\eta_j = Y(\sigma_j, \sigma_j + 2^{-k})$. On the set $\{\sigma_j = t\}$, one has $E[\eta_j \mid \mathscr{F}^1_{\sigma_j}] = E[Y(t, t + 2^{-k}) \mid \mathscr{F}^1_t]$. Hence, $R(k, j) \leq t$ on $\{\sigma_j = t\} \cap \{E[\eta_j \mid \mathscr{F}^1_{\sigma_j}] > \varepsilon\}$, and

$$E[\eta_j] \leq \varepsilon + P[R(k, j) \leq \sigma_j] \leq \varepsilon + P[R(k) \leq \sigma_j],$$

since the sequence $(R(k, j), j \geq 0)$ decreases to $R(k)$. Let $j \to \infty$; since the horizontal processes are continuous uniformly in the second coordinate, $E[\eta_j]$ converges to $E[Y(\sigma, \sigma + 2^{-k})]$, which proves the inequality announced above. Hence

$$E[f(X_\gamma, X_{\sigma[m]})] \leq \varepsilon + P[R(m-1) \leq 1],$$

and similarly

$$E[f(X_\gamma, X_{\tau[m]})] \leq \varepsilon + P[S(m-1) \leq 1].$$

A similar argument shows that

$$E[f(X_{\sigma[m]}, X_{T[m]})] \leq \varepsilon + P[S(m-1) \leq 1],$$

and

$$E[f(X_{\tau(m)}, X_{T[m]})] \leq \varepsilon + P[R(m-1) \leq 1].$$

Hence for any tactic T and $m \geq 1$, one has

$$E[f(X_T, X_{T[m]})] \leq \alpha + 2[2\varepsilon + P(S(m-1) \leq 1) + P(R(m-1) \leq 1)],$$

which yields the desired inequality when $\alpha \to 0$. □

Finally, the following theorem gives the existence of optimal stopping points for right-continuous processes of class (D) which are quasi-continuous along sequences of stopping points for the filtrations $(\mathscr{F}_z^1)$ and $(\mathscr{F}_z^2)$. Hence this theorem extends known results of existence of optimal stopping times for one-parameter processes (see, e.g., [10]). However we do not obtain a constructive way to find an optimal stopping point. The conditions (i) and (ii) below can be interpreted as 1- and 2-quasicontinuity of the process (X_z).

THEOREM 3.5. *Let $(X_z, z \in \mathbb{R}_+^2)$ be a process of class (D) with almost surely right-continuous trajectories. Suppose that the conditions* (i) *and* (ii) *are satisfied:*

(i) [(ii)] *Let $(\sigma(n), n \geq 1)$ $[(\tau(n), n \geq 1)]$ be a sequence of stopping points for the filtration $(\mathscr{F}_z^1)$ $[(\mathscr{F}_z^2)]$, taking on finitely many values and converging to σ $[\tau]$. The sequence $EX_{\sigma(n)}$ $[EX_{\tau(n)}]$ converges when $\sigma(n)_1 > \sigma_1$ and $\sigma(n)_2 < \sigma_2$ $[\tau(n)_2 > \tau_2$ and $\tau(n)_1 < \tau_1]$, and this sequence converges to EX_σ $[EX_\tau]$ when $\sigma(n)_1 < \sigma_1$ $[\tau(n)_2 < \tau_2]$.*

Then there exists an optimal stopping point τ^*, i.e., such that $EX_{\tau^*} = \sup\{EX_\tau$: τ stopping point$\}$.

PROOF. It suffices to notice that Propositions 2.2 and 2.3 in [21] show that the assumptions of Lemma 3.3 are satisfied by (X_z). Lemma 3.3—its analog obtained by exchanging the first and the second planar coordinate—Lemma 3.4 and Theorem 3.2 yield the existence of an optimal stopping point. □

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