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OPTIMAL PREDICTION OF LEVEL CROSSINGS IN GAUSSIAN PROCESSES AND SEQUENCES

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Let $\eta(t)$ be stationary, Gaussian, and suppose one wants to predict the future upcrossings of a certain level u. The paper investigates criteria for a good level crossing predictor, and restates in simple form a result by de Maré on optimal prediction. Write $\hat{\eta}_t(t+h)$ for the mean square predictor of $\eta(t+h)$ at time t, and let $\hat{\zeta}_t(t+h)$ be the conditional expectation of $\eta'(t+h)$ given observed data and given $\eta(t+h) = u$. It is shown that an alarm which predicts an upcrossing at $t+h$ if $\hat{\eta}_t(t+h)$ differs from u by a quantity that is a certain function of $\hat{\zeta}_t(t+h)$ is optimal in the sense that it maximizes the detection probability for a fixed total alarm time. Explicit formulas are given for the upcrossing risks at alarm, detection probability, and total alarm time. In an example the optimal alarm is compared to a naive alarm which predicts an upcrossing if $\hat{\eta}_t(t+h)$ differs from u by a fixed proportion of the residual standard deviation. The optimal alarm locates the upcrossings more precisely and at an earlier stage than the naive alarm, which has a tendency to give late alarms.

1. Principles of level crossing prediction. To predict a stationary stochastic process $\eta(t)$ is in principle nothing but to calculate the conditional distribution of the entire future of the process given all available data. If the process is Gaussian, with known mean and covariance function, the conditional mean value function $\hat{\eta}_t(t+h)$, $h > 0$, calculated at time t, completely determines the distribution of the future (together, of course, with the conditional covariances, which do not depend on the data), and is therefore the perfect way to summarize data. Furthermore $\hat{\eta}_t(t+h)$ is optimal in the mean square error sense if the object is to give a single value to approximate $\eta(t+h)$, i.e. to make *value prediction.*

However, if the object is to predict whether or not $\eta(t+h)$ will exceed a specified level u for some h, $0 < h < h_0$ one can not only use the single value $\hat{\eta}_t(t+h)$ for each particular h, but must also consider the predicted change rate, in a way to be specified later. Furthermore, and this is important for this type of *event prediction*, the goodness of the prediction principle must be judged by its ability to detect the level crossings, to locate them correctly in time, and to make few false alarms, or to make alarm for as short time as possible.

The naive way to predict upcrossings of a level $u > E(\eta(t))$ is to fix an alarm level $\hat{u}$, and to foretell that $\eta(t+h)$ will upcross u if $\hat{\eta}_t(t+h)$ upcrosses $\hat{u}$, and since $V(\hat{\eta}_t(t+h)) < V(\eta(t+h))$ it seems also reasonable that one should take $\hat{u}$ less than u in order to obtain sufficient detection probability. A basic result by

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de Maré (1980) states that one should predict an upcrossing of the level u by $\eta(t+h)$ if

$$\left(\frac{u-\hat{\eta}_t(t+h)}{\sigma_h}\right)^2 \le 2\log\Psi\left(\frac{\hat{\zeta}_t(t+h)}{\sigma_h'}\right) + 2\log(\sigma_h'/\sigma_h) + K_h. \tag{1}$$

Here $\hat{\zeta}_t(t+h)$ is the conditional expectation of $\eta'(t+h)$ given available data and also given that $\eta(t+h)=u$; σ_h^2 and $\sigma_h'^2$ are the residual variances of $\eta(t+h)$ and $\eta'(t+h)$ given data, and for $\sigma_h'^2$ given also that $\eta(t+h)=u$. Further, the function Ψ is defined by

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\int_{z=-\infty}^{y}\exp(-z^2/2)\,dz\,dy = \phi(x) + x\Phi(x),$$

and K_h is an arbitrary constant that determines the detection probability.

Taking the lower limit

$$\hat{u} = u - \sigma_h\{2\log\Psi(\hat{\zeta}_t(t+h)/\sigma_h') + 2\log(\sigma_h'/\sigma_h) + K_h\}^{1/2}$$

one obtains a *variable* alarm level that adjusts according to the expected growth rate of the process.

The alarm principle (1) is *optimal* in the sense that it has the highest detection probability of all alarm principles with the same overall alarm probability. Further, it predicts the *time* for an upcrossing better than other possible alarm principles.

The paper is organized as follows. In Section 2 is defined precisely what is meant by alarm probability (or alarm size) and detection probability, and in Section 3 are introduced the concepts of specific and total risks at alarms. By this we mean the conditional risks for upcrossing given that an alarm has actually been given; the specific risk is then defined for each set of available data which gives rise to alarm, while the total risk is the average of all specific risks.

The optimal alarm is derived in Section 4, by comparison of the conditional distribution of data given an upcrossing, and the unconditional distributions. Explicit formulas for its specific and total risks are given in Section 5.

By considering the costs for giving alarm, and the losses incurred by unpredicted upcrossings, one can find the best balance between detection probability and total alarm probability; this is done in Section 6. Finally, an example of an optimal level crossing predictor is given in Section 7, and compared to a naive predictor.

The literature on level crossing prediction is sparse. Some ideas on detection probabilities and risk functions were developed in Lindgren (1975) and (1979). The likelihood ratio principle was used by de Maré (1980) to derive an expression for the optimal alarm which is equivalent, but not as explicit as the one given here. The formulas for the risk functions are based on results in Lindgren (1980).

2. Alarm size and detection probability. Let $\{\eta(t), t \in T\}$ be a stationary zero mean Gaussian sequence or continuous time process with covariance function $r_\eta(t) = E(\eta(s)\eta(s+t))$. Suppose that, at time t, prediction may be based on some data $\xi(t)$, finite or infinite dimensional, which may include $\eta(s)$, $s \le t$, as well as other information available at time t. It is important that $\xi(t)$ contains all information that may be used for prediction at time t, and this precludes, e.g., the use of any derivative component of $\xi(t)$ which is not already in $\xi(t)$. Examples of possible data sets are $\xi(t) = (\eta(t), \eta(t-1), \cdots, \eta(t-p))^T$ in both discrete and continuous time, and $\xi(t) = (\eta(t), \eta'(t))^T$ if $\eta(t)$ is a differentiable continuous time process. We assume for simplicity of notation that $\xi(t)$ is finite dimensional, $\xi(t) \in R^p$. We also make the basic assumption that $\{\eta(t)\}$ and $\{\xi(t)\}$ are jointly stationary, ergodic and Gaussian.

Denote by $\{t_k\}$ the upcrossings of a predetermined level u, i.e. the times when

$$\eta(t_k - 1) < u \le \eta(t_k), \quad \text{in discrete time,} \tag{2}$$

$$\eta(t_k) = u, \quad \text{upcrossing, in continuous time.} \tag{3}$$

We write μ for the upcrossing intensity, i.e., the expected number of upcrossings per time unit,

$$\mu = \begin{cases} P(\eta(-1) < u \le \eta(0)), & \text{in discrete time,} \\ E(\#\{t_k \in (0, 1]\}), & \text{in continuous time,} \end{cases}$$

and suppose $\mu < \infty$, so there are only a finite number of t_k's in any finite interval. Since $\eta(t)$ is Gaussian a sufficient condition for this in continuous time is $-r_\eta''(0) < \infty$. In that case

$$\mu = f_{\eta(0)}(u) E(\eta'(0)^+ \mid \eta(0) = u) = f_{\eta(0)}(u) \int_{z=0}^{\infty} z f_{\eta'(0) \mid \eta(0)=u}(z)\, dz, \tag{4}$$

where $x^+ = \max(0, x)$, and f_η, $f_{\eta' \mid \eta = u}$ denote densities and conditional densities. General aspects on upcrossings in continuous time are developed in Leadbetter et al. (1983), Chapter 7.

Denote by $C_t^{(u)}$ the event that $\eta(\cdot)$ has an upcrossing of the level u at t and write $P^0(\cdot \mid C_t^{(u)})$ for the Palm probabilities (for $\eta(\cdot)$ and $\xi(\cdot) \in R^p$) given a u-upcrossing in $\eta(\cdot)$ at t. The Palm probabilities describe statistical properties around the upcrossings provided $\{\eta(t)\}$ and $\{\xi(t)\}$ are jointly ergodic. For example, for any Borel set $B \subset R^p$, with probability one,

$$\begin{aligned} P^0(\xi(t-h) \in B \mid C_t^{(u)}) &= lim_{T\to\infty} \frac{\#\{t_k \in (0, T];\ \xi(t_k - h) \in B\}}{\#\{t_k \in (0, T]\}} \\ &= \frac{E(\#\{t_k \in (0, 1];\ \xi(t_k - h) \in B\})}{E(\#\{t_k \in (0, 1]\})}. \end{aligned} \tag{5}$$

For a discrete time sequence the Palm probabilities are simply conditional probabilities, e.g.,

$$P^0(\xi(t) \in B \mid C_0^{(u)}) = P(\xi(t) \in B \mid \eta(-1) < u \le \eta(0)), \tag{6}$$

while for a continuous time differentiable $\eta(t)$ we have the following lemma.

LEMMA 2.1. *If $\{\eta(t)\}$ and $\{\xi(t)\}$ are jointly stationary, Gaussian and ergodic, $\eta(t)$ mean square differentiable and $\xi(t)$ continuous, and $\xi(t)$, $\eta(0)$, $\eta'(0)$ have a* *nonsingular distribution, then*

$$P^0(\xi(t) \in B \mid C_0^{(u)}) = \mu^{-1} \int_{z=0}^{\infty} z f_{\eta(0),\eta'(0)}(u, z) P(\xi(t) \in B \mid \eta(0) = u, \eta'(0) = z)\, dz.$$

PROOF. This follows in the same way as Lemma 7.5.2 and Theorem 10.3.1 in Leadbetter et al. (1983). □

By a *crossing predictor with warning time h* we mean a Borel set $\Gamma_h \subset R^p$, also called an *h-alarm* or simply, an *alarm region*, such that any time when $\xi(t)$ belongs to Γ_h, we consider it likely that $\eta(t + h)$ will have an upcrossing of the level u, i.e., $C_{t+h}^{(u)}$ will occur. We then say that the alarm is set at time t for an upcrossing at time $t + h$.

By our definition, any Borel set $\Gamma \subset R^p$ may be used as an alarm region, and obviously we need some measure of how well it will work as such. The following probabilities and conditional (Palm) probabilities describe the relevant properties of Γ_h as an alarm region,

(7) $$\alpha_h = P(\xi(t) \in \Gamma_h),$$

(8) $$\gamma_h = P^0(\xi(t) \in \Gamma_h \mid C_{t+h}^{(u)}).$$

We shall call α_h the *size* of the alarm region or the alarm probability, while γ_h is the *detection probability* with warning time h. Thus, α_h is the proportion of the total time $\xi(t)$ spends in the alarm region, while γ_h is the (long run) probability that the h-alarm is set exactly h time units before the upcrossings; cf. (5).

A further quantity of interest is

(9) $$\gamma_{h,h'} = P^0(\xi(t) \in \Gamma_{h'} \mid C_{t+h}^{(u)}),$$

which describes the "timing" of the alarm considered as a function of h'.

Each warning time $h > 0$ has its own alarm region Γ_h. An *alarm policy* is simply any family $\{\Gamma_h\}_{h>0}$ of alarm regions. At any time t the alarm can be set for different times in the future. Write

(10) $$H_t = \{h; \xi(t) \in \Gamma_h\},$$

so that $t + H_t$ is the (random) set of times for which, at time t, the alarm is set for an upcrossing. As time goes by, this set changes, points or intervals of high risk may appear or disappear, or may be shifted back and forth. This time dependence is well illustrated in a diagram showing the set

$$\{(t + h, h) \in R^2; t \in R, h \in H_t\}.$$

The fact that a point (s, h) belongs to this set just means that at time $s - h$ the alarm is set for an upcrossing at time s; see the example in Section 7.

It is clear that in case of an upcrossing at time t, the conditional probability $\gamma_h = P^0(\xi(t - h) \in \Gamma_h \mid C_t^{(u)})$ should be as large as possible for each warning time h. Further, since γ_h is increasing in Γ_h, i.e., (with obvious notation) $\Gamma_h \subset \Gamma_h'$ implies $\gamma_h \le \gamma_h'$, it is also clear that it is only meaningful to compare alarm policies with the same α_h-functions. If one is willing to accept a larger total alarm time one can always increase the detection probability by increasing Γ_h.

By an *optimal alarm policy of size* $\{\alpha_h\}_{h>0}$ is meant a family of alarm sets $\{\Gamma_h\}_{h>0}$, such that

$$P(\xi(t) \in \Gamma_h) = \alpha_h, \quad h > 0,$$
$$P^0(\xi(t) \in \Gamma_h \mid C_{t+h}^{(u)}) = \sup_{B\in\mathscr{S}} P^0(\xi(t) \in B \mid C_{t+h}^{(u)}),$$

where the sup is over all Borel sets B in R^p, such that

$$P(\xi(t) \in B) \leq \alpha_h.$$

An optimal policy can also be described as the policy which, for a given detection probability γ_h, spends the shortest total time in alarm state. (Note that this is not the same as having the smallest number of false alarms.)

In practice the total alarm time α_h must be balanced against what can be obtained in detection probability. In order to get a reasonable γ_h for some particular h it may be necessary to accept a long total alarm time for that h-value. Since both the *cost* of keeping up an alarm and the *loss* incurred by not giving it, may depend on the warning time h, here is a possibility to affect total costs by trading detection probability for total alarm time for some values of h. The minimal cost and best choice of α_h, γ_h is derived in Section 6 under some simple assumptions.

3. Specific and total risks at alarm. Besides the total alarm time α_h and the detection probability γ_h there are a few more quantities which are of interest for the description of an alarm policy.

Consider an alarm region Γ with smooth boundary $\partial\Gamma$, defined by a continuously differentiable function $\chi(x)$, $x \in R^p$, such that

$$x \in \Gamma \Leftrightarrow \chi(x) < 0.$$

When time is continuous we further assume that the data $\xi(t)$ form a continuously differentiable process.

An alarm from the alarm region Γ then starts any time $\xi(t)$ enters Γ. Let $\{s_k\}$ be the alarm times for some specific Γ, i.e.,

$$\xi(s_k - 1) \notin \Gamma, \quad \xi(s_k) \in \Gamma, \quad \text{in discrete time,} \tag{11}$$

$$\xi(s_k) \in \partial\Gamma, \quad \text{entering } \Gamma, \quad \text{in continuous time,} \tag{12}$$

and assume that the mean number of alarms per time unit is finite, so there are with probability one only a finite number of s_k in any finite interval.

Then write $B_t^{(\Gamma)}$ for the event that $\xi(t)$ enters the set Γ at time t, and as usual, $P^0(\bullet \mid B_t^{(\Gamma)})$ for the conditional (Palm) probabilities given an entrance at time t. Expectations with respect to P^0 are denoted by E^0.

The following lemma specifies the stochastic variation of $\xi(s_k)$ and its derivative at the alarm times in continuous time; for a proof, see Lindgren (1980).

LEMMA 3.1. (a). *The long run distribution of $\xi(s_k)$ at alarms s_k, i.e., the distribution of $\xi(0)$ under $P^0(\bullet \mid B_0^{(\Gamma)})$, has the density* (*over* $\partial\Gamma$)

$$f_{\xi(0)}^0(x) = c^{-1} f_{\xi(0)}(x) E((n_x \cdot \xi'(0))^+ \mid \xi(0) = x),$$

where n_x *is the unit normal to* $\partial\Gamma$ *at* x *pointing into* Γ, *and* $n_x \cdot \xi'(0) = \sum n_x^{(i)}\xi_i'(0)$. *The normalizing constant c is equal to the mean number of alarms per time unit.*

(b). *The long run joint distribution of* $\xi(s_k)$ *and* $\xi'(s_k)$ *has density*

$$f^0_{\xi(0)}(x)f^0_{\xi'(0)\mid\xi(0)=x}(z) = c^{-1}(n_x \cdot z)^+ f_{\xi(0)}(x)f_{\xi'(0)\mid\xi(0)=x}(z)$$

over $\partial\Gamma \times R^p$.

Now, for any $\Gamma \subset R^p$ with smooth boundary $\partial\Gamma$, let $\mu_h^{(\Gamma)}$ be the *conditional u-upcrossing intensity* for $\eta(t+h)$ given $B_t^{(\Gamma)}$, i.e., the upcrossing intensity at time $t+h$ given an alarm at time t. This intensity is a function of h such that, for any interval I,

$$E^0(\#\{u\text{-upcrossings by } \eta(t+h),\ h \in I\}) = \int_{h\in I} \mu_h^{(\Gamma)}\, dh \tag{13}$$

is the long run (considered over $\{s_k\}$) mean number of u-upcrossings in the intervals $s_k + I$ following the alarms.

For $x \in \partial\Gamma$ we also introduce the conditional u-upcrossing intensity $\mu_h^{(\Gamma)}(x)$ for $\eta(t+h)$ given $B_t^{(\Gamma)}$ and given that $\xi(t) = x$. With the density $f^0_{\xi(0)}(x)$ for the alarm point x, given in Lemma 3.1(a) one then has

$$\mu_h^{(\Gamma)} = \int_{x\in\partial\Gamma} \mu_h^{(\Gamma)}(x) f^0_{\xi(0)}(x)\, ds(x),$$

where $ds(x)$ is the surface element on $\partial\Gamma$.

We shall call $\mu_h^{(\Gamma)}$ the *total upcrossing risk* at alarm, and $\mu_h^{(\Gamma}(x)$ the *specific* risk at x. Note that every time an alarm is given one knows the value of $\xi(t)$ and can calculate the corresponding specific risk $\mu_h^{(\Gamma)}(x)$.

Further, let

$$\beta_h^{(\Gamma)} = P^0(\eta(t+h) > u \mid B_t^{(\Gamma)}),$$

$$\beta_h^{(\Gamma)}(x) = P^0(\eta(t+h) > u \mid B_t^{(\Gamma)},\ \xi(t) = x),$$

be the *total* and *specific exceedance risks* after alarms, satisfying

$$\beta_h^{(\Gamma)} = \int_{x\in\partial\Gamma} \beta_h^{(\Gamma)}(x) f^0_{\xi(0)}(x)\, ds(x).$$

For a fixed alarm region Γ, both $\beta_h^{(\Gamma)}$ and $\mu_h^{(\Gamma)}$ describe the risk for an exceedance at time h after the alarm. Then $\beta_h^{(\Gamma)}$ is the total risk that the value exceeds u exactly at time $t+h$, while $\mu_h^{(\Gamma)}$ gives a feeling for when the upcrossing is likely to occur. We shall return to the total and specific risks in Sections 5 and 7 and see how, for an optimal alarm, they can be calculated as simple or double integrals of elementary functions.

4. Optimal alarm regions. To obtain an optimal level crossing predictor or alarm policy one can use an analogy with "most powerful tests", and consider the likelihood ratio between the conditional (Palm) distribution of $\xi(t)$ given an upcrossing $\eta(t+h)=u$, and its unconditional distribution. As is clear from the definition of optimality, and α_h, γ_h, no other distributions can be of interest in this respect.

Write $p(x)$ for the (unconditional) density $f_{\xi(t)}(x)$ of $\xi(t)$, and denote by $p_h^{(u)}(x)$ the density of $\xi(t)$ under the Palm distribution $P^0(\bullet \mid C_{t+h}^{(u)})$, i.e., given a u-upcrossing $\eta(t+h)=u$. Thus $p_h^{(u)}(x)/p(x)$ is the likelihood ratio between the conditional and unconditional distributions.

LEMMA 4.1. *The alarm policy* $\{\Gamma_h\}_{h>0}$ *with*

$$\Gamma_h = \{x \in R^p;\ p_h^{(u)}(x)/p(x) \ge k_h\}, \tag{14}$$

k_h *nonnegative constants, is optimal of size*

$$\alpha_h = P(\xi(-h) \in \Gamma_h),$$

i.e.,

$$P^0(\xi(-h) \in \Gamma_h \mid C_0^{(u)}) = \sup_{B\in\mathscr{F}} P^0(\xi(-h) \in B \mid C_0^{(u)})$$

where the sup *is taken over all Borel sets B such that* $P(\xi(-h) \in B) \lesssim \alpha_h$.

PROOF. (This is similar to Theorem 2.1 in de Maré (1980), but the interpretation is slightly different.) Let Γ_h be defined by (14). As in the proof of the Neyman-Pearson Lemma, for any $B \in \mathscr{F}$ such that

$$P(\xi(-h) \in B) \le P(\xi(-h) \in \Gamma_h)$$

one has

$$P^0(\xi(-h) \in B \mid C_0^{(u)}) = \int_B \frac{p_h^{(u)}(x)}{p(x)}\, dP_{\xi(-h)}(x) = \int_{B\cap\Gamma_h} + \int_{B\cap\Gamma_h^c} \le \int_{B\cap\Gamma_h} + \int_{B^c\cap\Gamma_h}$$
$$= \int_{\Gamma_h} \frac{p_h^{(u)}(x)}{p(x)}\, dP_{\xi(-h)}(x) = p^0(\xi(-h) \in \Gamma_h \mid C_0^{(u)}),$$

since

$$\int_{B\cap\Gamma_h^c} dP_{\xi(-h)}(x) \le \int_{B^c\cap\Gamma_h} dP_{\xi(-h)}(x)$$

and $p_h^{(u)}(x)/p(x) \ge k_h$ on Γ_h, $p_h^{(u)}(x)/p(x) < k_h$ on Γ_h^c. □

To express the optimal alarm region Γ_h in more explicit form we have to introduce a further conditional upcrossing intensity,

$\mu_h(x)$ = the conditional u-upcrossing intensity for $\eta(t+h)$ given $\xi(t)=x$.

In continuous time this is a function of h such that, for any interval I,

$$E(\#\{u\text{-upcrossing by } \eta(t+h),\ h \in I\} \mid \xi(t)=x) = \int_{h\in I} \mu_h(x)\, dh, \tag{15}$$

(cf. (13)), while in discrete time $\mu_h(x)$ is a simple conditional probability. Note that $E(\bullet \mid \xi(t) = x)$ is a regular conditional expectation, and that, e.g., the unconditional upcrossing intensity is

$$\mu = \int_{R^p} \mu_h(x) p(x)\, dx.$$

One should not confound $\mu_h(x)$ with the specific risk $\mu_h^{(\Gamma)}(x)$, which is the upcrossing intensity given that $\xi(t)$ enters Γ at x.

The conditional intensity $\mu_h(x)$ can be expressed explicitly as in the following lemma, the proof of which is standard crossing theory; see, e.g., Leadbetter et al. (1983), Chapter 7.

LEMMA 4.2. *If $\xi(-h)$, $\eta(0)$, $\eta'(0)$ have a nonsingular distribution, then* (*cf.* (4))

$$\begin{aligned}\mu_h(x) &= f_{\eta(0)\mid\xi(-h)=x}(u) E(\eta'(0)^+ \mid \xi(-h) = x,\ \eta(0) = u)\\ &= f_{\eta(0)\mid\xi(-h)=x}(u) \int_{z=0}^{\infty} z f_{\eta'(0)\mid\xi(-h)=x,\eta(0)=u}(z)\, dz.\end{aligned} \tag{16}$$

In discrete time

$$\mu_h(x) = P(\eta(-1) < u \le \eta(0) \mid \xi(-h) = x).$$

The optimal alarm region consists of precisely those x for which $\mu_h(x)$ takes its largest values.

THEOREM 4.3. *The optimal alarm region Γ_h of size α_h for alarm with warning time h is given by*

$$\Gamma_h = \{x \in R^p;\ \mu_h(x) \ge k_h \mu\},$$

when the constant k_h is such that $P(\xi(-h) \in \Gamma_h) = \alpha_h$.

PROOF. By Lemma 4.1 the optimal region consists of those x for which

$$p_h^{(u)}(x)/p(x) \ge k_h.$$

Here $p(x) = f_{\xi(-h)}(x)$, and under the present conditions, by Lemma 2.1,

$$p_h^{(u)}(x) = \mu^{-1} \int_{z=0}^{\infty} z f_{\eta(0),\eta'(0)}(u, z) f_{\xi(-h)\mid\eta(0)=u,\eta'(0)=z}(x)\, dz.$$

By changing the order of conditioning this is seen to be equal to

$$\mu^{-1} f_{\xi(-h)}(x) f_{\eta(0)\mid\xi(-h)=x}(u) \int_{z=0}^{\infty} z f_{\eta'(0)\mid\xi(-h)=x,\eta(0)=u}(z)\, dz,$$

which in turn, by Lemma 4.2, is equal to

$$\mu^{-1} f_{\xi(-h)}(x) \mu_h(x).$$

Hence

$$p_h^{(u)}(x)/p(x) = \mu^{-1}\mu_h(x)$$

in the continuous case.

Since the same relation holds in the discrete case (as is easily seen by integrating both sides over any $B \in \mathscr{F}$) we can use Lemma 4.1 to get the statement of the theorem. □

REMARK. The normal assumption is not essential in Theorem 4.3, and the same optimal alarm region applies for all processes for which the conditional upcrossing intensity is given by (16).

We shall now investigate the conditional crossing intensity $\mu_h(x)$ further to get an explicit and intuitively appealing rule for when to give alarm, defined in terms of the mean square predictor

$$\hat{\eta}_t(t+h) = E(\eta(t+h) \mid \xi(t))$$

and the (conditional) expected growth rate

$$\hat{\zeta}_t(t+h) = E(\eta'(t+h) \mid \xi(t), \eta(t+h) = u).$$

To be precise, by $E(\eta'(t+h) \mid \xi(t), \eta(t+h) = u)$ we mean the random variable $g_h(\xi(t), u)$ where $g_h(\xi(t), \eta(t+h)) = E(\eta'(t+h) \mid \xi(t), \eta(t+h))$.

From now on, we concentrate on the continuous case; discrete time prediction, although conceptually simpler, is hindered by heavier notation.

Let, for the rest of the section, time be continuous and $\eta(t)$ and $\xi(t) \in R^p$ be continuously differentiable, zero mean stationary Gaussian processes with $V(\eta(0)) = 1$, $V(\eta'(0)) = \lambda_2$ and let the covariance matrix of $\eta(0)$, $\eta'(0)$, $\xi(t) = (\xi_1(t), \cdots, \xi_p(t))^T$ be

$$\Sigma_t = \begin{pmatrix} 1 & 0 & \Sigma_{13}(t) \\ 0 & \lambda_2 & \Sigma_{23}(t) \\ \Sigma_{31}(t) & \Sigma_{32}(t) & \Sigma_{33} \end{pmatrix}$$

where

$$\Sigma_{13}(t) = \Sigma_{31}(t)^T = r_{\eta\xi}(t) = E(\eta(0)\xi(t)^T),$$

$$\Sigma_{23}(t) = \Sigma_{32}(t)^T = -r'_{\eta\xi}(t) = E(\eta'(0)\xi(t)^T),$$

$$\Sigma_{33} = r_{\xi\xi}(0) = E(\xi(0)\xi(0)^T).$$

As earlier we assume that $\eta(0)$, $\eta'(0)$, $\xi(t)$ have a nonsingular distribution, so that in particular Σ_{33} is invertible.

LEMMA 4.4. *The conditional normal distribution of*

(a) $\xi(t)$ *given* $\eta(0) = u$, $\eta'(0) = z$ *has mean*

$$m_{\xi\cdot\eta\eta'} = u\Sigma_{31}(t) + z\lambda_2^{-1}\Sigma_{32}(t)$$

and covariance matrix

$$\Lambda_t = \Sigma_{33} - \Sigma_{31}(t)\Sigma_{13}(t) - \lambda_2^{-1}\Sigma_{32}(t)\Sigma_{23}(t),$$

(b) $\eta(0)$ *given* $\xi(t) = x$ *has mean*

$$m_{\eta\cdot\xi} = \Sigma_{13}(t)\Sigma_{33}^{-1}x$$

and variance

$$\sigma_{\eta\cdot\xi}^2 = 1 - \Sigma_{13}(t)\Sigma_{33}^{-1}\Sigma_{31}(t),$$

(c) $\eta'(0)$ *given* $\xi(t) = x$, $\eta(0) = u$ *has mean*

$$m_{\eta'\cdot\xi\eta} = \sigma_{\eta'\cdot\xi\eta}^2\lambda_2^{-1}\Sigma_{23}(t)\Lambda_t^{-1}(x - u\Sigma_{31}(t))$$

and variance

$$\sigma_{\eta'\cdot\xi\eta}^2 = \frac{\lambda_2^2}{\lambda_2 + \Sigma_{23}(t)\Lambda_t^{-1}\Sigma_{32}(t)}.$$

PROOF. (a) and (b) are standard, while (c) perhaps needs a comment. The simplest way to obtain this form of conditional mean and variance is to split the joint density of $\eta(0)$, $\eta'(0)$, $\xi(t)$ into products of conditional normal densities,

$$f_{\eta\eta'\xi} = f_\eta f_{\eta'} f_{\xi|\eta\eta'} = f_\xi f_{\eta|\xi} f_{\eta'|\xi\eta}$$

and then identify coefficients in the exponents, i.e.,

$$(u\ z\ x^T)\Sigma_t^{-1}\begin{pmatrix} u \\ z \\ x \end{pmatrix} = u^2 + \lambda_2^{-1}z^2 + (x - m_{\xi\cdot\eta\eta'})^T\Lambda_t^{-1}(x - m_{\xi\cdot\eta\eta'})$$

$$= x^T\Sigma_{33}^{-1}x + \sigma_{n\cdot\xi}^{-2}(u - m_{\eta\cdot\xi})^2 + \sigma_{\eta'\cdot\xi\eta}^{-2}(z - m_{\eta'\cdot\xi\eta})^2. \quad \square$$

Now define the $p \times 1$ matrices

$$A_h = \Sigma_{13}(-h)\Sigma_{33}^{-1},$$

$$B_h = \sigma_{\eta'\cdot\xi\eta}^2\lambda_2^{-1}\Sigma_{23}(-h)\Lambda_{-h}^{-1},$$

and the constant

$$C_h = \sigma_{\eta'\cdot\xi\eta}^2\lambda_2^{-1}\Sigma_{23}(-h)\Lambda_{-h}^{-1}\Sigma_{31}(-h),$$

so that

$$m_{\eta\cdot\xi} = A_h x, \quad m_{\eta'\cdot\xi\eta} = B_h x - C_h u.$$

Thus the mean square predictor of $\eta(t + h)$ at time t based on $\xi(t)$ is

$$\hat{\eta}_t(t + h) = E(\eta(t + h) \mid \xi(t)) = A_h\xi(t), \tag{17}$$

while the conditional predictor of $\eta'(t + h)$ given data $\xi(t)$ and given the hypothetical event $\eta(t + h) = u$, is

$$\hat{\zeta}_t(t + h) = E(\eta'(t + h) \mid \xi(t), \eta(t + h) = u) = B_h\xi(t) - C_h u. \tag{18}$$

We shall occasionally use the summarizing notation

$$\Xi = \Xi(t) = \begin{pmatrix} \hat{\eta}_t(t+h) \\ \hat{\zeta}_t(t+h) \end{pmatrix} = \begin{pmatrix} A_h \\ B_h \end{pmatrix} \xi(t) - \begin{pmatrix} 0 \\ C_h u \end{pmatrix}$$

for the predicted value and the (conditionally) predicted derivative.

THEOREM 4.5. *If $\eta(0)$, $\eta'(0)$, $\xi(-h) \in R^p$ have a nonsingular distribution, the optimal alarm region of size α_h for u-upcrossing by $\eta(t+h)$, based on $\xi(t)$, is*

$$\Gamma_h = \left\{ x \in R^p; \left(\frac{u - A_h x}{\sigma_{\eta\cdot\xi}}\right)^2 - 2\log\Psi\left(\frac{B_h x - C_h u}{\sigma_{\eta'\cdot\xi\eta}}\right) - 2\log\frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} \le K_h \right\},$$

where K_h is such that $P(\xi(-h) \in \Gamma_h) = \alpha_h$ and $\Psi(x) = \phi(x) + x\Phi(x)$, ϕ and Φ being the standardized normal density and distribution functions.

PROOF. By Theorem 4.3 and Lemma 4.2 the optimal alarm region consists of those x for which

$$\mu^{-1}\mu_h(x) = \mu^{-1} f_{\eta(t+h)\mid\xi(t)=x}(u) E(\eta'(t+h)^+ \mid \xi(t) = x, \eta(t+h) = u) \ge k_h.$$

Here $\mu = (2\pi)^{-1}\sqrt{\lambda_2}\exp(-u^2/2)$, and since for any normal variable ζ with mean m and variance σ^2,

$$E(\zeta^+) = \sigma\Psi(m/\sigma),$$

we have

$$\mu_h(u) = \frac{1}{\sigma_{\eta\cdot\xi}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{u - A_h x}{\sigma_{\eta\cdot\xi}}\right)^2\right) \sigma_{\eta'\cdot\xi\eta} \Psi\left(\frac{B_h x - C_h u}{\sigma_{\eta'\cdot\xi\eta}}\right),$$

so that

$$2\log\mu^{-1}\mu_h(x) = u^2 - \left(\frac{u - A_h x}{\sigma_{\eta\cdot\xi}}\right)^2 + 2\log\Psi\left(\frac{B_h x - C_h u}{\sigma_{\eta'\cdot\xi\eta}}\right) + 2\log\frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} - c$$

where $c = \log(2\pi/\lambda_2)$ is independent of x and u, which proves the statement of the theorem. □

The optimal alarm region Γ_h is of course the same as that derived by de Maré (1980), Corollary 3.7, but it is written here in a slightly different form to allow easy comparison with the mean square predictor $\hat{\eta}_t(t+h)$.

COROLLARY 4.6. *The optimal alarm for a u-upcrossing, with warning time h, starts any time the mean square predictor $\hat{\eta}_t(t+h)$ exceeds the varying lower alarm level*

$$\hat{u} = u - \sigma_{\eta\cdot\xi}\left(2\log\Psi\left(\frac{\hat{\zeta}_t(t+h)}{\sigma_{\eta'\cdot\xi\eta}}\right) + 2\log\frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} + K_h\right)^{1/2}.$$

The alarm stops when the predictor returns below the level, meaning that the risk for an exceedance is low, or when it exceeds the upper level

$$u + \sigma_{\eta\cdot\xi}(2 \log \Psi(\hat{\zeta}_t(t+h)/\sigma_{\eta'\cdot\xi\eta}) + 2 \log(\sigma_{\eta'\cdot\xi\eta}/\sigma_{\eta\cdot\xi}) + K_h)^{1/2},$$

in which case the risk for an upcrossing is low but when one expects that $\eta(t+h)$ is well above u already.

The optimal crossing predictor depends only on $\hat{\eta}_t(t+h)$ and $\hat{\zeta}_t(t+h)$ and one can therefore always reduce data to these two variables and define the reduced optimal alarm region $\tilde{\Gamma}_h$ for $\Xi = (\hat{\eta}_t(t+h), \hat{\zeta}_t(t+h))^T$ by

$$(19)\quad \tilde{\Gamma}_h = \left\{(m, m') \in R^2; \left(\frac{u-m}{\sigma_{\eta\cdot\xi}}\right)^2 - 2 \log \Psi\left(\frac{m'}{\sigma_{\eta'\cdot\xi\eta}}\right) - 2 \log \frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} \le K_h\right\}.$$

5. Size, detection probability and risk for the optimal alarm. The optimal alarm depends only on two linear functions of available data and is therefore easy to analyse for normal processes. As shall now be shown, the probabilities α_h, γ_h and the total and specific risks $\mu_h^{(\Gamma)}$, $\mu_h^{(\Gamma)}(x)$ and $\beta_h^{(\Gamma)}$, $\beta_h^{(\Gamma)}(x)$ defined in Section 3, can all be expressed as integrals of simple functions, and easily evaluated numerically.

Recall the notation

$$\Xi = \Xi(t) = (\hat{\eta}_t(t+h), \hat{\zeta}_t(t+h))^T = \begin{pmatrix} A_h \\ B_h \end{pmatrix} \xi(t) - \begin{pmatrix} 0 \\ C_h u \end{pmatrix}$$

introduced in Section 4 and write for short $H = (\eta(t+h), \eta'(t+h))^T$; think of Ξ as an estimator of H. Obviously, Ξ and H are jointly normal and we use the following notations for means and covariance matrices (unconditional and conditional):

$$m_\Xi = E(\Xi), \quad \Sigma_\Xi = \mathrm{Cov}(\Xi),$$

and

$$m_{\Xi\cdot H}(z) = E(\Xi \mid \eta(t+h) = u, \eta'(t+h) = z),$$
$$\Sigma_{\Xi\cdot H} = \mathrm{Cov}(\Xi \mid \eta(t+h), \eta'(t+h)).$$

The size α_h of the optimal alarm with reduced alarm region $\tilde{\Gamma}_h$, defined by (19), is

$$\alpha_h = P(\Xi \in \tilde{\Gamma}_h).$$

For any bivariate normal distribution of random variables X_1, X_2 with mean

$m = (m_1, m_2)^T$ and covariance matrix Σ (with $V(X_i) = \sigma_i^2$), define

$$\begin{aligned} \alpha_K(m, \Sigma) & \\ &= P\left(\left(\frac{u - X_1}{\sigma_{\eta\cdot\xi}}\right)^2 \le 2\log\Psi\left(\frac{X_2}{\sigma_{\eta'\cdot\xi\eta}}\right) + 2\log\frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} + K\right) \\ &= \int_{x_2=\sigma_{\eta'\cdot\xi\eta}\Psi^{-1}(e^{K/2})}^{\infty} \frac{1}{\sigma_2}\phi\left(\frac{x_2 - m_2}{\sigma_2}\right) \\ &\quad \cdot P\left(\left(\frac{u - X_1}{\sigma_{\eta\cdot\xi}}\right)^2 \le 2\log\Psi\left(\frac{x_2}{\sigma_{\eta'\cdot\xi\eta}}\right) + 2\log\frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} + K \;\middle|\; X_2 = x_2\right) dx_2, \end{aligned} \tag{20}$$

where Ψ^{-1} is the inverse function of Ψ.

THEOREM 5.1. *The optimal alarm region with constant K_h in $\tilde{\Gamma}_h$ has size*

$$\alpha_h = \alpha_{K_h}(m_\Xi, \Sigma_\Xi)$$

and detection probability

$$\gamma_h = \int_{z=0}^{\infty} \frac{z}{\lambda_2} \exp(-z^2/2\lambda_2)\alpha_{K_h}(m_{\Xi\cdot H}(z), \Sigma_{\Xi\cdot H})\, dz,$$

where $\alpha_K(m, \Sigma)$ is defined by (20) *and Ξ is $N(m_\Xi, \Sigma_\Xi)$ and $\Xi \mid H = (u, z)^T$ is $N(m_{\Xi\cdot H}(z), \Sigma_{\Xi\cdot H})$.*

PROOF. The statement for α_h is just the definition (20), while for γ_h we have to note that the derivative $\eta'(t + h)$ at a u-upcrossing ($\eta(t + h) = u$) has a Rayleigh distribution, so that

$$\gamma_h = \int_{z=0}^{\infty} \frac{z}{\lambda_2} \exp(-z^2/2\lambda_2)P(\Xi \in \tilde{\Gamma}_h \mid \eta(t + h) = u, \eta'(t + h) = z)\, dz,$$

and hence the result follows. □

This theorem makes it possible to calculate α_h and γ_h numerically as functions of K and hence select a suitable combination of size and detection ability; for an example, see Section 7.

We now turn to the more complicated specific and total risks, defined in Section 3, which describe the exceedance and crossing risks under the condition that an alarm has been given for some specific warning time h_0, i.e., that $\xi(t)$ has entered Γ_{h_0}, or equivalently that $\Xi(t) = ((\hat{\eta}_t + h_0), \hat{\zeta}_t(t + h_0))^T$ has entered $\tilde{\Gamma}_{h_0}$ at time t.

Thus we fix the warning time $h_0 > 0$, the alarm region $\tilde{\Gamma}_{h_0}$, and consider alarms given by the alarm process $\Xi(t) = (\hat{\eta}_t(t + h), \hat{\zeta}_t(t + h))^T$. We shall then describe the conditional behaviour of $\eta(t + h)$ for various values of h after the alarm. The conditional (Palm) distributions of $\eta(t + h)$ then depend not only on where Ξ

enters $\tilde{\Gamma}_{h_0}$, but also on the derivative

$$\Xi' = \Xi'(t) = (\hat{\eta}_t'(t+h), \hat{\zeta}_t'(t+h))^T = \begin{pmatrix} A_{h_0} \\ B_{h_0} \end{pmatrix} \xi'(t).$$

Write $y = (y_1, y_2)^T$, $z = (z_1, z_2)^T$ and let

$$\mu_h(y, z) = f_{\eta(t+h)|\Xi=y}(u) E(\eta'(t+h)^+ \mid \Xi = y, \Xi' = z, \eta(t+h) = u) \tag{21}$$

be the conditional upcrossing intensity for $\eta(t+h)$, $h > 0$, given $\Xi = y$, $\Xi' = z$.

Since all involved variables are normal, the conditional distributions in (21) are also normal. We shall use the following notation for conditional means, covariance and variances,

$$\begin{cases} m_{\Xi'\cdot\Xi}(y) = E(\Xi' \mid \Xi = y), \\ \Sigma_{\Xi'\cdot\Xi} = \operatorname{Cov}(\Xi' \mid \Xi), \end{cases}$$

$$\begin{cases} m_{\eta\cdot\Xi\Xi'}(y, z) = E(\eta(t+h) \mid \Xi = y, \Xi' = z), \\ \sigma^2_{\eta\cdot\Xi\Xi'} = V(\eta(t+h) \mid \Xi, \Xi'), \end{cases}$$

$$\begin{cases} m_{\eta'\cdot\Xi\Xi'\eta}(y, z) = E(\eta'(t+h) \mid \Xi = y, \Xi' = z, \eta(t+h) = u), \\ \sigma^2_{\eta'\cdot\Xi\Xi'\eta} = V(\eta'(t+h) \mid \Xi, \Xi', \eta(t+h)), \end{cases}$$

which are easily obtained from the covariance matrix of $\eta(t+h)$, $\eta'(t+h)$, $\xi(t)$, $\xi'(t)$.

Note that the conditional upcrossing intensity (21) then is

$$\mu_h(y, z) = \frac{1}{\sigma_{\eta\cdot\Xi\Xi'}} \phi\left(\frac{m_{\eta\cdot\Xi\Xi'}(y, z)}{\sigma_{\eta\cdot\Xi\Xi'}}\right) \cdot \sigma_{\eta'\cdot\Xi\Xi'\eta} \Psi\left(\frac{m_{\eta'\cdot\Xi\Xi'\eta}(y, z)}{\sigma_{\eta'\cdot\Xi\Xi'\eta}}\right). \tag{22}$$

The next step in the derivation of the risk functions is to describe the distribution of $\Xi = (\hat{\eta}_t(t+h_0), \hat{\zeta}_t(t+h_0))^T$ over the boundary $\partial\tilde{\Gamma}_{h_0}$ and of the derivative Ξ' at the entrance points of the alarm region $\tilde{\Gamma}_{h_0}$ for the fixed warning time h_0.

By Lemma 3.1(b) we then need the unit normal n_y of $\partial\tilde{\Gamma}_{h_0}$ at the entrance point y. Since $\partial\tilde{\Gamma}_{h_0}$ is defined as a level curve of the function

$$\chi(y_1, y_2) = \left(\frac{u - y_1}{\sigma_{\eta\cdot\xi}}\right)^2 - 2\log\Psi\left(\frac{y_2}{\sigma_{\eta'\cdot\xi\eta}}\right) - 2\log\frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} - K$$

we have (depending on h_0)

$$n_y = n_y^{(\Gamma_{h_0})} = -\left(\frac{\partial\chi}{\partial y_1}, \frac{\partial\chi}{\partial y_2}\right) \Big/ \left(\left(\frac{\partial\chi}{\partial y_1}\right)^2 + \left(\frac{\partial\chi}{\partial y_2}\right)^2\right)^{1/2}, \tag{23}$$

where

$$\frac{\partial\chi}{\partial y_1} = -\frac{2}{\sigma_{\eta\cdot\xi}}\left(\frac{u - y_1}{\sigma_{\eta\cdot\xi}}\right), \quad \frac{\partial\chi}{\partial y_2} = -\frac{2}{\sigma_{\eta'\cdot\xi\eta}}\frac{\Phi}{\Psi}\left(\frac{y_2}{\sigma_{\eta'\cdot\xi\eta}}\right).$$

LEMMA 5.2. *The long run conditional distribution of $\Xi'(s_k)$ at the alarm points s_k where $\Xi(s_k) = y \in \partial\tilde{\Gamma}_{h_0}$ has density*

$$f^0_{\Xi'|\Xi=y}(z) = c_y^{-1}(n_y \cdot z)^+ f_{\Xi'|\Xi=y}(z) \tag{24}$$

where $f_{\Xi'|\Xi=y}$ is normal and specified above and the normalizing constant

$$c_y = E((n_y \cdot \Xi')^+ \mid \Xi = y) = \tilde{\sigma}(y)\Psi(\tilde{m}(y)/\tilde{\sigma}(y))$$

with

$$\tilde{m}(y) = n_y \cdot m_{\Xi'\cdot\Xi}(y), \quad \tilde{\sigma}^2(y) = n_y \Sigma_{\Xi'\cdot\Xi} n_y^T.$$

PROOF. Since $\Xi(t)$ is a stationary bivariate Gaussian process we can apply Lemma 3.1 to find the distribution of the values of $\Xi(s_k)$ and $\Xi'(s_k)$ at the points of entrance into $\tilde{\Gamma}_{h_0}$ as

$$f^0_{\Xi}(y) f^0_{\Xi'|\Xi=y}(z) = c^{-1}(n_y \cdot z)^+ f_{\Xi}(x) f_{\Xi'|\Xi=y}(z)$$

and hence the expression (24).

Since the conditional distribution of $n_y \cdot \Xi'$, given $\Xi = y$, is normal with mean $\tilde{m}(y)$ and variance $\tilde{\sigma}^2(y)$ we get the result. □

THEOREM 5.3. *The optimal alarm region $\Gamma_0 = \Gamma_{h_0}$ has specific upcrossing risk*

$$\mu_h^{(\Gamma_0)}(x) = c_y^{-1} \int\int_{z\in R^2} (n_y^{(\Gamma_0)} \cdot z)^+ f_{\Xi'|\Xi=y}(z)\mu_h(y, z)\, dz$$

where $y = (A_{h_0}x, B_{h_0}x - C_{h_0}u)^T$, and $\mu_h(y, z)$ is given by (22). The specific exceedance risk is

$$\beta_h^{(\Gamma_0)}(x) = c_y^{-1} \int\int_{z\in R^2} (n_y^{(\Gamma_0)} \cdot z)^+ f_{\Xi'|\Xi=y}(z) \cdot \left(1 - \Phi\left(\frac{u - m_{\eta\cdot\Xi\Xi'}(y, z)}{\sigma_{\eta\cdot\Xi\Xi'}}\right)\right) dz$$

where $\eta(t+h) \mid \Xi = y, \Xi' = z$ is $N(m_{\eta\cdot\Xi\Xi'}(y, z), \sigma^2_{\eta\cdot\Xi\Xi'})$.

The total upcrossing risk is

$$\mu_h^{(\Gamma_0)} = c^{-1} \int_{y\in\partial\tilde{\Gamma}_h} \int_{z\in R^2} f_{\Xi}(y)(n_y^{(\Gamma_0)} \cdot z)^+ f_{\Xi'|\Xi=y}(z) \cdot \mu_h(y, z)\, dz\, ds(y)$$

and similarly for $\beta_h^{(\Gamma_0)}$; here $ds(y)$ denotes integration along the curve $\partial\tilde{\Gamma}_{h_0}$ and the normalizing constant c is defined in Lemma 3.1.

6. Balance between alarm size and detection probability. As mentioned in Section 2 the total alarm time α_h must be balanced against what can be obtained in detection probability γ_h. The likelihood ratio method in Section 4 assures that for given γ_h one gets the smallest possible α_h, and what remains is to decide what is an acceptable alarm time and what detection probability should be strived for. This is a complicated task and here shall only be mentioned a few components that ought to be embodied in a complete theory. To get a compelling terminology we shall use the word "catastrophe" to denote an upcrossing of a level u by the process $\eta(t)$.

The *alarm costs* consist of monetary, human, and other expenditures in order to reduce losses from an expected catastrophe. They should include possible costs for standstill in production and psychological losses caused by false alarms, perhaps leading to less confidence in the alarm system.

All these costs can depend on the warning time h and on the vagueness of the prediction, e.g., measured in terms of the width of the interval H_t specifying the set of likely catastrophe times; see Figures 1 and 2 in Section 7.

The *catastrophe costs* are similarly monetary, human, cultural and other losses due to catastrophe. They depend on whether or not any alarm was given and on the warning time and vagueness of the alarm.

We shall here see how simple consideration of alarm and catastrophe costs can be of some help when selecting a reasonable balance between α_h and γ_h. The cost structure is overly simplified and the result should only be used as illustration of the influence of cost consideration.

Suppose one can specify a cost per time unit $a(h)$ to maintain an alarm with warning time h. The expected cost of h-alarms per time unit is then $a(h)\alpha_h$, and the total expected alarm cost

$$\int_0^\infty a(h)\alpha_h \, dh. \tag{25}$$

Further, suppose the total cost of a catastrophe which was alarmed exactly h time units ahead, causes a loss of the amount $G(h)$ and write

$$G(h) = G(\infty) + \int_h^\infty g(x)\, dx.$$

Here $G(\infty)$ is the inevitable loss which is independent of alarms, while $g(h)$ represents the reduction of losses attained by an alarm given a time h ahead. The total loss caused by a catastrophe at time 0 is then

$$G(\infty) + \int_0^\infty g(h) I\{\xi(-h) \notin \Gamma_h\}\, dh$$

and the expected loss per catastrophe is

$$G(\infty) + \int_0^\infty g(h)(1 - \gamma_h)\, dh. \tag{26}$$

Since there are on the average μ catastrophes per time unit, for each warning time h the average alarm cost minus the average savings is

$$a(h)\alpha_h - \mu g(h)\gamma_h, \tag{27}$$

and this can be minimized for each h by a correct choice of α_h and γ_h. The total expected cost per time unit, for all possible warning times is

$$\int_0^\infty a(h)\alpha_h\, dh + \mu G(\infty) + \mu \int_0^\infty g(h)(1 - \gamma_h)\, dh$$
$$= \mu\left(G(\infty) + \int_0^\infty g(h)\, dh\right) + \int_0^\infty (a(h)\alpha_h - \mu g(h)\gamma_h)\, dh.$$

THEOREM 6.1. *If the expected alarm cost per time unit is*

$$\int_0^\infty a(h)\alpha_h \, dh$$

and the expected loss per catastrophe

$$G(\infty) + \int_0^\infty g(h)(1 - \gamma_h) \, dh,$$

then the optimal alarm region Γ_h, *giving the long run minimal costs, is defined by*

$$\Gamma_h = \{x \in R^p; \ \mu_h(x) \geq a(h)/g(h)\}$$

where the conditional upcrossing intensity $\mu_h(x)$ *is defined by* (16).

PROOF. Recalling the notation $p(x)$ and $p_h^{(u)}(x)$ for the unconditional and conditional densities of $\xi(t)$ given a u-upcrossing $\eta(t + h) = u$, the alarm dependent cost term (27) at warning time h is

$$a(h)\alpha_h - \mu g(h)\gamma_h = \int_{x \in \Gamma} \{a(h)p(x) - \mu g(h)p_h^{(u)}(x)\} \, dx.$$

This is obviously minimized when

$$\Gamma = \{z \in R^p; \ a(h)p(x) - \mu g(h)p_h^{(u)}(x) \leq 0\}$$

and since $p_h^{(u)}(x) = \mu^{-1}p(x)\mu_h(x)$ we have found the optimal alarm region

$$\Gamma_h = \{x \in R^p; p(x)(a(h) - g(h)\mu_h(x)) \leq 0\}$$

proving the result. □

Of the two cost structures for alarms and catastrophe, respectively, (26) is probably the most realistic one, even if it does not account for the positive effects of displaced alarms such as those in Figure 2 in Section 7. In any case, it could well be possible in practice to specify the function $g(h)$. The form (25) is more questionable, since it does not consider the number of alarms, nor whether the alarm has been in effect for a long time or not. It might be a suitable structure when alarm costs mainly consist of loss of production due to standstill.

7. An example. To illustrate the characteristic features of the optimal level crossing predictor and to compare it with another natural alarm criterion we have simulated a stationary normal process $\eta(t)$ as the solution of

$$\eta'''(t) + a_2\eta''(t) + a_1\eta'(t) + a_0\eta(t) = \sigma W'(t)$$

where $W'(t)$ is white noise, $E((dW(t))^2) = dt$. With $a_0 = 0.034$, $a_1 = 0.210$, $a_2 = 0.400$, and σ^2 chosen so that $V(\eta(t)) = 1$ we have $\lambda_2 = 0.085$.

The process $\eta(t)$ was simulated exactly by sampling the vector $(\eta(t), \eta'(t), \eta''(t))$ at a small sampling distance; see Åström (1970), Section 3.10.

The predictor was based on five values of $\eta(\bullet)$, a distance 2 apart,

$\xi(t) = (\eta(t), \eta(t-2), \cdots, \eta(t-8))^T$, continuously updated, and to obtain a sufficient number of upcrossings and alarms in the illustrations the level u was chosen as low as $u = 2$.

At each simulated point t the mean square predictors

$$\hat{\eta}_t(t+h) = E(\eta(t+h) \mid \xi(t)) = A_h \xi(t),$$

$$\hat{\zeta}_t(t+h) = E(\eta'(t+h) \mid \xi(t), \eta(t+h) = u) = B_h \xi(t) - C_h u$$

were calculated for h between 0.5 and 10, in steps of 0.5. The obtained residual standard deviations $\sigma_{\eta\cdot\xi}$ and $\sigma_{\eta'\cdot\xi\eta}$, given in Lemma 4.4, can be found in Table 1.

For each simulation step t the optimal alarm boundaries $u_{\text{lower}}^{\text{opt}}$ and $u_{\text{upper}}^{\text{opt}}$ from Corollary 4.6, i.e., (with $K_h = 6$)

$$\left.\begin{matrix} u_{\text{upper}}^{\text{opt}} \\ u_{\text{lower}}^{\text{opt}} \end{matrix}\right\} = u \pm \sigma_{\eta\cdot\xi}\left(2 \log \Psi\left(\frac{\hat{\zeta}_t(t+h)}{\sigma_{\eta'\cdot\xi\eta}}\right) + 2 \log \frac{\sigma_{\eta'\cdot\xi\eta}}{\sigma_{\eta\cdot\xi}} + 6\right)^{1/2}, \tag{28}$$

were calculated, and if $u_{\text{lower}}^{\text{opt}} < \hat{\eta}_t(t+h) < u_{\text{upper}}^{\text{opt}}$ an alarm was set for an upcrossing at $t + h$.

To see how an alarm developes and changes with time, the points

$$\{(t+h, h);\ \text{alarm at } t \text{ for upcrossing at } t+h\} = \{(t+h, h);\ \xi(t) \in \Gamma_h\}$$

were plotted in a separate diagram; see Figure 1 which also shows $\eta(t)$ for $0 \le t \le 100$.

As was shown in Theorem 4.5 the optimal alarm region Γ_h^{opt} has the shortest total alarm time

$$\alpha_h = P(\xi(t) \in \Gamma_h)$$

of all alarm regions Γ_h with the same detection probability

$$\gamma_h = \gamma_h^{\text{opt}} = P(\xi(t) \in \Gamma_h^{\text{opt}} \mid \eta(t+h) = u, \text{upcrossing}).$$

To see how this fact is reflected in the *timing* of the alarms we have also

TABLE 1
Residual standard deviations when predicting $\eta(t+h)$ and $\eta'(t)$ from $\xi(t) = (\eta(t), \eta(t-2), \cdots, \eta(t-8))^T$ (and conditioned on $\eta(t+h) = u$)

h	$\sigma_{\eta\cdot\xi}$	$\sigma_{\eta'\cdot\xi\eta}$	h	$\sigma_{\eta\cdot\xi}$	$\sigma_{\eta'\cdot\xi\eta}$
0.5	.035	.012	5.5	.772	.157
1.0	.085	.026	6.0	.826	.172
1.5	.149	.040	6.5	.871	.186
2.0	.222	.054	7.0	.906	.200
2.5	.303	.068	7.5	.933	.214
3.0	.388	.083	8.0	.952	.227
3.5	.473	.097	8.5	.964	.238
4.0	.557	.112	9.0	.970	.249
4.5	.636	.127	9.5	.973	.258
5.0	.708	.142	10.0	.974	.266

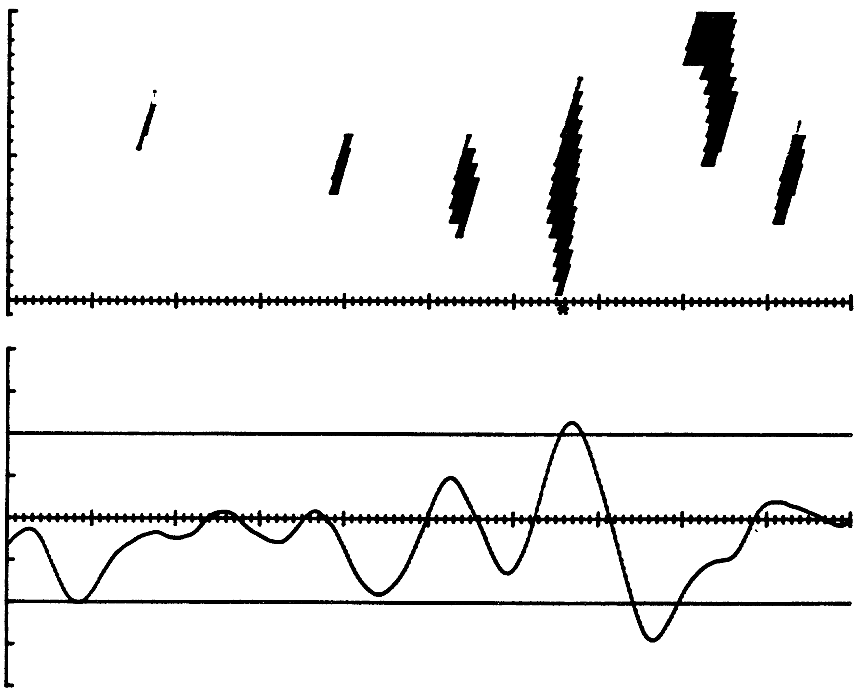

FIG. 1. (a) *Process* $\eta(t)$, $0 \le t \le 100$ *and levels* $u = \pm 2$. (b) *Sets* $\{(t+h, h);\ \xi(t) \in \Gamma_h\}$ *showing where alarm was given by the optimal alarm* (28); $*$ *indicates the occurred u-upcrossing.*

TABLE 2
Alarm size and detection probability for the optimal alarm levels (28) *and the naive levels* (29); $K_h = 6$ *and* $\lambda_\alpha = 1.64$

h	α_h^{opt}	γ_h^{opt}	α_h^{naive}	γ_h^{naive}
0.5	.005	.998	.006	.876
1.0	.014	.992	.015	.840
1.5	.027	.979	.026	.790
2.0	.043	.953	.039	.724
2.5	.060	.911	.053	.648
3.0	.074	.849	.067	.567
3.5	.086	.765	.082	.486
4.0	.093	.660	.095	.409
4.5	.089	.540	.107	.340
5.0	.079	.418	.117	.280
5.5	.064	.307	.124	.231
6.0	.049	.218	.126	.190
6.5	.037	.155	.122	.158
7.0	.028	.115	.112	.131
7.5	.023	.094	.096	.110
8.0	.021	.085	.077	.093
8.5	.021	.083	.058	.083
9.0	.021	.082	.046	.078
9.5	.020	.079	.039	.080
10.0	.018	.072	.037	.085

considered the naive alarm levels

$$\left.\begin{matrix} u_{\text{upper}} \\ u_{\text{lower}} \end{matrix}\right\} = u \pm \lambda_\alpha \sigma_{\eta\cdot\xi}, \tag{29}$$

where $1 - \Phi(\lambda_\alpha) = \alpha$. Since $\sigma^2_{\eta\cdot\xi}$ is the residual variance of $\eta(t+h)$ given $\hat{\eta}_t(t+h)$ we have that if $\hat{\eta}_t(t+h) > u_{\text{lower}}$ then

$$\begin{aligned} &P(\eta(t+h) > u \mid \hat{\eta}_t(t+h)) \\ &\quad = 1 - \Phi\left(\frac{u - \hat{\eta}_t(t+h)}{\sigma_{\eta\cdot\xi}}\right) \geq 1 - \Phi\left(\frac{u - u_{\text{lower}}}{\sigma_{\eta\cdot\xi}}\right) = 1 - \Phi(\lambda_\alpha) = \alpha, \end{aligned}$$

meaning that the alarm is set as soon as the (regular) conditional probability of an exceedance is at least α.

Since $\hat{\eta}_t(t+h)$ is normal with mean zero and variance $\sigma^2_{\hat{\eta}}$ the total alarm time for the naive alarm region is

$$\alpha_h = P(u_{\text{lower}} < \hat{\eta}_t(t+h) < u_{\text{upper}}) = \Phi(u_{\text{upper}}/\sigma_{\hat{\eta}}) - \Phi(u_{\text{lower}}/\sigma_{\hat{\eta}}).$$

Further, the conditional distribution of $\hat{\eta}_t(t+h)$ given $\eta(t+h) = u$, $\eta'(t+h) = z$ is normal with mean $m_{\hat{\eta}\cdot\eta\eta'}(z)$ and variance $\sigma^2_{\hat{\eta}\cdot\eta\eta'}$ and we can get

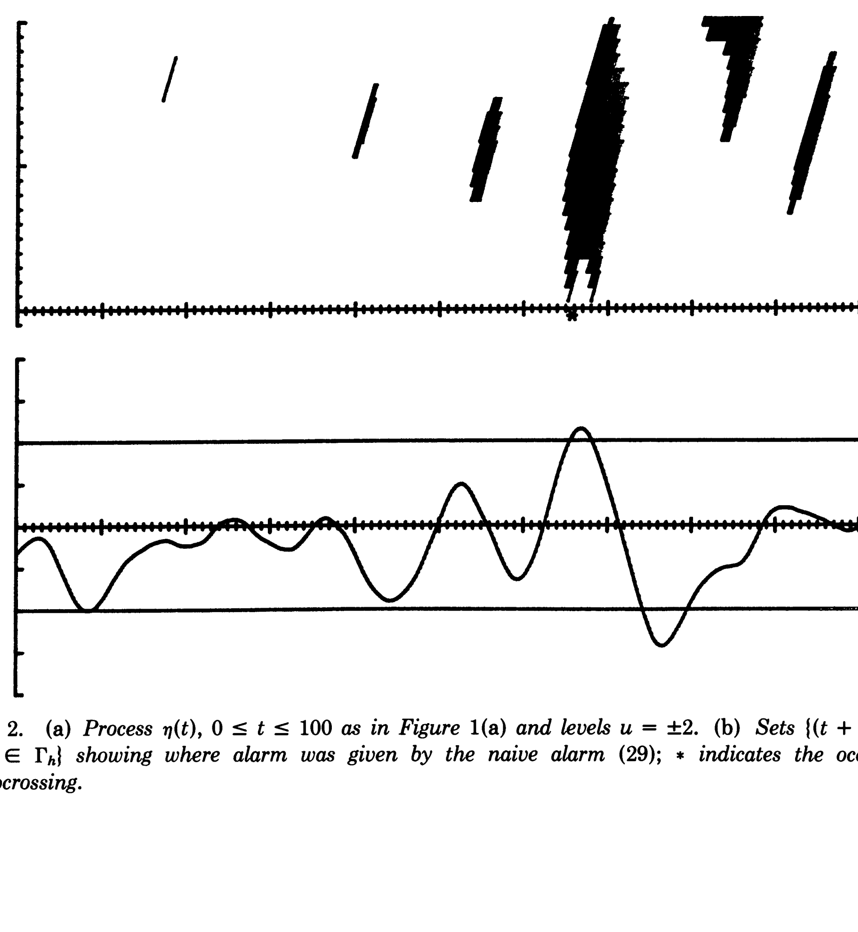

FIG. 2. (a) *Process* $\eta(t)$, $0 \leq t \leq 100$ *as in Figure* 1(a) *and levels* $u = \pm 2$. (b) *Sets* $\{(t+h, h);$ $\xi(t) \in \Gamma_h\}$ *showing where alarm was given by the naive alarm* (29); $*$ *indicates the occurred* u*-upcrossing.*

the detection probability

$$\gamma_h = \int_0^\infty \frac{z}{\lambda_2} \exp(-z^2/2\lambda_2) P(u_{\text{lower}} < \hat{\eta}_t(t+h) < u_{\text{upper}} \mid \eta(t+h) = u, \eta'(t+h) = z)\, dz,$$

where the probability is given by the normal distribution.

The quantities α_h^{opt}, γ_h^{opt} for the optimal alarm region, and α_h, γ_h for the naive region defined by (29) are given in Table 2 for the special case $K_h = 6$ illustrated in Figure 1 and with $\lambda_\alpha = 1.64$.

For comparison with the optimal alarms in Figure 1 we have plotted the resulting alarms for the naive predictor in Figure 2, showing the set $\{(t+h, h); \xi(t) \in \Gamma_h\}$ for the same section of the process as in Figure 1. As is seen the optimal alarm locates the upcrossing correctly in time at an earlier stage, while the naive predictor has a tendency to give later alarms. Also note from Table 2 that for this choice of parameters, the naive predictor either has much smaller probability of being in the alarm state at the proper time, or spends much longer total time there.

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