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RECURRENCE CLASSIFICATION AND INVARIANT MEASURE FOR REFLECTED BROWNIAN MOTION IN A WEDGE

BY R. J. WILLIAMS

University of California at San Diego

The object of study in this paper is reflected Brownian motion in a two-dimensional wedge with constant direction of reflection on each side of the wedge. The following questions are considered. Is the process recurrent? If it is recurrent, what is its invariant measure? Let ξ be the angle of the wedge $(0 < \xi < 2\pi)$ and let θ_1 and θ_2 be the angles of reflection on the two sides of the wedge, measured from the inward normals towards the directions of reflection, with positive angles being toward the corner $(-\pi/2 < \theta_1, \theta_2 < \pi/2)$. Set $\alpha = (\theta_1 + \theta_2)/\xi$.

Varadhan and Williams (1985) have shown that the process exists and is unique, in the sense that it solves a certain submartingale problem, when $\alpha < 2$. It is shown here that if $\alpha < 0$, the process is transient (to infinity). If $0 \le \alpha < 2$, the process is shown to be (finely) recurrent and to have a unique (up to a scalar multiple) σ-finite invariant measure. It is further proved that the density for this invariant measure is given in polar coordinates by $p(r, \theta) = r^{-\alpha}\cos(\alpha\theta - \theta_1)$.

1. Introduction. In [7], Varadhan and Williams resolved the question of the existence and uniqueness of a strong Markov process with continuous sample paths that loosely speaking has the following three properties.

(1.1) The state space is an infinite two-dimensional wedge, and the process behaves in the interior of the wedge like ordinary Brownian motion.

(1.2) The process reflects instantaneously at the boundary of the wedge, the direction of reflection being constant along each side.

(1.3) The amount of time that the process spends at the corner of the wedge is zero (in the sense of Lebesgue measure).

Under those conditions for which the process exists and is unique, the following questions are answered in this paper. Is the process recurrent? If it is recurrent, what is its invariant measure?

A summary of the pertinent results in [7] (including the precise mathematical characterization of the process), and of the main results of this paper, is given below. For this, the following notation is needed.

The wedge state space is given in polar coordinates by

$$S = \{(r, \theta): 0 \le \theta \le \xi,\ r \ge 0\},$$

where $\xi \in (0, 2\pi)$ is the angle of the wedge. The two sides of the wedge are

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denoted by $\partial S_1 = \{(r, \theta): \theta = 0, r \geq 0\}$ and $\partial S_2 = \{(r, \theta): \theta = \xi, r \geq 0\}$. The origin 0 is the corner of the wedge. The directions of reflection on the two sides of the wedge are specified by constant vectors v_1 and v_2, normalized such that for $j = 1, 2$, $v_j \cdot n_j = 1$, where n_j is the unit normal vector to ∂S_j that points into S. For each j, define the angle of reflection θ_j to be the angle between n_j and v_j, such that θ_j is positive if and only if v_j points towards the origin. Note that $-\pi/2 < \theta_j < \pi/2$. Define $\alpha = (\theta_1 + \theta_2)/\xi$.

Let C_S denote the space of continuous functions w: $[0, \infty) \to S$. For each $t \geq 0$, let $\mathcal{M}_t \equiv \sigma\{w(s): 0 \leq s \leq t\}$, the σ-algebra of subsets of C_S generated by the coordinate maps $w \to w(s)$ for $0 \leq s \leq t$. Similarly, let $\mathcal{M} \equiv \sigma\{w(s): 0 \leq s < \infty\}$. For each $n \in \{0, 1, 2, \cdots\}$ and $F \subset \mathbb{R}^2$, let $C^n(F)$ denote the set of real-valued functions that are n-times continuously differentiable in some domain containing F. Let $C_b^n(F)$ denote the set of functions in $C^n(F)$ that together with their partial derivatives up to and including those of order n are bounded on F. If $n = 0$, the superscript n will be omitted. Define the differential operators

$$D_j = v_j \cdot \nabla, \quad \text{for} \quad j = 1, 2,$$

and let Δ be the Laplacian operator.

The precise mathematical formulation of the question of existence and uniqueness of a process that heuristically satisfies (1.1)–(1.3) is in terms of a submartingale problem. Given $x \in S$, a *solution of the submartingale problem starting from* x is a probability measure P_x on $(C_S, \mathcal{M})$ that satisfies (i)–(iii) below.

(i) $P_x(w(0) = x) = 1$.

(ii) $E^{P_x}(\int_0^\infty 1_{\{0\}}(w(s))\, ds) = 0$.

(iii) For each $f \in C_b^2(S)$,

$$f(w(t)) - \frac{1}{2}\int_0^t \Delta f(w(s))\, ds \tag{1.4}$$

is a P_x-submartingale on $(C_S, \mathcal{M}, \{\mathcal{M}_t\})$, whenever f is constant in a neighborhood of the origin and satisfies

$$D_j f \geq 0 \quad \text{on } \partial S_j, \quad \text{for} \quad j = 1, 2. \tag{1.5}$$

A family $\{P_x, x \in S\}$, where P_x is a solution of the submartingale problem starting from x, is simply called a *solution of the submartingale problem*.

The basic problem considered in [7] was the question of the existence and uniqueness of a solution of the submartingale problem. A summary of the major results obtained there follows. If $\alpha < 2$, then there is a unique solution $\{P_x, x \in S\}$ of the submartingale problem. Furthermore, for $x \neq 0$, $w(\cdot)$ reaches the corner of the wedge with P_x-probability zero if $\alpha \leq 0$, or with P_x-probability one if $0 < \alpha < 2$. If $\alpha \geq 2$, there is no solution of the submartingale problem starting from any $x \in S$. However, in this case, for each $x \in S$, there is a unique P_x satisfying (i) and (iii); it is concentrated on those paths that reach the corner and terminate there (corresponding to the process with absorption at the corner). In either case ($\alpha < 2$ or $\alpha \geq 2$), $\{P_x, x \in S\}$ has the strong Markov property and is Feller continuous (i.e., $P_{x_n} \to P_x$ weakly if $x_n \to x$).

For $\alpha \geq 2$, since the process associated with $\{P_x, x \in S\}$ is absorbed at the corner, it has the Dirac delta function at the corner as its invariant measure. On the other hand, if $\alpha < 2$, it is a nontrivial problem to determine the recurrence classification and the invariant measure (when one exists) of the process associated with the unique solution of the submartingale problem. The way in which this problem is resolved in this paper is outlined below. Henceforth, it is assumed that $\alpha < 2$.

For each $t \geq 0$ and $w \in C_S$, define

$$Z(t, w) = w(t). \tag{1.6}$$

We shall often write $Z(t)$ for $Z(t, \cdot)$, following the usual convention. To ensure that standard terminology and results on recurrence and invariant measures can be employed, it is shown in Section 2 that Z and the associated family $\{P_x, x \in S\}$ defines a *Hunt process*. More precisely, it is shown that there are augmentations $\mathscr{F}$ and $\mathscr{F}_t$ of the σ-algebras $\mathscr{M}$ and $\mathscr{M}_t$ such that

$$(C_S, \mathscr{F}, \mathscr{F}_t, Z(t), \hat{\theta}_t, P_x) \tag{1.7}$$

is a Hunt process with state space $(S, \mathscr{B}_S)$, where $\mathscr{B}_S$ denotes the Borel σ-algebra on S, and $\hat{\theta}_t$ is the usual shift operator given by

$$Z(\cdot, \hat{\theta}_t(w)) = Z(\cdot + t, w). \tag{1.8}$$

For the definition of a Hunt process and notational conventions see Blumenthal and Getoor [3; pages 20, 45]. In Section 3 this Hunt process is shown to be *transient* (to infinity) if $\alpha < 0$, and in Section 4 it is shown to be (finely) *recurrent* if $0 \leq \alpha < 2$. The latter property is used in Section 5, together with results of Azéma, Kaplan-Duflo and Revuz [1, 2], to show that up to a scalar multiple there is a unique σ-finite invariant measure for the process when $0 \leq \alpha < 2$. In Section 6, the density (with respect to Lebesgue measure) for this invariant measure is shown to be given in polar coordinates by

$$p(r, \theta) = r^{-\alpha}\cos(\alpha\theta - \theta_1). \tag{1.9}$$

Note that p is constant if $\alpha = 0$ and that p is integrable in each bounded neighborhood of the corner because $\alpha < 2$. Indeed, for $\alpha \geq 2$, $r^{-\alpha}\cos(\alpha\theta - \theta_1)$ fails to be integrable in any neighborhood of the corner; a fact which gives intuitive support to the finding in [7] that there is no solution of the submartingale problem when $\alpha \geq 2$.

2. Hunt process. If μ is a finite measure on $(S, \mathscr{B}_S)$, define the finite measure P_μ on $(C_S, \mathscr{M})$ by

$$P_\mu(A) = \int_S \mu(dx)P_x(A) \quad \text{for each} \quad A \in \mathscr{M}. \tag{2.1}$$

The right member above is well-defined, since $x \to P_x(A)$ is $\mathscr{B}_S$-measurable (cf. [7]). Let $(C_S, \mathscr{M}^\mu, P_\mu)$ denote the completion of $(C_S, \mathscr{M}, P_\mu)$ and $\mathscr{M}_t^\mu$ denote the augmentation of $\mathscr{M}_t$ with respect to $(C_S, \mathscr{M}^\mu, P_\mu)$. Define

$$\mathscr{F} = \wedge_\mu \mathscr{M}^\mu \tag{2.2}$$

and for each $t \geq 0$ let

(2.3) $$\mathscr{F}_t = \wedge_\mu \mathscr{M}_t^\mu.$$

In both (2.2) and (2.3), μ ranges over all finite measures on $\mathscr{B}_S$. It follows from the Feller continuity and strong Markov property of $\{P_x, x \in S\}$, and Blumenthal and Getoor [3; pages 20–21, 25–29, 42–45], that $\{\mathscr{F}_t, t \geq 0\}$ is right continuous and

$$\mathscr{Z} = (C_S, \mathscr{F}, \mathscr{F}_t, Z(t), \hat{\theta}_t, P_x)$$

is a Hunt process on $(S, \mathscr{B}_S)$. (The killing time ζ for $\mathscr{Z}$ is not explicitly mentioned since $P_x(\zeta = +\infty) = 1$ for all $x \in S$.) Some notions related to the recurrence classification of the Hunt process $\mathscr{Z}$ are recalled below.

A set $B \subset S$ is called *nearly Borel* if for each finite measure μ on $\mathscr{B}_S$ there are two Borel sets B_1 and B_2, depending on μ, such that $B_1 \subset B \subset B_2$ and

$$P_\mu\{\exists t \in [0, \infty): Z(t) \in B_2 \backslash B_1\} = 0.$$

If B is nearly Borel then for $B^c \equiv S \backslash B$,

$$T_B \equiv \inf\{t > 0: Z(t) \in B\}$$

and

$$T_{B^c} \equiv \inf\{t > 0: Z(t) \notin B\}$$

are $\{\mathscr{F}_t\}$-stopping times (see Chung [4, page 96]). A set $V \subset S$ is called *finely open* (for $\mathscr{Z}$) if for each $x \in V$ there exists a nearly Borel set $B \subset S$ such that $x \in B \subset V$ and

$$P_x(T_{B^c} > 0) = 1.$$

The collection of all finely open subsets of S is a topology. It is called the *fine topology* on S. For each $x \in S$, let $\mathscr{B}_f(x)$ denote the collection of all finely open nearly Borel sets in S that contain x.

For a nearly Borel set B in S, define $R_B \subset C_S$ by

$$R_B = \{\limsup_{t \to \infty} 1_B(Z(t)) = 1\}.$$

A point $x \in S$ is called *finely recurrent* (for $\mathscr{Z}$) if for each $B \in \mathscr{B}_f(x)$, $P_x(R_B) = 1$. It is called *finely transient* if there is a $B \in \mathscr{B}_f(x)$ such that $P_x(R_B) = 0$. A point $x \in S$ is either finely recurrent or finely transient (see Azéma, Kaplan-Duflo and Revuz [1, page 188] for a proof).

The Hunt process Z will be called *finely recurrent/finely transient* if each point $x \in S$ is finely recurrent/finely transient. A priori, it is possible that $\mathscr{Z}$ could be neither finely recurrent nor finely transient; however, it will be shown in the sequel that it is either one or the other.

The functions Φ and Ψ, defined below, were introduced in [7]. They will be used often in the following sections. In polar coordinates (r, θ), let

(2.4) $$\Phi(r, \theta) = \begin{cases} r^\alpha \cos(\alpha\theta - \theta_1), & \text{if } \alpha \neq 0 \\ \ln r + \theta \tan \theta_1, & \text{if } \alpha = 0. \end{cases}$$

In [7], Φ was shown to satisfy

(2.5) $$\Delta\Phi = 0 \quad \text{in} \quad S\backslash\{0\}$$

and

(2.6) $$D_j\Phi \equiv v_j \cdot \nabla\Phi = 0 \quad \text{on} \quad \partial S_j\backslash\{0\} \quad \text{for} \quad j = 1, 2.$$

Define Ψ on $S\backslash\{0\}$ by

(2.7) $$\Psi = \begin{cases} \Phi & \text{if} \quad \alpha > 0 \\ e^{\Phi} & \text{if} \quad \alpha = 0 \\ 1/\Phi & \text{if} \quad \alpha < 0, \end{cases}$$

and let $\Psi(0) = 0$. Two important properties of Ψ are that it is continuous on S and $\Psi(r, \theta)$ is increasing with r for each fixed value of θ.

It was shown in [7] that for each $x \in S\backslash\{0\}$, $0 < \eta < \Psi(x) < K < \infty$ and $\tau = \inf\{t \geq 0: \Psi(Z(t)) = \eta \text{ or } K\}$, we have

(2.8) $$P_x(\tau < \infty) = 1.$$

Then, by applying the submartingale property (1.4) to a suitable extension of Φ outside $\{z \in S: \eta \leq \Psi(z) \leq K\}$ and using (2.5)–(2.6) together with Doob's stopping theorem, we obtain:

(2.9) $$E^{P_x}[\Phi(Z(\tau))] = \Phi(x).$$

3. Transient if $\alpha < 0$. The next theorem shows that $\mathscr{Z}$ is transient to infinity if $\alpha < 0$. It follows from this that $\mathscr{Z}$ is finely transient, because each open set is finely open.

THEOREM 3.1. *Suppose $\alpha < 0$, $\eta > 0$ and $x \in S$. Then*

(3.1) $$P_x(\liminf_{t\to\infty} \Psi(Z(t)) < \eta) = 0.$$

PROOF. For each $r \geq 0$, define $\tau_r = \inf\{t \geq 0: \Psi(Z(t)) = r\}$. First it is shown that

(3.2) $$P_z(\tau_r < \infty) = 1$$

whenever $0 \leq \Psi(z) \leq r$. This clearly holds if $\Psi(z) = r$. If $0 < \Psi(z) < r$, then it follows from Varadhan and Williams [7] that $\tau_0 = \infty$, P_z-a.s. and

(3.3) $$E^{P_z}[\tau_r] \leq (2 - \alpha)^{-1}(r^{-2/\alpha} - \Phi(z)^{2/\alpha}).$$

To obtain the above from [7], the fact that $(\cos(\alpha\theta - \theta_1))^{(2/\alpha)-2} \geq 1$ for $\theta \in [0, \xi]$ and $\alpha < 0$ has been used. Now (3.2) follows from (3.3). Finally, if $0 = \Psi(z) < r$, then $z = 0$ and using the Feller continuity of $\{P_y, y \in S\}$ at $y = 0$, we obtain

(3.4) $$E^{P_0}[\tau_r] \leq \liminf_{y\to 0} E^{P_y}[\tau_r].$$

By (3.3), since $\Phi(y) \to \infty$ as $y \to 0$ and $\alpha < 0$, the right member of (3.4) is dominated by $(2 - \alpha)^{-1}r^{-2/\alpha}$ and (3.2) follows.

Let $r > \Psi(x) \vee \eta$. Since $\Psi = 1/\Phi$ for $\alpha < 0$, by setting $x = y$ in (2.9), we obtain for each $K > r$ and $y \in S$: $\Psi(y) = r$,

$$P_y(\tau_\eta < \tau_K) = (\Phi(y) - K^{-1})/(\eta^{-1} - K^{-1}). \tag{3.5}$$

Since $\tau_K \to \infty$, P_y – a.s. as $K \to \infty$, it follows from (3.5) that

$$P_y(\tau_\eta < \infty) = \eta\Phi(y),$$

where the right member equals $\eta r^{-1} < 1$, by the choice of y and r. By combining this with (3.2), and the strong Markov property of $\{P_z, z \in S\}$, and induction on n, we obtain for each $n \in \mathbb{N}$:

$$P_x(\liminf_{t\to\infty} \Psi(Z(t)) < \eta) \leq (\eta r^{-1})^n.$$

The desired result (3.1) follows by letting $n \to \infty$ in the above. □

4. Finely recurrent if $0 \leq \alpha < 2$.

DEFINITION. For $x, y \in S$, y is said to lead to x (denoted $y \to x$) iff for each $B \in \mathscr{B}_f(x)$, $P_y(T_B < \infty) > 0$. It is said that x communicates with y iff $y \to x$ and $x \to y$. "Communicates with" is an equivalence relation.

In this section, it is shown that if $0 \leq \alpha < 2$, then the Hunt process $\mathscr{Z}$ is finely recurrent and for each $x \in S$ the equivalence class

$$\mathscr{C}(x) \equiv \{y \in S\colon x \to y \text{ and } y \to x\} \tag{4.1}$$

is all of S. The fine recurrence of the points in $S^0 \equiv S\backslash\partial S$ is proved first.

THEOREM 4.1. *Suppose $0 \leq \alpha < 2$ and $x \in S^0$. Then each $y \in S$ leads to x and x is finely recurrent.*

PROOF. Let $B \in \mathscr{B}_f(x)$, and U and V be nonempty open balls centered at x such that $U \subset \bar{U} \subset V \subset S^0$. Then $\tilde{B} \equiv B \cap U \in \mathscr{B}_f(x)$. For each $z \in V$, Z behaves like Brownian motion under P_z until it hits $V^c \equiv S\backslash V \supset \partial S$. Since $\tilde{B} \subset U \subset V$, it follows that $\tilde{B}$ is a finely open nearly Borel set for Brownian motion and from the fine recurrence properties of the latter that

$$c_1 \equiv \inf_{z\in\bar{U}} P_z(T_{\tilde{B}} < T_{V^c}) > 0. \tag{4.2}$$

For each $r \geq 0$, let

$$\tau_r = \inf\{t \geq 0\colon \Psi(Z(t)) = r\}.$$

Then for each $r > 0$ and $z \in S$,

$$P_z(\tau_r < \infty) = 1. \tag{4.3}$$

If $\alpha > 0$, this can be proved using the strong Markov property and the properties $P_z(\tau_0 < \infty) = 1$ and $P_0(\tau_r < \infty) = 1$ (cf. Varadhan and Williams [7]). If $\alpha = 0$ and $0 \leq \Psi(z) \leq r$, (4.3) follows by the same reasoning as used to prove (3.2) above. If $\alpha = 0$ and $\Psi(z) > r$, by similar reasoning to that which led to (3.5), recalling that

$\Psi = e^{\Phi}$ when $\alpha = 0$, we have for each $K > \Psi(z)$:

$$P_z(\tau_r < \tau_K) = (\ln K - \Phi(z))/(\ln K - \ln r).$$

One obtains (4.3) from this by letting $K \to \infty$.

Since the distance from $\bar{U}$ to ∂S is strictly positive, there is $\varepsilon > 0$ such that $\bar{U} \subset \{z \in S: \Psi(z) > \varepsilon\}$ (see Figure 4.1). It is shown below that for $W \equiv \{z \in S: \Psi(z) = \varepsilon\}$,

$$c_2 \equiv \inf_{z \in W} P_z(T_{\bar{U}} < \tau_{\varepsilon/2}) > 0. \tag{4.4}$$

For each $z \in S$ and closed set F in S, let $d(z, F)$ denote the distance from z to F. For each $z \in S\backslash\{0\}$, let arg z denote the polar angle θ of $z = (r, \theta)$. Let $d_0 > 0$ such that $W_1 \equiv \{z \in W: d(z, \partial S_1) \le d_0\} \subset \{z \in W: 0 \le \arg z \le \frac{1}{3}\xi\}$ and $W_2 \equiv \{z \in W: d(z, \partial S_2) \le d_0\} \subset \{z \in W: \frac{2}{3}\xi \le \arg z \le \xi\}$. Define $W_0 = W\backslash(W_1 \cup W_2)$.

Let $\sigma_0 = T_{\partial S} \wedge \tau_{\varepsilon/2}$. Since W_0 is disjoint from $\partial S \cup \Psi^{-1}(\varepsilon/2)$ and for $z \in W_0$, Z behaves like Brownian motion under P_z until the time σ_0, it follows from the fine recurrence properties of Brownian motion that

$$\inf_{z \in W_0} P_z(T_{\bar{U}} < \sigma_0) > 0. \tag{4.5}$$

Let $H_2 = \partial S_2 \cup \{z \in S\backslash\{0\}: \arg z = \pi\}$. Let $\sigma_1 = T_{H_2} \wedge \tau_{\varepsilon/2}$. Now W_1 is disjoint from $H_2 \cup \Psi^{-1}(\varepsilon/2)$. It follows from [8] that for each $z \in W_1$, $Z(\cdot \wedge \sigma_1)$ under P_z is equivalent in law to $Z^0(\cdot \wedge \sigma_1^0)$ where

$$Z^0(t) = X(t) + v_1(-\min_{0 \le s \le t} X_2(s))^+ \quad \text{for all} \quad t \ge 0,$$

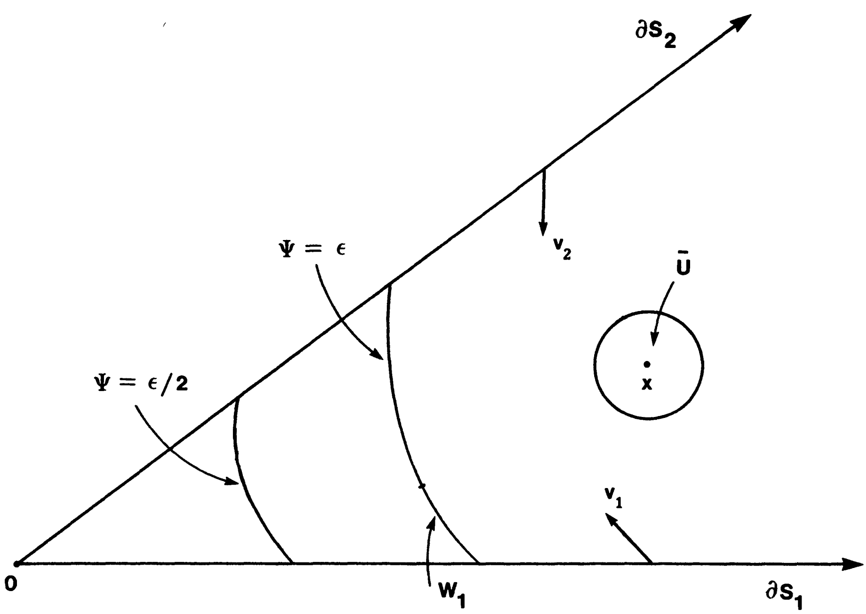

FIG. 4.1.

for a two-dimensional Brownian motion X starting from z, and

$$\sigma_1^0 = \inf\{t \geq 0: Z^0(t) \in H_2 \cup \Psi^{-1}(\varepsilon/2)\}.$$

The probability that Z^0 hits the set $W_0 \cap \{z: \xi/3 < \arg z < \xi/2\}$ before hitting $H_2 \cup \Psi^{-1}(\varepsilon/2)$ is strictly positive and continuous as a function of $z \in W_1$. Hence,

$$\inf_{z \in W_1} P_z(T_{W_0} < \sigma_1) > 0. \tag{4.6}$$

Similarly, for $H_1 \equiv \partial S_1 \cup \{z \in S\backslash\{0\}: \arg z = \xi - \pi\}$ and $\sigma_2 \equiv T_{H_1} \wedge \tau_{\varepsilon/2}$, we have

$$\inf_{z \in W_2} P_z(T_{W_0} < \sigma_2) > 0. \tag{4.7}$$

Since $W = W_0 \cup W_1 \cup W_2$, (4.4) follows from (4.5)–(4.7) and the strong Markov property. Now by (4.2), (4.4), and the strong Markov property, we have

$$\inf_{z \in W} P_z(T_{\tilde{B}} < \tau_{\varepsilon/2}) \geq c_0 \equiv c_1 c_2 > 0. \tag{4.8}$$

By a standard iterative argument, using (4.3) with $r = \varepsilon$ and $\varepsilon/2$, together with the uniform bound (4.8) and the strong Markov property, we have for each $y \in S$:

$$P_y(T_{\tilde{B}} < \infty) = 1, \tag{4.9}$$

and consequently, since $B \supset \tilde{B}$,

$$P_y(T_B < \infty) = 1. \tag{4.10}$$

This proves that each $y \in S$ leads to x, because B was an arbitrary member of $\mathscr{B}_f(x)$.

A simple proof by contradiction now shows that x is finely recurrent. For if x is not finely recurrent, there is $B \in \mathscr{B}_f(x)$ and $t^* > 0$ such that

$$P_x(Z(t) \in B \text{ for some } t > t^*) < 1.$$

Since the left member above may be rewritten as $P_x(P_{Z(t^*)}\{T_B < \infty\})$, this contradicts (4.10). □

COROLLARY 4.1. *Suppose $0 \leq \alpha < 2$ and $x \in S^0$. Then $\mathscr{C}(x)$ defined by* (4.1) *contains S^0 and is finely closed.*

PROOF. It is an immediate consequence of Theorem 4.1 that $\mathscr{C}(x)$ contains S^0. Since $x \in S^0$ is finely recurrent, it follows from Azéma, Kaplan-Duflo and Revuz [1, pages 199–200] that $\mathscr{C}(x)$ is finely closed. □

The next two results will be used to prove that each point of ∂S is finely recurrent.

LEMMA 4.1. *Suppose B is a nearly Borel subset of S such that*

$$E^{P_x}\left[\int_0^\infty 1_B(Z(t))\,dt\right] = 0, \quad \textit{for all} \quad x \in S.$$

Then $B^c = S\backslash B$ is finely dense in S.

PROOF. See Chung [4; Proposition 4, page 109]. □

LEMMA 4.2. *Suppose* $\alpha < 2$ *and* $x \in S$. *Then*

$$E^{P_x}\left[\int_0^\infty 1_{\partial S}(Z(t))\,dt\right] = 0. \tag{4.11}$$

PROOF. By condition (ii) of the submartingale problem, P_x-a.s., Z spends zero time (in the sense of Lebesgue measure) at the corner of the wedge. Therefore, it suffices to prove that for each $\varepsilon > 0$,

$$E^{P_x}\left[\int_0^\infty 1_{\partial S_\varepsilon}(Z(t))\,dt\right] = 0 \tag{4.12}$$

where $\partial S_\varepsilon = \{z \in \partial S: \Psi(z) > \varepsilon\}$. But this follows from successively applying the strong Markov property, together with the result of [7] that

$$E^{P_z}\left[\int_0^{\tau_{\varepsilon/2}} 1_{\partial S\setminus\{0\}}(Z(s))\,ds\right] = 0 \tag{4.13}$$

for each $z \in S$: $\Psi(z) \geq \varepsilon/2$ and $\tau_{\varepsilon/2} = \inf\{t \geq 0: \Psi(Z(t)) = \varepsilon/2\}$. □

COROLLARY 4.2. *Suppose* $0 \leq \alpha < 2$ *and* $x \in S^0$. *Then* $\mathscr{C}(x) = S$.

PROOF. By Lemmas 4.1 and 4.2, $S^0 \equiv S\setminus\partial S$ is finely dense in S. By Corollary 4.1, $\mathscr{C}(x)$ contains the fine closure of S^0. Thus, $S = (S^0)^f \subset \mathscr{C}(x) \subset S$, where the superscript f denotes the fine closure. □

THEOREM 4.2. *Suppose* $0 \leq \alpha < 2$. *Each* $x \in S$ *is finely recurrent and satisfies* $\mathscr{C}(x) = S$.

PROOF. For $x \in S^0$, this follows from Theorem 4.1 and Corollary 4.2. By [1, page 200], a finely recurrent point can only lead to a finely recurrent point. Since each $y \in S^0$ is finely recurrent and leads to every $x \in S$, it follows that each $x \in S$ is finely recurrent. Furthermore, since "communicates with" is an equivalence relation and $\mathscr{C}(y) = S$ holds for any $y \in S^0$, it follows that $\mathscr{C}(x) = S$ for all $x \in S$. □

Since each $x \in S$ is finely recurrent, $\mathscr{Z}$ is finely recurrent.

5. Invariant measure-existence and uniqueness. A σ-finite measure μ on $(S, \mathscr{B}_S)$ is an invariant measure for $\mathscr{Z}$ iff for each nonnegative bounded Borel measurable function h on S:

$$\int_S \mu(dx) E^{P_x}[h(Z(t))] = \int_S \mu(dx) h(x), \quad \text{for all} \quad t \geq 0. \tag{5.1}$$

A σ-finite invariant measure for $\mathscr{Z}$ will be called *unique* iff it and its positive scalar multiples are the only σ-finite invariant measures for $\mathscr{Z}$.

REMARK. The integrals in (5.1) are nonnegative, but may be infinite for some functions h.

THEOREM 5.1. *Suppose $0 \le \alpha < 2$. Then there is a unique σ-finite invariant measure for $\mathscr{Z}$.*

PROOF. By Theorem 4.2, S is the only equivalence class of points in S under the relation "communicates with" and all points in it are finely recurrent. In the terminology of Azéma, Kaplan-Duflo and Revuz [2, page 158], S is *the* "conservative class" for $\mathscr{Z}$. The existence and uniqueness of a σ-finite invariant measure for $\mathscr{Z}$ on S then follows from Theorem I.3 of [2, page 162]. □

6. Invariant measure-calculation. Throughout this section, it is assumed that $0 \le \alpha < 2$. For $x = (r, \theta) \in S\backslash\{0\}$, define

$$p(x) = r^{-\alpha}\cos(\alpha\theta - \theta_1), \tag{6.1}$$

and define p arbitrarily at the corner of S. Let μ be the measure defined in $(S, \mathscr{B}_S)$ by

$$\mu(B) = \int_B p(x)\, dx, \quad \text{for all} \quad B \in \mathscr{B}_S. \tag{6.2}$$

The right member of (6.2) will sometimes be written in polar coordinates as $\int_{B\backslash\{0\}} p(r, \theta) r\, dr\, d\theta$. The measure μ is σ-finite because $\mu(B)$ is finite for each compact set B. In particular, since $\alpha < 2$, this applies to any compact set containing the origin.

In this section, it is shown that for $0 \le \alpha < 2$, μ is an invariant measure for $\mathscr{Z}$ and hence by Theorem 5.1 it is the unique (up to a scalar multiple) σ-finite invariant measure for $\mathscr{Z}$.

Define vectors v_j^*, $j = 1, 2$, in polar coordinates by

$$v_1^* = (\tan\theta_1, 1) \quad \text{and} \quad v_2^* = (\tan\theta_2, -1). \tag{6.3}$$

Here the first component of v_j^* is in the radial direction and the second component is in the angular direction of the rotating polar coordinate frame of reference. For $j = 1, 2$, let

$$D_j^* = v_j^* \cdot \nabla. \tag{6.4}$$

For each j, v_j^* is the vector v_j "flipped" around the normal to ∂S_j and D_j^* is the adjoint boundary operator to D_j.

Since p is a linear combination of the real and imaginary parts of the function $z^{-\alpha}$ for $z = (r, \theta)$, and ∇p is given in polar coordinates by

$$\nabla p = (\partial p/\partial r, (1/r)(\partial p/\partial\theta)) = -\alpha r^{-\alpha-1}(\cos(\alpha\theta - \theta_1), \sin(\alpha\theta - \theta_1)), \tag{6.5}$$

it follows as in Varadhan and Williams [7] that p, defined by (6.1) in a domain containing $S\backslash\{0\}$, satisfies the following *adjoint boundary value problem*:

$$\Delta p = 0 \quad \text{in} \quad S\backslash\{0\} \tag{6.6}$$

$$D_j^* p = 0 \quad \text{on} \quad \partial S_j\backslash\{0\} \quad \text{for} \quad j = 1, 2. \tag{6.7}$$

This is the formal *adjoint* problem to that associated with the operators Δ and $\{D_j, j = 1, 2\}$, which appear in the submartingale characterization of the process. When $0 < \alpha < 2$, p is not the unique (up to a scalar multiple) positive solution of (6.6)–(6.7). For instance, the function that is identically one is also a positive solution of these equations. To ensure uniqueness, an additional condition at the corner or at infinity is needed. Even when the appropriate uniqueness condition is satisfied, it is not trivial to verify that the solution is the density of an invariant measure for the process, because of the discontinuity in the directions of reflection and in the smoothness of the boundary at the corner of the wedge. Nonetheless, finding positive solutions of (6.6)–(6.7) is a good way to obtain candidates for the density of the invariant measure. If it can be verified that the measure associated with one of these candidates is an invariant measure, then it follows by the uniqueness established previously that it is *the* invariant measure.

The following procedure is used in Theorem 6.1 to verify that μ is an invariant measure for $\mathscr{Z}$. A sequence of smooth bounded domains with associated smooth vector fields on their boundaries are chosen to approximate S and its associated vector fields v_j on $\partial S_j \setminus \{0\}$, $j = 1, 2$. The sequence is chosen such that for each domain, the Brownian motion with oblique reflection in the direction of the associated vector field at the boundary has μ as an invariant measure, and such that when appropriate weak limits are taken, the invariance of μ follows for $\mathscr{Z}$. The adjoint boundary conditions associated with these smoothed domains and vector fields involve the derivative of the tangential component of the direction of reflection. For $\alpha \neq 0$, this derivative is not identically zero, i.e., the tangential component of the direction of reflection is not the same on ∂S_1 and ∂S_2 and therefore cannot be kept the same on an arc joining ∂S_1 smoothly to ∂S_2. As a consequence, the function that is identically one cannot be an invariant density for these reflected processes on smoothed domains when $\alpha \neq 0$. The appropriate "corner condition" on p that emerges from this and pertains even when $\alpha = 0$ is as follows. For any sufficiently smooth arc Σ lying in $S \setminus \partial S$ and joining ∂S_1 smoothly to ∂S_2, we have

$$(6.8) \qquad \int_\Sigma \frac{\partial p}{\partial n}\, d\ell = \tan(-\theta_2) p(b) - (\tan \theta_1) p(a),$$

where $d\ell$ denotes the infinitesimal element of arc length along Σ and $\partial/\partial n$ denotes differentiation in the direction of the normal to Σ that points to the right as Σ is traversed from ∂S_1 to ∂S_2, and a and b are the end points of Σ that respectively lie in ∂S_1 and ∂S_2.

THEOREM 6.1. *Suppose $0 \le \alpha < 2$. The σ-finite measure μ given by* (6.1)–(6.2) *is an invariant measure for $\mathscr{Z}$.*

PROOF. Since the density p of μ is integrable with respect to Lebesgue measure on each compact set, it follows by approximation that to show μ is invariant, it suffices to prove that (5.1) holds for all nonnegative $h \in C_c(S)$, the space of all continuous functions on S that have compact support.

Let $0 < \varepsilon < 1 < K < \infty$. Define

(6.9) $$F_{\varepsilon K} = \{z \in S \colon \varepsilon \le \Psi(z) \le K\}$$

and let $U_{\varepsilon K}$ be a bounded domain in $\mathbb{R}^2$ having the following properties (a)–(d).

(a) $U_{\varepsilon K} = \{z \in \mathbb{R}^2 \colon \phi(z) > 0\}$ and $\partial U_{\varepsilon K} = \{z \in \mathbb{R}^2 \colon \phi(z) = 0\}$ for some $\phi \in C_b^3(\mathbb{R}^2)$ such that for some $\beta > 0$, $|\nabla\phi| \ge \beta > 0$ on $\partial U_{\varepsilon K}$.
(b) $F_{\varepsilon K} \subset \bar{U}_{\varepsilon K} \subset \{z \in S \colon \varepsilon/2 \le \Psi(z) \le K + 1\}$.
(c) $\partial S_j \cap \partial U_{\varepsilon K}$ is connected for $j = 1, 2$.
(d) For each fixed $K > 1$, $\partial U_{\varepsilon K} \cap \{z \in S \colon \Psi(z) \ge K\}$ is the same for all $0 < \varepsilon < 1$.

The function p will be used to define a vector field $v_{\varepsilon K}$ on $\partial U_{\varepsilon K}$ that equals v_j on $\partial S_j \cap \partial U_{\varepsilon K}$ for $j = 1, 2$ and satisfies (6.13) below. For fixed ε and K, let n denote the unit normal vector field on $\partial U_{\varepsilon K}$ that points into $U_{\varepsilon K}$ and let ℓ denote the unit tangent vector field to $\partial U_{\varepsilon K}$ oriented such that $U_{\varepsilon K}$ is on the right as one moves in the direction of ℓ on $\partial U_{\varepsilon K}$ (see Figure 6.1). Let $\partial/\partial n \equiv n \cdot \nabla$ and $\partial/\partial\ell \equiv \ell \cdot \nabla$. Choose $z_0 \in \partial S_1 \cap \partial U_{\varepsilon K}$. Define

(6.10) $$v_{\varepsilon K}^n \equiv v_{\varepsilon K} \cdot n = 1 \quad \text{on} \quad \partial U_{\varepsilon K},$$

(6.11) $$v_{\varepsilon K}^{\ell}(z) \equiv (v_{\varepsilon K} \cdot \ell)(z) = \frac{(\tan\theta_1)p(z_0) + \int_{z_0}^{z} (\partial p/\partial n)\, d\ell}{p(z)},$$

for all $z \in \partial U\varepsilon K$, where the integral with respect to $d\ell$ is the line integral along

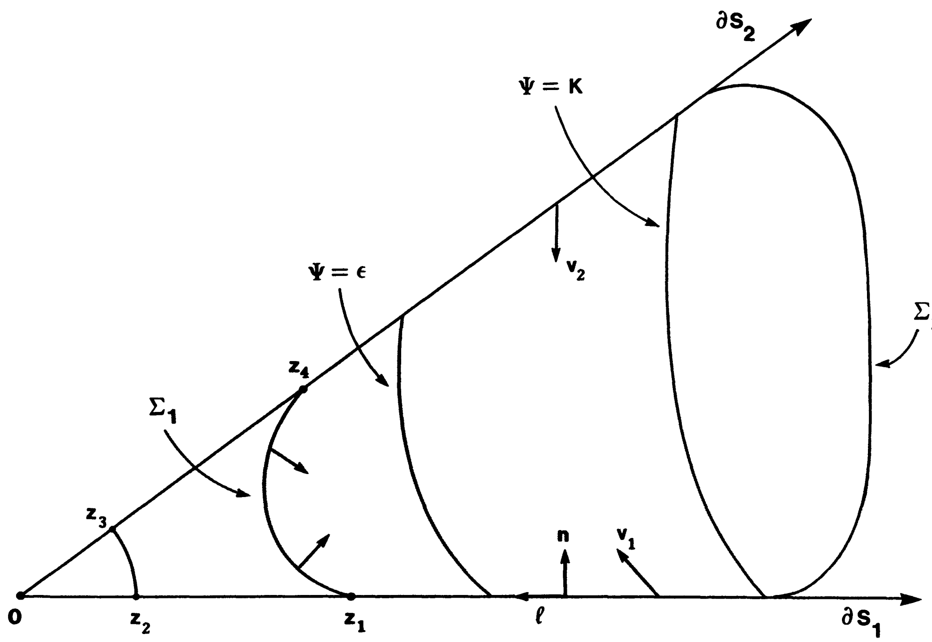

FIG. 6.1.

$\partial U_{\varepsilon K}$ with respect to arc length in the direction of the unit tangent vector ℓ. By Gauss' theorem and (6.6):

$$\int_{\partial U_{\varepsilon K}} \frac{\partial p}{\partial n}\, d\ell = 0.$$

Thus the integral with respect to $d\ell$ in (6.11) is well-defined. Furthermore, it is readily verified from (6.11) that for any $z^* \in \partial U_{\varepsilon K}$:

$$\text{(6.12)} \qquad v_{\varepsilon K}^{\ell}(z) = \frac{v_{\varepsilon K}^{\ell}(z^*)p(z^*) + \int_{z^*}^{z} (\partial p/\partial n)\, d\ell}{p(z)}, \quad \text{for all} \quad z \in \partial U_{\varepsilon K}.$$

The vector field $v_{\varepsilon K}$ was defined so as to satisfy the "adjoint boundary condition"

$$\text{(6.13)} \qquad (\partial/\partial \ell)(v_{\varepsilon K}^{\ell} p) = \partial p/\partial n \quad \text{on} \quad \partial U_{\varepsilon K}.$$

It is verified below that $v_{\varepsilon K}$ equals v_j on $\Gamma_j \equiv \partial S_j \cap \partial U_{\varepsilon K}$ for $j = 1, 2$. The remainder of the boundary of $U_{\varepsilon K}$ is composed of the two arcs

$$\Sigma_1 = \partial U_{\varepsilon K} \cap \{z \in S^0 \colon \Psi(z) \le \varepsilon\}$$

and

$$\Sigma_2 = \partial U_{\varepsilon K} \cap \{z \in S^0 \colon \Psi(z) \ge K\}.$$

By condition (d) above on $\partial U_{\varepsilon K}$, for each fixed $K > 1$, the arc Σ_2 is the same for all $0 < \varepsilon < 1$. By (6.7), on Γ_1 we have $v_1^* \cdot \nabla p = 0$ or equivalently,

$$\text{(6.14)} \qquad \partial p/\partial n = (\tan \theta_1)(\partial p/\partial \ell).$$

Thus, by (6.11), $v_{\varepsilon K}^{\ell}(z) = \tan \theta_1$ for all $z \in \Gamma_1$, and since $v_1^n \equiv v_1 \cdot n = 1$, it follows that $v_{\varepsilon K} = v_1$ on Γ_1.

For the proof that $v_{\varepsilon K} = v_2$ on Γ_2, let z_1 be the closest point of Γ_1 to the corner and let z_4 be the closest point of Γ_2 to the corner. Choose r_0: $0 < r_0 < |z_1| \wedge |z_4|$ and such that $\{(r_0, \theta)\colon 0 \le \theta \le \xi\}$ does not intersect $\bar{U}_{\varepsilon K}$. Let $z_2 = (r_0, 0)$ and $z_3 = (r_0, \xi)$ (see Figure 6.1). Since $v_{\varepsilon K}^{\ell}(z) = \tan \theta_1$ on Γ_1, we have by (6.12):

$$\text{(6.15)} \qquad v_{\varepsilon K}^{\ell}(z_4) = \frac{(\tan \theta_1)p(z_1) + \int_{z_1}^{z_4} (\partial p/\partial n)\, d\ell}{p(z_4)}.$$

The integral from z_1 to z_4 here is along the arc Σ_1. By Gauss' theorem and (6.6), this can be replaced by the integral along the piecewise smooth curve $\Lambda \equiv \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, where Λ_1 is the closed line segment in ∂S_1 from z_1 to z_2, $\Lambda_2 = \{(r_0, \theta)\colon 0 \le \theta \le \xi\}$, Λ_3 is the closed line segment in ∂S_2 from z_3 to z_4, and $\partial p/\partial n$ is defined except at z_1, z_2, z_3 and z_4, as the derivative of p in the direction of the normal pointing into the domain bounded by $\Lambda \cup (-\Sigma_1)$. Then in the same way that (6.12) follows from (6.11), letting $d\ell$ denote the infinitesimal element of arc length, even off $\partial U_{\varepsilon K}$, we obtain

$$\text{(6.16)} \qquad v_{\varepsilon K}^{\ell}(z_4) = \frac{v_{\varepsilon K}^{\ell}(z_3)p(z_3) + \int_{\Lambda_3} (\partial p/\partial n)\, d\ell}{p(z_4)}$$

where

$$\text{(6.17)} \qquad v_{\varepsilon K}^{\ell}(z_3) \equiv \frac{v_{\varepsilon K}^{\ell}(z_2)p(z_2) + \int_{\Lambda_2} (\partial p/\partial n)\, d\ell}{p(z_3)},$$

and

$$v'_{\epsilon K}(z_2) \equiv \frac{(\tan\theta_1)p(z_1) + \int_{\Lambda_1}(\partial p/\partial n)\,d\ell}{p(z_2)}. \tag{6.18}$$

It is readily verified using (6.7) and (6.18) that $v'_{\epsilon K}(z_2) = \tan\theta_1$. By substituting this and $z_2 = (r_0, 0)$ in (6.17), and using $\partial p/\partial n = \partial p/\partial r = -\alpha r^{-\alpha-1}\cos(\alpha\theta - \theta_1)$ on Λ_2, we obtain

$$\begin{aligned} v'_{\epsilon K}(z_3) &= \frac{(\tan\theta_1)r_0^{-\alpha}\cos(\theta_1) - \alpha r_0^{-\alpha-1}\int_0^{\xi}\cos(\alpha\theta - \theta_1)r_0\,d\theta}{r_0^{-\alpha}\cos(\theta_2)} \\ &= -\tan\theta_2. \end{aligned} \tag{6.19}$$

Then, substituting (6.19) into (6.16) and using

$$0 = v_2^* \cdot \nabla p = \partial p/\partial n + \tan\theta_2(\partial p/\partial\ell) \quad \text{on} \quad \Lambda_3, \tag{6.20}$$

and (6.12), we obtain

$$v'_{\epsilon K}(z) = -\tan\theta_2, \quad \text{for all} \quad z \in \Gamma_2. \tag{6.21}$$

Note that we have just verified that p satisfies the corner condition (6.8).

Since the domain $U_{\epsilon K}$ and vector field $v_{\epsilon K}$ on $\partial U_{\epsilon K}$ are sufficiently smooth for the theory of Stroock and Varadhan [6] to apply, it follows that for each $x \in \bar{U}_{\epsilon K}$ there is a unique probability measure $P_x^{\epsilon K}$ on $(C_S, \mathscr{M})$ satisfying (i)′–(iii)′ below.

(i)′ $P_x^{\epsilon K}(w(0) = x) = 1$.
(ii)′ $P_x^{\epsilon K}(w(t) \in \bar{U}_{\epsilon K}$ for all $t \ge 0) = 1$.
(iii)′ For any $f \in C_b^{1,2}([0, \infty) \times \mathbb{R}^2)$ satisfying $v_{\epsilon K} \cdot \nabla f \ge 0$ on $[0, \infty) \times \partial U_{\epsilon K}$, we have

$$f(t, w(t)) - \int_0^t 1_{U_{\epsilon K}}(w(s))\left(\frac{\partial f}{\partial s} + \frac{1}{2}\Delta f\right)(s, w(s))\,ds$$

is a $P_x^{\epsilon K}$-submartingale.

Here $C_b^{1,2}([0, \infty) \times \mathbb{R}^2)$ denotes the set of functions $f(t, x)$ that together with their first t-derivative and first two x-derivatives are continuous and bounded on $[0, \infty) \times \mathbb{R}^2$.

It follows immediately from substituting suitable modifications of $f(t, \cdot) \equiv -t$ in condition (iii)′ above that:

$$E^{P_x^{\epsilon K}}\left[\int_0^\infty 1_{\partial U_{\epsilon K}}(w(s))\,ds\right] = 0. \tag{6.22}$$

Moreover, a consequence of the uniqueness of $\{P_x^{\epsilon K}, x \in \bar{U}_{\epsilon K}\}$ is that (cf. [6, page 196] and [7]):

$$x \to P_x^{\epsilon K}(A) \quad \text{is Borel measurable on } \bar{U}_{\epsilon K} \quad \text{for each} \quad A \in \mathscr{M}.$$

Next it is shown that for each bounded Borel measurable function h on $\bar{U}_{\epsilon K}$:

$$\int_{\bar{U}_{\epsilon K}} E^{P_x^{\epsilon K}}[h(w(t))]p(x)\,dx = \int_{\bar{U}_{\epsilon K}} h(x)p(x)\,dx, \quad \text{for all} \quad t \ge 0. \tag{6.23}$$

For this, by approximation, Fubini's theorem, and the uniqueness of the Laplace transform, it suffices to show that for each $\gamma > 0$ and $h \in C(\bar{U}_{\varepsilon K})$:

$$(6.24) \qquad \int_{\bar{U}_{\varepsilon K}} E^{P_x^{\varepsilon K}}\left[\int_0^\infty e^{-\gamma t} h(w(t))\, dt\right] p(x)\, dx = \frac{1}{\gamma}\int_{\bar{U}_{\varepsilon K}} h(x)p(x)\, dx.$$

For such γ and h, since $\partial U_{\varepsilon K}$ and $v_{\varepsilon K}$ are sufficiently smooth, it follows from Gilbarg and Trudinger [5; pages 122–124, 130–131] that there is a function $g \in C_b^2(\mathbb{R}^2)$ satisfying

$$(6.25) \qquad (\tfrac{1}{2}\Delta - \gamma)g = -h \quad \text{in} \quad U_{\varepsilon K},$$

$$(6.26) \qquad v_{\varepsilon K} \cdot \nabla g = 0 \quad \text{on} \quad \partial U_{\varepsilon K}.$$

By applying the submartingale property (iii)$'$ of $P_x^{\varepsilon K}$ to $f(t, x) = e^{-\gamma t} g(x)$ and $-f(t, x)$, taking expectations, letting $t \to \infty$ and using (6.22), it follows that

$$(6.27) \qquad g(x) = E^{P_x^{\varepsilon K}}\left[\int_0^\infty e^{-\gamma t} h(w(t))\, dt\right], \quad \text{for all} \quad x \in \bar{U}_{\varepsilon K}.$$

Since the integrals in (6.24) are the same over $U_{\varepsilon K}$ and $\bar{U}_{\varepsilon K}$, it follows from (6.25) and (6.27) that (6.24) holds if and only if

$$(6.28) \qquad \int_{U_{\varepsilon K}} \Delta g(x) p(x)\, dx = 0.$$

But this holds because, by the choice of $v_{\varepsilon K}$, p satisfies the adjoint boundary value problem to that satisfied by g, namely, $\Delta p = 0$ in $U_{\varepsilon K}$ and (6.13) holds on $\partial U_{\varepsilon K}$. The detailed verification of (6.28) using Green's second identity and integration by parts on $\partial U_{\varepsilon K}$ is left to the reader. Thus (6.23) holds, which means that p is the density of an invariant measure for the strong Markov process associated with the family $\{P_x^{\varepsilon K}, x \in \bar{U}_{\varepsilon K}\}$. To deduce from this that μ is an invariant measure for $\mathscr{Z}$, the next lemma on weak convergence is needed. For this, let

$$F_K = \{z \in S: 0 \le \Psi(z) \le K\},$$

$$U_K = \{z \in S^0: 0 \le \Psi(z) \le K\} \cup U_{\varepsilon K},$$

and v_K be a vector field defined on $\partial U_K \setminus \{0\}$ by

$$v_K = \begin{cases} v_j & \text{on} \quad (\partial S_j \setminus \{0\}) \cap \partial F_K, \quad j = 1, 2 \\ v_{\varepsilon K} & \text{on} \quad \partial U_K \cap \{z \in S: \Psi(z) \ge K\} \end{cases}$$

where v_K is independent of $0 < \varepsilon < 1$, by condition (d) on $\partial U_{\varepsilon K}$.

For each $0 < \varepsilon < 1 < K$, define the probability measure $P_\mu^{\varepsilon K}$ on $(C_S, \mathscr{M})$ by

$$(6.29) \qquad P_\mu^{\varepsilon K}(A) = \frac{1}{\mu(\bar{U}_{\varepsilon K})}\int_{\bar{U}_{\varepsilon K}} P_x^{\varepsilon K}(A) p(x)\, dx, \quad \text{for all} \quad A \in \mathscr{M}.$$

Note that by setting $h = 1_B$ in (6.23) and dividing by $\mu(\bar{U}_{\varepsilon K})$, it follows that for any $B \in \mathscr{B}_S$:

$$P_\mu^{\varepsilon K}(w(t) \in B) = P_\mu^{\varepsilon K}(w(0)) \tag{6.30}$$
$$= \frac{1}{\mu(\bar{U}_{\varepsilon K})} \int_{\bar{U}_{\varepsilon K}} 1_B(x) p(x)\, dx, \quad \text{for all} \quad t \ge 0.$$

LEMMA 6.1. *As $\varepsilon \downarrow 0$, the family $\{P_\mu^{\varepsilon K}, 0 < \varepsilon < 1\}$ converges weakly on $(C_S, \mathscr{M})$ to the unique probability measure P_μ^K on $(C_S, \mathscr{M})$ that satisfies* (I)–(IV) *below.*

(I) $P_\mu^K(w(0) \in B) = \mu(B)/\mu(\bar{U}_K)$ *for all Borel sets* $B \subset \bar{U}_K$.
(II) $P_\mu^K(w(t) \in \bar{U}_K$ *for all* $t \ge 0) = 1$.
(III) $E^{P_\mu^K}[\int_0^\infty 1_{\{0\}}(w(s))\, ds] = 0$.
(IV) *For each* $f \in C_b^2(S)$,

$$f(w(t)) - \frac{1}{2} \int_0^t \Delta f(w(s))\, ds$$

is a P_μ^K-submartingale whenever f is constant in a neighborhood of the origin and satisfies

$$v_K \cdot \nabla f \ge 0 \quad on \quad \partial U_K \backslash \{0\}.$$

PROOF. The uniqueness of a probability measure P_μ^K on $(C_S, \mathscr{M})$ that satisfies (I)–(IV) is proved as follows. Let P_w^0 denote a regular conditional probability distribution (r.c.p.d.) of $P_\mu^K \mid \mathscr{M}_0$ and let $\sigma = \inf\{t \ge 0\colon \Psi(\omega(t)) \ge K\}$. Then it follows from the uniqueness of the solution $P_{w(0)}$ of the submartingale problem starting from $w(0)$ that for P_μ^K-almost every w, we have $P_w^0 = P_{w(0)}$ on $\mathscr{M}_\sigma$ (cf. [7] and Stroock and Varadhan [6; Theorem 5.6, page 193]). By combining this with (I), we see that P_μ^K is uniquely determined on $\mathscr{M}_{\sigma_1}$, where $\sigma_1 = \inf\{t \ge 0\colon \Psi(w(t)) \ge K\}$. Let $P_w^{\sigma_1}$ denote an r.c.p.d. of $P_\mu^K \mid \mathscr{M}_{\sigma_1}$ and for each $w \in \{\sigma_1 < \infty\}$ define $\hat{P}_w^{\sigma_1}$ on $(C_S, \mathscr{M})$ by

$$\hat{P}_w^{\sigma_1}(A) = P_w^{\sigma_1}(\omega\colon \omega(\cdot + \sigma_1) \in A), \quad \text{for all} \quad A \in \mathscr{M}. \tag{6.31}$$

Define $\tau_1 = \inf\{t \ge \sigma_1\colon \Psi(w(t)) \le 1\}$. By using the technique employed in [8], for extending the submartingale property from functions $f \in C_b^2(S)$ to functions $f \in C_b^{1,2}([0, \infty) \times \mathbb{R}^2)$, and using the uniqueness of the solution of the submartingale problem associated with (i)′–(iii)′ above, we may conclude that there is a P_μ^K-null set $N_{\sigma_1} \in \mathscr{M}_{\sigma_1}$ such that for $w \notin N_{\sigma_1} \cup \{\sigma_1 = \infty\}$ and $\tau(\omega) = \inf\{t \ge 0\colon \Psi(\omega(t)) \le 1\}$, we have $\hat{P}_w^{\sigma_1} = P_{w(\sigma_1)}^{\varepsilon K}$ on $\mathscr{M}_\tau$. Here $P_{w(\sigma_1)}^{\varepsilon K}$ on $\mathscr{M}_\tau$ is the same for all $0 < \varepsilon < 1$, by condition (d) on $U_{\varepsilon K}$ and the remark following the definition of v_K. By combining the above, we conclude that P_μ^K is uniquely determined on $\mathscr{M}_{\tau_1}$. Continuing in this manner, it follows that for the increasing sequence of stopping times $\sigma_1, \tau_1, \sigma_2, \tau_2, \cdots$ at which w alternately hits the arcs $\Psi^{-1}(K)$ and $\Psi^{-1}(1)$, P_μ^K is uniquely determined on $\mathscr{M}_{\sigma_n}$ and $\mathscr{M}_{\tau_n}$ for all n. Since $\tau_n > \sigma_n \uparrow \infty$ as $n \to \infty$, by the continuity of $w(\cdot)$, it follows that P_μ^K is uniquely determined on $\mathscr{M}$.

In view of the above uniqueness and the weak metrizability of the set of probability measures on $(C_S, \mathscr{M})$, to show that $\{P_\mu^{\varepsilon K}, 0 < \varepsilon < 1\}$ converges weakly to a P_μ^K satisfying (I)–(IV), it suffices to show that given $\varepsilon_n \downarrow 0$ as $n \to \infty$, $\{P_\mu^{\varepsilon_n K}\}_{n=1}^\infty$ is weakly relatively compact and any weak accumulation point satisfies properties (I)–(IV).

The weak relative compactness follows in a similar manner to that in [7] (see also Theorem 4.2 and Lemma 7.3 of [8]). Properties (I), (II), and (IV), are readily verified to hold for any weak accumulation point P_μ^K of $\{P_\mu^{\varepsilon_n K}\}_{n=1}^\infty$. To verify property (III) for such a P_μ^K, let $0 < \eta < 1$. By setting $B = \{z \in S: 0 \le \Psi(z) < \eta\}$ in (6.30), then integrating over a finite time interval and using Fubini's theorem, we obtain for all $t \ge 0$:

$$E^{P_\mu^{\varepsilon K}}\left[\int_0^t 1_{[0,\eta)}(\Psi(w(s)))\, ds\right] = \frac{t}{\mu(\bar{U}_{\varepsilon K})}\int_{\bar{U}_{\varepsilon K}} 1_{[0,\eta)}(\Psi(x))p(x)\, dx. \tag{6.32}$$

By letting $\varepsilon \downarrow 0$ through a sequence such that the associated $P_\mu^{\varepsilon K}$ converge weakly to P_μ^K, it follows that

$$E^{P_\mu^{K}}\left[\int_0^t 1_{[0,\eta)}(\Psi(w(s)))\, ds\right] \le \frac{t}{\mu(\bar{U}_{K})}\int_{\bar{U}_{K}} 1_{[0,\eta)}(\Psi(x))p(x)\, dx. \tag{6.33}$$

Then by letting $\eta \downarrow 0$ and invoking Fatou's lemma on the left and monotone convergence on the right, we obtain:

$$E^{P_\mu^{K}}\left[\int_0^t 1_{\{0\}}(\Psi(w(s)))\, ds\right] \le 0. \tag{6.34}$$

Since $\Psi(x) = 0$ if and only if $x = 0$, and t was arbitrary, property (III) follows. □

We shall now show that (5.1) holds for each nonnegative $h \in C_c(S)$. In deducing this from (6.23), some care is needed, because μ is only a σ-finite measure, not a finite measure on S. It follows from the invariance property (6.23), together with the definition (6.29) of $P_\mu^{\varepsilon K}$ and the weak convergence established in Lemma 6.1, that for any given nonnegative $h \in C_c(S)$:

$$\mu(\bar{U}_K)E^{P_\mu^K}[h(w(t))] = \int_{\bar{U}_K} h(x)p(x)\, dx. \tag{6.35}$$

Define

$$\sigma_K = \inf\{t \ge 0: \Psi(w(t)) \ge K\}.$$

By the uniqueness argument in the proof of Lemma 6.1 we have $P_w^0 = P_{w(0)}$ on $\mathscr{M}_{\sigma_K}$, where P_w^0 denotes an r.c.p.d. of $P_\mu^K \mid \mathscr{M}_0$. Thus, the left member of (6.35) is equal to

$$\begin{aligned} &\mu(\bar{U}_K)E^{P_\mu^K}[h(w(t))(1_{\{t<\sigma_K\}} + 1_{\{t\ge\sigma_K\}})] \\ &\qquad = \int_{\bar{U}_K} 1_{[0,K]}(\Psi(x))E^{P_x}[h(w(t));\, t < \sigma_K]p(x)\, dx \\ &\qquad\quad + \mu(\bar{U}_K)E^{P_\mu^K}[h(w(t));\, t \ge \sigma_K]. \end{aligned} \tag{6.36}$$

By monotone convergence and the definition of μ, to complete the proof of (5.1), we must show that the last term in (6.36) tends to zero as $K \to \infty$. By conditioning, this term equals

$$\mu(\bar{U}_K)E^{P_\mu^K}[\sigma_K \le t;\ E^{P_\mu^K}(h(w(t)) \mid \mathscr{M}_{\sigma_K})]. \tag{6.37}$$

Since h has compact support, there is $M > 1$ such that $h = 0$ on $\{z \in S: \Psi(z) \ge M\}$. Suppose henceforth that $K > M$. Then, using similar reasoning to that in the proof of uniqueness in Lemma 6.1, it follows that (6.37) is dominated by

$$\int_{\bar{U}_{\varepsilon K}} E^{P_x}[\sigma_K \le t;\ P_{w(\sigma_K)}^{\varepsilon K}\{\tau_M \le t\} \| h \|]p(x)\, dx \tag{6.38}$$

where $\tau_M(\omega) \equiv \inf\{t \ge 0: \Psi(\omega(t)) \le M\}$,

$$\| h \| \equiv \sup_{z \in S} | h(z) | = \sup_{\Psi(z) < M} | h(z) | < \infty,$$

and (6.38) is independent of $0 < \varepsilon < 1$. The following lemma is used to verify that (6.38) tends to zero as $K \to \infty$.

LEMMA 6.2. *For each $t \ge 0$ and $x \in \{z \in \bar{U}_K: \Psi(z) \ge M\}$, we have*

$$E^{P_x^{\varepsilon K}}\left[\exp\left(-\int_0^{t \wedge \tau_M} q(w(s))\, ds\right);\ \tau_M \le t\right] \le g(\Psi(x)), \tag{6.39}$$

where for $(r, \theta) \in S \setminus \{0\}$,

$$q(r, \theta) = \begin{cases} \tfrac{1}{2}\alpha^2(\cos(\alpha\theta - \theta_1))^{(2/\alpha)-2} & \text{if } 0 < \alpha < 2 \\ \tfrac{1}{2}e^{2\theta \tan\theta_1}(1 + \tan^2\theta_1) & \text{if } \alpha = 0, \end{cases} \tag{6.40}$$

and for $M \le y < \infty$ and $\nu = \alpha/2$,

$$g(y) = \begin{cases} \dfrac{y^{1/2}[\mathscr{K}_{\nu-1}(\alpha K^{1/\alpha})\mathscr{I}_\nu(\alpha(y \wedge K)^{1/\alpha}) + \mathscr{I}_{\nu-1}(\alpha K^{1/\alpha})\mathscr{K}_\nu(\alpha(y \wedge K)^{1/\alpha})]}{M^{1/2}[\mathscr{K}_{\nu-1}(\alpha K^{1/\alpha})\mathscr{I}_\nu(\alpha M^{1/\alpha}) + \mathscr{I}_{\nu-1}(\alpha K^{1/\alpha})\mathscr{K}_\nu(\alpha M^{1/\alpha})]}, & \text{if } 0 < \alpha < 2 \\ \dfrac{\mathscr{K}_{-1}(K)\mathscr{I}_0(y \wedge K) + \mathscr{I}_{-1}(K)\mathscr{K}_0(y \wedge K)}{\mathscr{K}_{-1}(K)\mathscr{I}_0(M) + \mathscr{I}_{-1}(K)\mathscr{K}_0(M)} & \text{if } \alpha = 0, \end{cases} \tag{6.41}$$

where $\mathscr{K}_\gamma$ and $\mathscr{I}_\gamma$ denote the modified Bessel functions of order γ.

PROOF. To obtain (6.39), a technique from [7] was used. Namely, functions of the form $f = g(\Psi)$ were considered, because they automatically satisfied the homogeneous boundary conditions:

$$v_j \cdot \nabla f = 0 \quad \text{on } \partial U_K \cap \partial S_j \quad \text{for } j = 1, 2. \tag{6.42}$$

Then g was chosen to satisfy a suitable one-dimensional boundary-value problem, such that through the submartingale property (iii)$'$ of $P_x^{\varepsilon K}$, the estimate (6.39) was obtained. Details of the verification of (6.39) are given below.

Using the properties of modified Bessel functions (see, for example, Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Functions*, National Bureau of Standards, 1972, pages 375–378) one can verify that $g \in C_b^2([M, K) \cup (K, \infty))$ satisfies the following differential equation if $0 < \alpha < 2$:

$$g''(y) = \begin{cases} y^{(2/\alpha)-2} g(y) & \text{for } M < y < K \\ 0 & \text{for } y > K, \end{cases} \tag{6.43}$$

or if $\alpha = 0$ it satisfies

$$g''(y) + y^{-1} g'(y) = \begin{cases} g(y) & \text{for } M < y < K \\ 0 & \text{for } y > K. \end{cases} \tag{6.44}$$

Furthermore, since $\mathscr{I}_\gamma(y)$ and $\mathscr{K}_\gamma(y)$ are positive for $y > 0$ and $\gamma \ge -1$, and

$$(d/dy)(y^\nu \mathscr{I}_\nu(y)) = y^\nu \mathscr{I}_{\nu-1}(y),$$

and

$$(d/dy)(y^\nu \mathscr{K}_\nu(y)) = -y^\nu \mathscr{K}_{\nu-1}(y),$$

it follows that $g \in C_b^1([M, \infty))$ and

$$g(M) = 1, \tag{6.45}$$

$$g(y) \ge 0 \qquad \text{for all } y \ge M, \tag{6.46}$$

$$g'(y) = g'(K) = 0 \qquad \text{for all } y \ge K, \tag{6.47}$$

$$g(y) = g(K) \qquad \text{for all } y \ge K. \tag{6.48}$$

From (6.42) and (6.48) we conclude that

$$v_{\epsilon K} \cdot \nabla f = 0 \quad \text{on} \quad \partial U_K \cap \{z \in S : \Psi(z) \ge M\}. \tag{6.49}$$

Moreover, by the definitions (2.4) and (2.7) of Φ and Ψ, on $\{z \in U_K : M < \Psi(z) < K \text{ or } K < \Psi(z)\}$ we have

$$\begin{aligned} \Delta f(z) &= g''(\Psi(z)) |\nabla \Psi(z)|^2 + g'(\Psi(z)) \Delta \Psi(z) \\ &= \begin{cases} g''(\Psi(z))(\Psi(z))^{2-2/\alpha} 2q(z) & \text{if } 0 < \alpha < 2 \\ (g''(\Psi(z)) + (\Psi(z))^{-1} g'(\Psi(z))) 2q(z) & \text{if } \alpha = 0. \end{cases} \end{aligned} \tag{6.50}$$

By combining (6.50) with (6.43)–(6.44), it follows that for $z \in U_K$:

$$\tfrac{1}{2} \Delta f(z) = \begin{cases} (fq)(z) & \text{for } M < \Psi(z) < K \\ 0 & \text{for } \Psi(z) > K. \end{cases} \tag{6.51}$$

Fix $x \in \{z \in \bar{U}_K : \Psi(z) \ge M\}$. Although f is not twice continuously differentiable across $\{z \in U_K : \Psi(z) = K\}$, the submartingale property (iii)$'$ of $P_x^{\epsilon K}$, with $t \wedge \tau_M$ and $1_{U_K \setminus \{\Psi = K\}}$ respectively in place of t and $1_{U_{\epsilon K}}$, does hold for f. This can be verified by a standard approximation argument, using suitable approximations $f_n \in C_b^2(\mathbb{R}^2)$ to f, together with Doob's stopping theorem and the property:

$$E^{P_x^{\epsilon K}}\left[\int_0^{\tau_M} 1_{U_K \cap \{\Psi = K\}}(w(s))\, ds\right] = 0.$$

The latter is a consequence of the fact that under $P_x^{\epsilon K}$, inside U_K, $w(\cdot)$ behaves like Brownian motion up to the time τ_M (cf. [7] or [8, Lemma 3.3]). Thus, in view of (6.49), we have

$$m(t) \equiv f(w(t \wedge \tau_M)) - \frac{1}{2}\int_0^{t\wedge\tau_M} (1_{U_{\varepsilon K}\setminus\{\Psi=K\}}\Delta f)(w(s))\, ds \tag{6.52}$$

is a $P_x^{\varepsilon K}$-martingale. Let

$$b(t) = \exp\left(-\int_0^{t\wedge\tau_M} q(w(s))\, ds\right). \tag{6.53}$$

Since $q \geq 0$ on $S\setminus\{0\}$, b is a bounded, decreasing process on $(C_S, \mathscr{M}, P_x^{\varepsilon K})$. By applying the product formula of stochastic calculus to $b(t)f(w(t \wedge \tau_M))$ and using (6.52), we obtain $P_x^{\varepsilon K}$-a.s. for all $t \geq 0$:

$$\begin{aligned} &b(t)f(w(t\wedge\tau_M)) - f(x) \\ &\quad= \int_0^{t\wedge\tau_M} b(s)\, dm(s) + \frac{1}{2}\int_0^{t\wedge\tau_M} b(s)(1_{U_{\varepsilon K}\setminus\{\Psi=K\}}\Delta f)(w(s))\, ds \\ &\qquad - \int_0^{t\wedge\tau_M} f(w(s))q(w(s))b(s)\, ds. \end{aligned} \tag{6.54}$$

The stochastic integral with respect to $dm(s)$ in (6.54) is a $P_x^{\varepsilon K}$-martingale and the sum of the terms following it is nonpositive, by (6.51) and the nonnegativity of f, q and b. Hence, $b(t)f(w(t \wedge \tau_M))$ is a $P_x^{\varepsilon K}$-supermartingale and therefore

$$E^{P_x^{\varepsilon K}}[b(t)f(w(t \wedge \tau_M))] \leq f(x).$$

The desired result (6.39) follows from this and (6.45)–(6.46). □

It is readily verified that for q given by (6.40),

$$q_0 \equiv \sup_{(r,\theta)\in S\setminus\{0\}} q(r, \theta) < \infty.$$

Hence, it follows from Lemma 6.2 and (6.48) that for each $t \geq 0$ and $x \in \{z \in \bar{U}_K: \Psi(z) \geq K\}$:

$$P_x^{\varepsilon K}(\tau_M \leq t) \leq e^{q_0 t} g(K). \tag{6.55}$$

The asymptotic properties of the modified Bessel functions are such that, as $y \to \infty$,

$$\mathscr{I}_\gamma(y) \sim \frac{e^y}{\sqrt{2\pi y}}\{1 + O(y^{-1})\}$$

$$\mathscr{K}_\gamma(y) \sim \sqrt{\frac{\pi}{2y}}\, e^{-y}\{1 + O(y^{-1})\}.$$

Hence, as $K \to \infty$,

$$g(K) = \begin{cases} O(K^{(1-1/\alpha)/2}\exp(-\alpha K^{1/\alpha})) & \text{if} \quad 0 < \alpha < 2 \\ O(K^{-1/2}e^{-K}) & \text{if} \quad \alpha = 0. \end{cases} \tag{6.56}$$

The final estimate that we need is:

$$\mu(\bar{U}_K) = \int_{\bar{U}_K} p(x)\, dx \le \int_{\bar{U}_K} r^{-\alpha} r\, dr\, d\theta \tag{6.57}$$
$$\le \begin{cases} \xi((K+1)C_\alpha)^{(2-\alpha)/\alpha}/(2-\alpha) & \text{if } 0<\alpha<2 \\ \tfrac{1}{2}\xi(K+1)^2 e^{2\xi|\tan\theta_1|} & \text{if } \alpha = 0, \end{cases}$$

where

$$C_\alpha \equiv (\min_{\theta\in[0,\xi]}\cos(\alpha\theta-\theta_1))^{-1}.$$

This follows from the definitions of $\bar{U}_K$ and Ψ, because $z = (r, \theta) \in \bar{U}_K$ implies $\Psi(z) \le K + 1$ and consequently

$$r^\alpha \le (K+1)C_\alpha \quad \text{if} \quad 0<\alpha<2$$

or

$$r \le (K+1)e^{\xi|\tan\theta_1|} \quad \text{if} \quad \alpha = 0.$$

By combining (6.55), (6.56), and (6.57), we see that (6.38) is dominated by

$$\mu(\bar{U}_K)e^{q_0 t}g(K)\,\|h\|,$$

and that this tends to zero as $K \to \infty$. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA-SAN DIEGO
LA JOLLA, CALIFORNIA 92093