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# APPROXIMATIONS OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY SEMIMARTINGALES[1]

BY PHILIP PROTTER

*Purdue University*

The classical results on the instability of the solutions of stochastic differential equations are extended in two directions: the coefficients are allowed to depend on the paths of the solutions, and arbitrary semimartingales (not simply continuous ones) are allowed as differentials. This extends the results of Wong and Zakai, McShane, Nakao and Yamato, etc.

**1. Introduction.** The instability of the solutions of stochastic differential equations due to the Itô calculus has been of interest to researchers since Wong and Zakai first discoverd it in 1965 [18]. It inspired McShane to develop his integral [10] and it has justified a continued interest in the Stratonovich integral (where the rules of classical calculus apply), which has been especially popular with engineers. Recent work with the stochastic calculus on manifolds, moreover (e.g., [14]), has shown the Stratonovich calculus to have a natural importance for stochastic differential equations.

The "classical" result, due to Nakao and Yamato [15] for continuous semimartingale differentials, can be stated in simplified form as follows:

$$dX_t^n = f(t, X_t^n, Z_t^n)dZ_t^n \tag{1.1}$$

$$dX_t = f(t, X_t, Z_t) \circ Z_t \tag{1.2}$$

$$dX_t = f(t, X_t, Z_t)dZ_t + \tfrac{1}{2}\{(f)(\partial f/\partial x) + (\partial f/\partial z)\}(t, X_t, Z_t)d[Z^c, Z^c]_t \tag{1.3}$$

where $Z^n$ are piecewise $\mathscr{C}^1$ approximations of $Z$; the small circle $\circ$ denotes the Stratonovich integral, with the standard notation meaning the Itô integral. As $Z^n$ tends to $Z$, the $X^n$ of (1.1) tend to the solution $X$ of (1.2), which is equal to the $X$ of (1.3) by the well-known relations between the Itô and Stratonovich integrals. Note that if $Z$ itself is $\mathscr{C}^1$ (or even simply of bounded variation), then $Z^c = 0$ and this result is no longer surprising.

We extend here Nakao and Yamato's result in two directions. Coefficients are allowed to depend on the paths of the solutions, and arbitrary semimartingales are permitted as differentials. The limiting equation remains that of the Stratonovich differential, giving it an enlarged usefulness. Our theorems ((3.1), (4.2), and (5.3)) are stated, however, in the notation and form of the semimartingale integral as presented in [14] or [2].

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Allowing general semimartingales (i.e., those with jumps) as differentials is presented here for the first time. Differentials with jumps have been considered previously: Marcus [11] considers them, but his hypotheses require that there be only a finite number of jumps on bounded intervals; Kushner [9] makes the same restrictions while considering a related problem, with a different perspective. These restrictions avoid the interesting (and only difficult) case where the semimartingale contains a "purely discontinuous" local martingale component with paths of infinite variation on compacts. Since the semimartingale differential $M$ in general contains two terms with paths of infinite variation (a continuous local martingale $N^c$ and a "purely discontinuous" local martingale $N^d$), it is desirable to approximate both $N^c$ and $N^d$ simultaneously by processes with paths of finite variation, and also to determine the limiting equation. This is the content of Theorem 5.3.

Coefficients that depend on the paths of the solutions have been useful in applications and date back to Itô and Nisio's classic paper [7]. In our context, pioneering work was done by Wong and Zakai [19], and more recently Doss and Priouret [3] have considered a particular coefficient that depends in some way on the paths of the solution. The case for continuous semimartingale differentials, with the technical restrictions imposed by Nakao and Yamato [15] having been removed, is presented in Theorems 3.1 and 4.2.

Care is taken to prove the results for systems and for an arbitrary number of semimartingale differentials, since historically some techniques did not work for systems [18], and others did not work for arbitrary numbers of differentials [17]. The recent book of Ikeda and Watanabe [6] presents the classical result for systems with simple coefficients and Brownian and Lebesgue differentials; it together with the article of Marcus [11] provides a bibliography of past work in this area. Between the first and second versions of this article, the results of Konecny [8] have appeared. He removes the technical restrictions on the continuous semimartingale differentials imposed by Nakao and Yamato, and he also considers semimartingale differentials with jumps, but only as in [11], where at most a finite number of jumps are permitted on compact time intervals.

**2. Preliminaries.** Throughout this article there will be a fixed underlying probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is $P$-complete, and there will be a right continuous filtration $(\mathscr{F}_t)_{t\geq 0}$, with $\mathscr{F}_0$ containing all $P$-null sets (the "usual hypotheses"). Semimartingales will always be taken to be right continuous. The notations and general assumed knowledge will be that of the book by Dellacherie and Meyer [2]; however for the more technical and less well-known results that we need, specific references with page numbers will be given.

DEFINITION 2.1. A process $V$ will be said to be an $FV$ process if it is adapted and if it a.s. has right continuous paths of finite variation on compacts.

A process $V$ will be called a *raw FV process* if the requirement that $V$ be adapted is dropped.

We let $\int_0^t |dV_s|$ and/or $|V|_t$ denote the total variation of the paths of $V$ on $[0, t]$. The $\mathscr{H}^P$ norm of an $FV$ process (or even a raw $FV$ process) is simply $\| \int_0^\infty |dV_s| \|_{L^p}$, but we will also use freely the notion of $\mathscr{H}^p$ semimartingales, $p \geq 1$ (cf. [2, page 30]).

DEFINITION 2.2. For $p \geq 1$, we denote by $\mathscr{R}^p$ (resp. raw $\mathscr{R}^p$) all adapted (resp. measurable) processes $H$ indexed by $[0, \infty[$, having right continuous paths with left limits, having a limit at $\infty$, and such that

$$\| H \|_{\mathscr{R}^p} = \| H^* \|_{L^p} < \infty,$$

where $H^* = \sup_t |H_t|$.

$\mathscr{R}^0$ (resp. raw $\mathscr{R}^0$) will denote finite valued adapted (resp. measurable) processes of the above type.

The question of existence and uniqueness of solutions of stochastic differential equations with semimartingale differentials is by now well studied. Emery [4] has established the existence and uniqueness of a solution of:

$$X_t = H_t + \int_0^t F(X)_{s-} \, dZ_s$$

where $H \in \mathscr{R}^0$, $Z$ is a semimartingale, and $F \in \mathrm{Lip}(K)$, which is defined as follows:

DEFINITION 2.3. An operator $F$ mapping $\mathscr{R}^0$ processes into itself is said to be in $\mathrm{Lip}(K)$ for a constant $K$ if the following two conditions hold: for $X, Y \in \mathscr{R}^0$,

(i) $X^{T-} = Y^{T-}$ implies $F(X)^{T-} = F(Y)^{T-}$ for any stopping time $T$;
(ii) $(FX - FY)_t^* \leq K(X - Y)_t^*$, $0 < t < \infty$, (where $X_t^* = \sup_{s<t} |X_s|$).

Emery's condition is not the most general one known, but it is a particularly simple one and sufficient for our purposes. We now introduce a new type of operator. Let $\mathscr{S}^d$ denote all processes $X = (X^1, \cdots, X^d)$ where each $X^i$ is a semimartingale. Let $\mathscr{V}^d$ (resp. raw $\mathscr{V}^d$) denote all processes $V = (V^1, \cdots, V^d)$ where each $V^i$ is an $FV$ process (resp. raw $FV$ process).

DEFINITION 2.4. An operator $\Lambda$ mapping $\mathscr{S}^d \oplus$ raw $\mathscr{V}^d$ into raw $\mathscr{V}^1$ processes will be said to be an $FV$ operator if the following three conditions are satisfied:

(i) $\Lambda$ restricted to $\mathscr{S}^d$ is in $\mathrm{Lip}(K)$, for some constant $K$.
(ii) If $(\mathbf{X}^n)$, $\mathbf{X}$ are in $\mathscr{S}^d \oplus$ raw $\mathscr{V}^d$, and $\lim_{n\to\infty} \| X^n - X \|_{\mathscr{R}^2} = 0$, then $\lim_{n\to\infty} |\Lambda X^n|_t = |\Lambda X|_t$ a.s., $0 < t < \infty$.
(iii) If $X \in \mathscr{S}^d$, then $\Lambda X \in \mathscr{V}^1$. (i.e., $X$ adapted implies $\Lambda X$ adapted).

Examples of $FV$ operators include:

(2.5) The *Itô-Nisio operator* [7]: for a diffuse measure $\mu$ finite on compacts and a Lipschitz function $g$, define

$$\Lambda X_t = \int_0^t g(X_s)\mu(ds).$$

(2.6) The operator $\Lambda X_t = X_t^*$.

(2.7) The operator of Doss and Priouret [3]:

$$\Lambda X_t = X_t - \inf_{s \le t}(X_s \wedge 0), \quad \text{for } X \text{ continuous.}$$

We are now ready to describe the coefficients we will use in our differential equations.

(2.8) Let $f: \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}$ be such that for $f(t, \omega, \mathbf{x}, \mathbf{z}, \lambda)$:

(i) $f$, its first-order partials, and the first-order partials of $\partial f/\partial x^\alpha$ and $\partial f/\partial z^i$ ($1 \le \alpha \le d$; $1 \le i \le r$) are all bounded on $[0, t]$, $0 < t < \infty$.
(ii) There exists a constant $K$ such that

$$\max\left\{ \left| f(t, \omega, \mathbf{x}, \mathbf{z}, \lambda) - f(s, \omega, \mathbf{y}, \mathbf{w}, \mu) \right|, \left| \frac{\partial f}{\partial x^\alpha}(t, \omega, \mathbf{x}, \mathbf{z}, \lambda) - \frac{\partial f}{\partial x^\alpha}(s, \omega, \mathbf{y}, \mathbf{w}, \mu) \right|, \left| \frac{\partial f}{\partial z^i}(t, \omega, \mathbf{x}, \mathbf{z}, \lambda) - \frac{\partial f}{\partial z^i}(s, \omega, \mathbf{y}, \mathbf{w}, \mu) \right| \right\}$$
$$\le K\{|t - s| + \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{z} - \mathbf{w}\| + |\lambda - \mu|\}. \quad (1 \le \alpha \le d; 1 \le i \le r).$$

We will call the above condition (2.8).

DEFINITION 2.9. $F$ is called an acceptable coefficient if there exists

(i) an $f = (f_i^\alpha)_{1 \le \alpha \le d; 1 \le i \le r}$ satisfying condition (2.8);
(ii) an $FV$ operator $\Lambda$ as defined in (2.4); and if $X \in \mathcal{S}^d \oplus \text{raw } \mathcal{V}^d$, $Z \in \mathcal{S}^r \oplus \text{raw } \mathcal{V}^r$,

then the following holds:

$$F(\mathbf{X}, \mathbf{Z})_s = f(s, \omega, \mathbf{X}_{s-}, \mathbf{Z}_{s-}, \Lambda(\mathbf{X})_{s-}). \tag{2.10}$$

When $\mathbf{Z}$ is fixed during a discussion, we will sometimes write $F(\mathbf{X})$ instead of $F(\mathbf{X}, \mathbf{Z})$.

DEFINITION 2.11. Let $F_i$ ($1 \le i \le r$) be acceptable coefficients as defined in (2.9), and let $f_i^\alpha$ be their associated functions (cf. (2.10)). For $X \in \mathcal{S}^d \oplus \text{raw } \mathcal{V}^d$, $Z \in \mathcal{S}^r \oplus \text{raw } \mathcal{V}^r$, let $G_{ij}^\alpha$ be given by:

$$G_{i,j}^\alpha(\mathbf{X}, \mathbf{Z})_s = \sum_{\beta=1,d} \left\{ f_j^\beta(s, \cdots) \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots) + \frac{\partial f_i^\alpha}{\partial z^j}(s, \cdots) \right\} \tag{2.12}$$

where $(s, \cdots)$ denotes $(s, \omega, \mathbf{X}_{s-}, \mathbf{Z}_{s-}, \Lambda(\mathbf{X})_{s-})$. Then we call $G_{ij}^\alpha$ the associated coefficients of $(F_i)$. When $\mathbf{Z}$ is fixed during a discussion, we sometimes write $G^{\alpha}_{ij}(\mathbf{X})_s$ for $G^{\alpha}_{i,j}(\mathbf{X}, \mathbf{Z})_s$.

PROPOSITION 2.13a. *Let $\mathbf{Z} \in \mathscr{S}^r$, $\mathbf{M} \in \mathscr{S}^r$ and let $F_i$ be acceptable coefficients, as defined in* (2.9). *Let $G^{\alpha}_{i,j}$ be the associated coefficients of $F_i$, and let $\mathbf{V} \in \mathscr{V}^r$. Then the system of equations* $(1 \le \alpha \le d)$

$$X^{\alpha}_t = x^{\alpha} + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F^{\alpha}_i(\mathbf{X}, \mathbf{Z})_s \, dM^{\alpha}_s + \textstyle\sum_{i,j=1,r} \displaystyle\int_0^t G^{\alpha}_{i,j}(\mathbf{X}, \mathbf{Z})_s \, dV^{\alpha}_s$$

*has a solution, and it is unique in $\mathscr{R}^0$.*

PROPOSITION 2.13b. *Let $\mathbf{Z} \in$ raw $\mathscr{V}^r$, $\mathbf{M} \in$ raw $\mathscr{V}^r$, and let $F$ be an acceptable coefficient, as defined in* (2.9). *Then the system of equations* $(1 \le \alpha \le d)$

$$X^{\alpha}_t = x^{\alpha} + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F^{\alpha}_i(\mathbf{X}, \mathbf{Z})_s \, dM^{\alpha}_s$$

*has a solution, and it is unique in* raw $\mathscr{R}^0$.

PROOFS. Proposition 2.13a is an immediate consequence of Emery's theorem (see the description preceding (2.3)). As for Proposition 2.13b, change the filtration to the trivial one $\mathscr{F}_t = \mathscr{F}$, for $t \ge 0$, and now $\mathbf{Z}$, $\mathbf{M}$ are both in $\mathscr{V}^r \subseteq \mathscr{S}^r$, and thus Proposition 2.13b is a corollary of Proposition 2.13a. □

DEFINITION 2.14. Given a sequence $(\Pi^n)$ of refining partitions of $[0, \infty[$ such that $\lim_{n\to\infty}\text{mesh}(\Pi^n) = 0$, and a continuous semimartingale $Z$ with $Z_0 = 0$, we define the $n$th polygonal approximation of $Z$ to be

$$Z^{(n)}_t = Z_{t_\nu} + ((Z_{t_{\nu+1}} - Z_{t_\nu})/(t_{\nu+1} - t_\nu))(t - t_\nu)$$

when $t_\nu \le t < t_{\nu+1}$ *and* $t_\nu$, $t_{\nu+1}$ are in $\Pi^n$. We write $\Delta^\nu Z$ for $Z_{t_{\nu+1}} - Z_{t_\nu}$ and $\Delta^\nu t$ for $t_{\nu+1} - t_\nu$. Also, we let $\Pi^n_t$ denote the restriction of the partition $\Pi^n$ to $[0, t]$.

**3. Continuous case: polygonal approximations.** This section begins with a statement of our main theorem when the semimartingale differentials have continuous paths and are approximated polygonally (cf. Definition 2.1).

Let $\Pi^n$ be a sequence of refining partitions of $[0, \infty)$ such that $\lim_{n\to\infty}\text{mesh}(\Pi^n) = 0$.

THEOREM 3.1. *Let $Z^i$, $1 \le i \le r$, be continuous semimartingales, $Z^i_0 = 0$; let $F^{\alpha}_i$ $(1 \le \alpha \le d)$ be acceptable coefficients as in* (2.9); *and let $Z^{(n)i}$ denote the* $n$th *polygonal approximation of $Z^i$ (cf.* (2.14)). *Let $\mathbf{x} = (x^{\alpha})_{1 \le \alpha \le d}$ be a point in $\mathbb{R}^d$, and let $\mathbf{X}^{(n)}_t$ denote the solution of* $(1 \le \alpha \le d)$:

$$X^{(n)\alpha}_t = x^{\alpha} + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F^{\alpha}_i(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)})_s \, dZ^{(n)i}_s. \tag{3.2}$$

*Further, let* $G^{\alpha}_{i,j}$ *be the associated coefficients* (2.12) *and let* $\mathbf{X}_t$ *denote the solution of* $(1 \le \alpha \le d)$;

$$(3.3)\quad X^{\alpha}_t = x^{\alpha} + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F^{\alpha}_i(\mathbf{X}, \mathbf{Z})_s \, dZ^i_s + \frac{1}{2}\textstyle\sum_{i,j=1,r} \displaystyle\int_0^t G^{\alpha}_{i,j}(\mathbf{X}, \mathbf{Z})_s \, d[Z^{i,c}, Z^{j,c}]_s.$$

*Then* $\mathbf{X}^{(n)}$ *converges in probability to* $\mathbf{X}$ *uniformly on compacts.*

While Theorem 3.1 is the chief object of interest in this section, we will prove instead the following theorem, which involves more work but is also an essential lemma for the case of arbitrary (i.e., right continuous) semimartingale differentials considered in Section 4. Theorem 3.1 is clearly a corollary of Theorem 3.4, since one need only take $V^i_s \equiv 0$, $1 \le i \le r$.

THEOREM 3.4. *Let* $V^i$, $1 \le i \le r$, *be FV processes (cf.* (2.1)). *Let the hypotheses of Theorem* 3.1 *hold with* $M^{(n)i} = Z^{(n)i} + V^i$; $M^i = Z^i + V^i$, *any* $\mathbf{Y}^{(n)}$ *denote the solution of* $(1 \le \alpha \le d)$:

$$(3.4a)\quad Y^{(n)\alpha}_t = x^{\alpha} + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F^{\alpha}_i(\mathbf{Y}^{(n)}, \mathbf{Z}^{(n)})_s \, dM^{(n)i}_s.$$

*Further, let* $\mathbf{Y}_t$ *denote the solution of* $(1 \le \alpha \le d)$:

$$(3.4b)\quad Y^{\alpha}_t = x^{\alpha} + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F^{\alpha}_i(\mathbf{Y}, \mathbf{Z})_s \, dM^i_s + \frac{1}{2}\textstyle\sum_{i,j=1,r} \displaystyle\int_0^t G^{\alpha}_{i,j}(\mathbf{Y}, \mathbf{Z})_s \, d[M^{i,c}, M^{j,c}]_s.$$

*Then* $\mathbf{Y}^{(n)}$ *converges in probability to* $\mathbf{Y}$ *uniformly on compacts.*

Before formally beginning the proof of Theorem 3.4, we state and prove a sequence of lemmas used in the proof. Several "well-known" technical results are needed, and the reader is referred to the excellent book of Dellacherie and Meyer [2]; we give the appropriate pages, when possible. The proof of Theorem 3.4 follows Corollary 3.41.

LEMMA 3.5. *If Theorem* 3.4 *holds for continuous semimartingales* $Z^i$ $(1 \le i \le r)$ *in* $\mathscr{H}^4$, *and FV processes* $V^i$ $(1 \le i \le r)$ *in* $\mathscr{H}^4$, *then it holds as well for arbitrary continuous semimartingales* $Z^i$ *and arbitrary processes* $V^i$ $(1 \le i \le r)$.

PROOF. A refinement of Stricker's Theorem ([2], no. 63 bis, pages 271–272) lets us change to an equivalent probability measure, $Q$, if necessary, so that the $Z^i$ and $V^i$ of Theorem 3.4 become, respectively, $\mathscr{H}^4$ continuous semimartingales and $\mathscr{H}^4$ $FV$ processes under $Q$ $(1 \le i \le r)$. Since the stochastic calculus is invariant under a change to an equivalent probability ([2, page 338]), and since convergence in probability is also invariant under such a change, the lemma is proved. □

At this stage, we need to make a technically simplifying assumption on the operator $\Lambda$, which we will later remove:

HYPOTHESIS 3.6. There exists an increasing process $K$ in raw $\mathscr{R}^0$ such that, for any process $H \in \mathscr{S}^d \oplus$ raw $\mathscr{V}^d$, one has $|\Lambda H|_t \le K_t$.

Let $\prod_{t_0}^n$ denote the partition $\Pi^n$ restricted to $[0, t_0]$, where $t_0$ is taken to be a partition point.

LEMMA 3.7. *Under the hypotheses and notations of Theorem* 3.4 *and under Hypothesis* 3.6, *and where* $M^{(n)i} = Z^{(n)i} + V^i$, *one has*

$$\begin{aligned}\int_0^t & F_i^\alpha(\mathbf{Y}^{(n)}, \mathbf{Z}^{(n)})_s \, dM_s^{(n)i} \\ &= \textstyle\sum_{t_\nu \in \Pi_t^n} F_i^\alpha(\mathbf{Y}^{(n)}, \mathbf{Z}^{(n)})_{t_\nu} \{\Delta^\nu Z^i + \Delta^\nu V^i\} \\ &\quad + \textstyle\sum_{t_\nu \in \Pi_t^n} \frac{1}{2} \sum_{j=1,r} G_{i,j}^\alpha(\mathbf{Y}^{(n)}, \mathbf{Z}^{(n)})_{t_\nu} \Delta^\nu Z^i \Delta^\nu Z^j + I_t^n,\end{aligned} \tag{3.8}$$

*where* $I_t^n$ *tends to* 0 *in probability uniformly on* $[0, t]$. *If, in addition,* $Z^i$ *and* $V^i$ *are taken to be in* $\mathscr{H}^4$, *then* $I^n$ *tends to* 0 *in* raw $\mathscr{R}^2$.

PROOF. For simplicity, we write $F_i^\alpha(\mathbf{Y}^{(n)})$ for $F_i^\alpha(\mathbf{Y}^{(n)}, \mathbf{Z}^{(n)})$.

$$\begin{aligned}\int_0^t & F_i^\alpha(\mathbf{Y}^{(n)})_s \, dV_s^i \\ &= \textstyle\sum_{t_\nu \in \pi_t^n} F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu} \Delta^\nu V^i + \sum_\nu \displaystyle\int_{t_\nu}^{t_{\nu+1}} [F_i(\mathbf{Y}^{(n)})_s - F_i(\mathbf{Y}^{(n)})_{t_\nu}] \, dV_s^i,\end{aligned} \tag{3.9}$$

where, in general,

$$\int_s^t dV_u \quad \text{denotes} \quad \int_{]s,t]} dV_u.$$

Also

$$\begin{aligned}|F_i^\alpha & (\mathbf{Y}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}| \\ &\le K_1\{|s - t_\nu| + 2\|\mathbf{Y}_\cdot^{(n)} - \mathbf{Y}_{\cdot\wedge t_\nu}^{(n)}\|_s^* + \|\mathbf{Z}_\cdot^{(n)} - \mathbf{Z}_{\cdot\wedge t_\nu}^{(n)}\|_s^*\} \\ &\le K_2\{|s - t_\nu| + \|\mathbf{Z}_\cdot - \mathbf{Z}_{\cdot\wedge t_\nu}\|_{t_{\nu+1}}^* + \|\mathbf{V}_\cdot - \mathbf{V}_{\cdot\wedge t_\nu}\|_{t_{\nu+1}}^*\},\end{aligned} \tag{3.10}$$

where in the second inequality we used that $\mathbf{Y}^{(n)}$ is the solution of equation (3.4a) and each $F_i^\alpha$ is bounded on compacts. Since the last term in (3.10) tends to 0 a.s. as mesh$(\Pi^n)$ tends to 0, the dominated convergence theorem tells us that the second term on the right in (3.9) tends to 0 a.s., and hence in probability, as $n \to \infty$.

By the above, we need only verify (3.8) for the continuous semimartingale approximations. *For ease of notation, let us fix a* $\nu$, *and set* $a = t_\nu$ *and* $b = t_{\nu+1}$. Then

$$\int_a^b F_i^\alpha(\mathbf{Y}^{(n)})_s \, dZ_s^{(n)i} = \frac{\Delta^\nu Z^i}{\Delta^\nu t} \int_a^b F_i^\alpha(\mathbf{Y}^{(n)})_s \, ds; \tag{3.11}$$

consider $\int_a^b F_i^\alpha(\mathbf{Y}^{(n)})_s\, ds$ and integrate by parts:

$$\text{(3.12)} \qquad \int_a^b F_i^\alpha(\mathbf{Y}^{(n)})_s\, ds$$

$$= \mathbf{F}_i^\alpha(\mathbf{Y}^{(n)})_a(b-a)$$

$$\text{(3.12)(i)} \qquad + \int_a^b (b-s)\frac{\partial f_i^\alpha}{\partial t}(s, \cdots)\, ds$$

$$\text{(3.12)(ii)} \qquad + \textstyle\sum_{\beta=1,d} \displaystyle\int_a^b (b-s)\frac{\partial f_i^\alpha}{\partial y^\beta}(s, \cdots)\left\{\textstyle\sum_{j=1,r} f_j^\beta(s, \cdots)\dfrac{\Delta^\nu Z^j}{\Delta^\nu t}\right\} ds$$

$$\text{(3.12)(iii)} \qquad + \textstyle\sum_{j=1,r} \displaystyle\int_a^b (b-s)\frac{\partial f_i^\alpha}{\partial z^j}(s, \cdots)\frac{\Delta^\nu Z^j}{\Delta^\nu t}\, ds$$

$$\text{(3.12)(iv)} \qquad + \int_b^a (b-s)\frac{\partial f_i^\alpha}{\partial \lambda}(s, \cdots)\, d\Lambda(\mathbf{Y}^n)_s,$$

where of course $(s, \cdots)$ denotes the argument $(s, \omega, \mathbf{Y}_s^{(n)}, \mathbf{Z}_s^{(n)i}, \Lambda(\mathbf{Y}^n)_s)$.

Consider first (3.12)(i):

$$\left| \textstyle\sum_{t_\nu\in\pi^n} \dfrac{\Delta^\nu Z^i}{\Delta^\nu t}\displaystyle\int_{t_\nu}^{t_{\nu+1}} (t_{\nu+1}-s)\frac{\partial f_i^\alpha}{\partial t}(s, \cdots)\, ds \right|$$

$$\le K(\omega)\sup_\nu|\Delta^\nu Z^i| \textstyle\sum_\nu \dfrac{1}{\Delta^\nu t}\displaystyle\int_{t_\nu}^{t_{\nu+1}} (t_{\nu+1}-s)\, ds$$

$$\le K(\omega)\sup_\nu|\Delta^\nu Z^i|\, t,$$

where $K(\omega)$ bounds $\partial f_i^\alpha/\partial s$ on $[0, t]$; the above tends to 0 a.s.

Consider next (3.12)(iv):

$$\left| \textstyle\sum_{t_\nu\in\pi^n} \dfrac{\Delta^\nu Z^i}{\Delta^\nu t}\displaystyle\int_{t_\nu}^{t_{\nu+1}} (t_{\nu-1}-s)\frac{\partial f_i^\alpha}{\partial \lambda}(s, \cdots)\, d\Lambda(\mathbf{Y}^{(n)})_s \right|$$

$$\le K(\omega)\sup_\nu|\Delta^\nu Z^i| \int |d\Lambda\mathbf{Y}_s^{(n)}|,$$

which also tends to 0 a.s. as $\text{mesh}(\pi^n)$ tends to 0, by Hypothesis 3.6.

Consider next (3.12)(ii): integrating by parts again gives

$$\textstyle\sum_{\beta=1,d} \displaystyle\int_a^b (b-s)\frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\left\{\textstyle\sum_{j=1,r} f_j^\beta(s, \cdots)\dfrac{\Delta^\nu Z^j}{\Delta^\nu t}\right\} ds$$

$$\text{(3.13)} \qquad = \textstyle\sum_{\beta=1,d}\left\{\dfrac{(b-a)^2}{2}\dfrac{\partial f_i}{\partial x^\beta}(a, \cdots)\textstyle\sum_{j=1,r} f_j^\beta(a, \cdots)\dfrac{\Delta^\nu Z^j}{\Delta^\nu t}\right\}$$

$$+ \textstyle\sum_{\beta=1,d}\displaystyle\int_a^b \frac{(b-s)^2}{2}\, d\left[\frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\left\{\textstyle\sum_{j=1,r} f_j^\beta(s, \cdots)\dfrac{\Delta^\nu Z_j^i}{\Delta^\nu t}\right\}\right]$$

and multiplying by $\Delta^\nu Z^i/\Delta^\nu t$ (from (3.11)), (3.13) becomes, letting $\{s, xxx\} := \{\sum_{j=1,r} f_j^\beta(s, \cdots)\}$ and *continuing with the notational convention that $\nu$ is fixed and*

$a = t_\nu$; $b = t_{\nu+1}$:

$$
\begin{aligned}
&\frac{\Delta^\nu Z^i \Delta^\nu Z^j}{2} \textstyle\sum_{\beta=1,d} \dfrac{\partial f_i^\alpha}{\partial x^\beta}(a, \cdots)\, \{\textstyle\sum_{j=1,r} f_j^\beta(a, \cdots)\} \\
&\quad + \frac{\Delta^\nu Z^i \Delta^\nu Z^j}{(\Delta^\nu t)^2} \textstyle\sum_{\beta=1,d} \displaystyle\int_a^b \frac{(b-s)^2}{2} \left[ \frac{\partial^2 f_i^\alpha}{\partial x^\beta \partial s}(s, \cdots)\{s, xxx\} \right. \\
&\qquad\qquad \left. + \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\, \{\textstyle\sum_{j=1,r} \dfrac{\partial f_j^\beta}{\partial s}(s, \cdots)\} \right] ds \\
&\quad + \frac{\Delta^\nu Z^i \Delta^\nu Z^j}{(\Delta^\nu t)^2} \textstyle\sum_{\beta=1,d} \displaystyle\int_a^b \frac{(b-s)^2}{2} \left[ \textstyle\sum^{\gamma=1,d} \dfrac{\partial^2 f_i^\alpha}{\partial x^\beta \partial x^\gamma}(s, \cdots)\{s, xxx\} \right. \\
&\qquad\qquad \left. + \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\left\{ \textstyle\sum_{j=1,r} \dfrac{\partial f_j^\beta}{\partial x^\gamma}(s, \cdots) \right\} \right] dY_s^{(n),\gamma} \qquad (3.14) \\
&\quad + \frac{\Delta^\nu Z^i \Delta^\nu Z^j}{(\Delta^\nu t)^2} \textstyle\sum_{\beta=1,d} \displaystyle\int_b^a \frac{(b-s)^2}{2} \left[ \textstyle\sum_{k=1,r} \dfrac{\partial^2 f_i^\alpha}{\partial x^\beta \partial z^k}(s, \cdots)\{s, xxx\} \right. \\
&\qquad\qquad \left. + \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\left\{ \textstyle\sum_{j=1,r} \dfrac{\partial f_j^\beta}{\partial z^k}(s, \cdots) \right\} \right] dZ_s^{(n)k} \\
&\quad + \frac{\Delta^\nu Z^i \Delta^\nu Z^j}{(\Delta^\nu t)^2} \textstyle\sum_{\beta=1,d} \displaystyle\int_a^b \frac{(b-s)^2}{2} \left[ \frac{\partial^2 f_i^\alpha}{\partial x^\beta \partial \lambda}(s, \cdots)\{s, xxx\} \right. \\
&\qquad\qquad \left. + \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\left\{ \textstyle\sum_{j=1,r} \dfrac{\partial f_j^\beta}{\partial \lambda}(s, \cdots) \right\} \right] d\Lambda \mathbf{Y}_s^{(n)}.
\end{aligned}
$$

Since all the $f_i^\alpha$ and its partials given above are assumed to be bounded on compact time sets, all but the first term in (3.14) above is bounded by

$$
\begin{aligned}
&\frac{|\Delta^\nu Z^i \Delta^\nu Z^j|}{(\Delta^\nu t)^2} \left\{ K_1(\omega)(\Delta^\nu t)^3 + K_2(\omega)(\Delta^\nu t)^3 \right. \\
&\qquad + K_3(\omega)(\Delta^\nu t)^2 \left( \textstyle\sum_{k=1,r} |\Delta^\nu Z^k| + \displaystyle\int_a^b |dV_s^k| \right) \\
&\qquad + K_4(\omega)(\Delta^\nu t)^2 (\textstyle\sum_{k=1,r} |\Delta^\nu Z^k|) \qquad (3.15) \\
&\qquad \left. + K_5(\omega)(\Delta^\nu t)^2 \displaystyle\int_a^b |d\Lambda(\mathbf{Y}^{(n)})_s| \right\}.
\end{aligned}
$$

By Hypothesis 3.6, $\int_0^t |d\Lambda(\mathbf{Y}^{(n)})_s|$ is bounded uniformly in $n$. One easily verifies that (3.15), summed over $\nu$, tends to 0 as mesh $(\Pi^n)$ tends to 0, provided $\sum_{t_\nu \in \Pi^n} |\Delta^\nu Z^i \Delta^\nu Z^j|$ stays bounded in probability. This is proved in Lemma 3.18. Thus, for (3.12)(ii),

$$
\begin{aligned}
&\textstyle\sum_\nu \dfrac{\Delta^\nu Z^i}{\Delta^\nu t} \textstyle\sum_{\beta=1,d} \displaystyle\int_a^b \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots)\left\{ \textstyle\sum_{j=1,r} f_j^\beta(s, \cdots) \dfrac{\Delta^\nu Z^j}{\Delta^\nu t} \right\} ds \\
&\qquad = \textstyle\sum_\nu \dfrac{\Delta^\nu Z^i \Delta^\nu Z^j}{2} \textstyle\sum_{\beta=1,d} \dfrac{\partial f_i^\alpha}{\partial x^\beta}(t_\nu, \cdots) \textstyle\sum_{j=1,r} f_j^\beta(t_\nu, \cdots) \qquad (3.16) \\
&\qquad\quad + \textstyle\sum_\nu \{\text{right side of (3.14)}\},
\end{aligned}
$$

where $\sum_\nu$ {right side of (3.14)} $\le \sum_\nu$ (3.15), which tends to 0 in probability as mesh$(\Pi^n) \to 0$. Finally, consider (3.12)(iii); integrating by parts a second time yields

$$\sum_{j=1,r} \int_a^b (b-s) \frac{\partial f_i^\alpha}{\partial z^j}(s, \cdots) \frac{\Delta^\nu Z^j}{\Delta^\nu t}\, ds$$

$$(3.17) \qquad \begin{aligned} &= \sum_{j=1,r} \frac{(b-a)^2}{2} \frac{\Delta^\nu Z^j}{\Delta^\nu t} \frac{\partial f_i^\alpha}{\partial z^j}(a, \cdots) \\ &\quad + \left\{ \sum_{j=1,r} \int_a^b \frac{(b-s)^2}{2} \frac{\Delta^\nu Z^j}{\Delta^\nu t} \left[ \frac{\partial^2 f_i^\alpha}{\partial z^j \partial s}(s, \cdots)\, ds \right.\right. \\ &\qquad\qquad + \sum_{\beta=1,d} \frac{\partial^2 f_i}{\partial z^j \partial x^\beta}(s, \cdots)\, dY_s^{(n)\beta} \\ &\quad \left.\left. + \sum_{k=1,d} \frac{\partial^2 f_i^\alpha}{\partial z^j \partial z^k}(s, \cdots)\, dZ_s^{(n)k} + \frac{\partial^2 f_i}{\partial z^j \partial \lambda}(s, \cdots) \right] d\Lambda(\mathbf{Y}^{(n)})_s \right\}. \end{aligned}$$

One can show by an argument exactly analogous to the preceding one (summarized in (3.16)) that the terms contained in the brackets "{ }" on the right side of (3.17) tend to 0 in probability when multiplied by $\Delta^\nu Z^i/\Delta^\nu t$ and summed over $\nu$, as mesh $(\Pi^n)$ tends to 0. Recalling from (2.12) that

$$G_{i,j}^\alpha(s, \cdots) = \frac{\partial f_i^\alpha}{\partial z^j}(s, \cdots) + \sum_{\beta=1,d} \frac{\partial f_i^\alpha}{\partial x^\beta}(s, \cdots) f_j^\beta(s, \cdots),$$

and noting that $b - a = \Delta^\nu t$ in this context, combining (3.12)(i) through (3.12)(iv) establishes Lemma 3.7. □

During the course of proving Lemma 3.7, we needed the following technical result, established here for completeness.

LEMMA 3.18. *Let $Z^i$ and $Z^j$ be any two semimartingales, and $(\Pi^n)$ a sequence of refining partitions, where* $\lim_{n\to\infty}$mesh$(\Pi^n) = 0$. *Then* $\sum_{t_\nu \in \Pi_t^n} |\Delta^\nu Z^i \Delta^\nu Z^j|$ *stays bounded in probability as* mesh$(\Pi^n)$ *tends to* 0 $(0 < t < \infty)$.

PROOF. "Bounded in probability" means that for any $\varepsilon > 0$ there exists a $K$ such that $\sup_n P(\sum_{t_\nu \in \Pi^n} |\Delta^\nu Z^i \Delta^\nu Z^j| > K) < \varepsilon$. We have $(\sum_\nu |\Delta^\nu Z^i \Delta^\nu Z^j|)^2 \le \sum_\nu (\Delta^\nu Z^i)^2 \sum_\nu (\Delta^\nu Z^j)^2$; but each of $\sum_\nu (\Delta^\nu Z^i)^2$ is convergent in probability (cf. [2, page 344]) and hence bounded in probability. □

LEMMA 3.19. *Let $Z^i$, $1 \le i \le r$ be continuous $\mathscr{H}^4$ semimartingales, and let $V^i$, $1 \le i \le r$, be FV processes which are also $\mathscr{H}^4$. Let the hypotheses and notations of Theorem* 3.4, *and also let Hypothesis* 3.6 *hold. Then there exists a sequence of finite-valued stopping times $(T^k)$ increasing a.s. to $\infty$ such that $(\mathbf{Y}_s^{(n)})^{T_k}$ is Cauchy in $\mathscr{R}^2$.*

PROOF. We first establish some notation. For $n > m$, the partition $\Pi^n$ is a refinement of $\Pi^m$ by hypothesis. For $t_\nu \in \Pi^n$, we write

$$[t_\nu]_m = \sup_{t_\mu \le t_\nu : t_\mu \in \Pi^m}(t_\mu).$$

When $n$ and $m$ are fixed and there is no danger of confusion, we will write $[t_\nu]$ for $[t_\nu]_m$. Thus for $n > m$ fixed, and for general semimartingales $Z$ and processes $H, J$,

$$\textstyle\sum_{t_\nu \in \Pi^n} H_{t_\nu} \Delta^\nu Z - \sum_{t_\mu \in \Pi^m} J_{t_\mu} \Delta^\mu Z = \sum_\nu \{H_{t_\nu} - J_{[t_\nu]}\} \Delta^\nu Z.$$

For $n > m$, by Lemma 3.7 we have

$$\begin{aligned} Y_t^{(n)\alpha} - Y_t^{(m)\alpha} &= \textstyle\sum_{i=1,r} \left\{ \int_0^t F_i^\alpha(\mathbf{Y}^{(n)})_s \, d(Z_s^{(n)i} + V_s^i) \right. \\ &\qquad \left. - \int_0^t F_i^\alpha(\mathbf{Y}^{(m)})_s \, d(Z_s^{(m)i} + V_s^i) \right\} \\ &= \textstyle\sum_{i=1,r} \sum_{\nu \in \Pi_t^n} \{F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(m)})_{[t_\nu]}\}(\Delta^\nu Z^i + \Delta^\nu V^i) \\ &\quad + \tfrac{1}{2} \textstyle\sum_{\nu \in \Pi_t^n} [\sum_{i,j=1,r} \{G_{i,j}^\alpha(\mathbf{Y}^{(n)})_{t_\nu} - G_{i,j}^\alpha(\mathbf{Y}^{(m)})_{[t_\nu]}\} \Delta^\nu Z^i \Delta^\nu Z^j] \\ &\quad + (I^n - I^m). \end{aligned} \tag{3.20}$$

With the added hypothesis that $Z^i$ and $V^i$ are in $\mathscr{H}^4$, we have by Lemma 3.7 that $I^n$ and $I^m$ tend to 0 as $n, m \to \infty$ in raw $\mathscr{R}^2$.

Now let $T$ be any stopping time bounded by a fixed $t_0 (0 < t_0 < \infty)$. Let

$$\| (\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)})_{T-}^* \| = \textstyle\sum_{\alpha=1,d} (Y^{(n)\alpha} - Y^{(m)\alpha})_{T-}^*.$$

Consider

$$\begin{aligned} E\{[(Y^{(n)\alpha} - Y^{(m)\alpha})_{T-}^*]^2\} &\le 5E\{\textstyle\sum_{i=1,r} (\sum_{\Pi_{T-}^n} \{F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(m)})_{[t_\nu]}\} \Delta^\nu Z^i)^{*2}\} \\ &\quad + 5E\{\textstyle\sum_{i=1,r} (\sum_{\Pi_{T-}^n} \{F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(m)})_{[t_\nu]} \Delta^\nu V^i)^{*2}\} \\ &\quad + 5/2E\{\textstyle\sum_{\Pi_{T-}^n} (\sum_{i,j=1,r} \{G_{i,j}^\alpha(\mathbf{Y}^{(n)})_{t_\nu} - G_{i,j}^\alpha(\mathbf{Y}^{(n)})_{[t_\nu]}\} \Delta^\nu Z^i \Delta^\nu Z^j)^{*2}\} \\ &\quad + 5E\{(I^m)^{*2}\} + 5E\{(I^n)^{*2}\}. \end{aligned} \tag{3.21}$$

Introduce the notation:

$$\hat{F}_i^\alpha(\mathbf{Y}^{(n)})_t = F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}, \quad t_\nu < t \le t_{\nu+1},$$

the $t_\nu$ running through the partition $\Pi^n$. Since $Z^i$ is a continuous semimartingale, let $Z^i = M^i + A^i$ denote its unique decomposition into a continuous martingale

$M^i$ and continuous $FV$ process $A^i$. Inequality (3.21) then yields

$$(3.22)\quad \begin{aligned} &E\{[(Y^{(n)\alpha} - Y^{(m)\alpha})^*_{T-}]^2\} \\ &\le 5E\left\{\left(\sum_{i=1,r} \int_0^T \hat{F}_i^\alpha(\mathbf{Y}^{(n)})_s - \hat{F}_i^\alpha(\mathbf{Y}^{(m)})_s\, dZ_s^i\right)^{*2}\right\} \\ &+ 5E\left\{\left(\sum_{i=1,r} \int_0^{T-} \hat{F}_i^\alpha(\mathbf{Y}^{(n)})_s - \hat{F}_i^\alpha(\mathbf{Y}^{(m)})_s\, dV_s^i\right)^{*2}\right\} \\ &+ 5E\left\{\left(\sum_{i,j=1,r} \int_0^T [\hat{G}_{i,j}^\alpha(\mathbf{Y}^{(n)})_s - \hat{G}_{i,j}^\alpha(\mathbf{Y}^{(m)})_s]\, d[M^i, M^j]_s\right)^2\right\} \\ &+ 5E((\sum_{i,j=1,r} \{\hat{G}_{i,j}^\alpha(\mathbf{Y}^{(n)})_s - \hat{G}_{ij}^\alpha(\mathbf{Y}^{(m)})_s\} \\ &\qquad \cdot\{\Delta^\nu M^i \Delta^\nu A^j + \Delta^\nu A^i \Delta^\nu M^j + \Delta^\nu A^i \Delta^\nu A^j\}^{*2}) + \varepsilon_{n,m}. \end{aligned}$$

Consider the second to last term on the right side of (3.22); in general we have, for example,

$$\begin{aligned} |\sum H_s^{(n)} \Delta^\nu M^i \Delta^\nu A^j| &\le \sup_\nu |\Delta^\nu M^i| \; |\sum H_s^{(n)} \Delta^\nu A^j| \\ &\le \sup_{\nu\in\Pi_t^n} |\Delta^\nu M^i| \; \|H^{(n)*}\|_{L^\infty} \int_0^t |dA_s^j| \end{aligned}$$

which tends to 0 a.s. as mesh$(\Pi^n)$ tends to 0. Reasoning analogously for $\Delta^\nu A^i \Delta^\nu M^j$ and $\Delta^\nu A^i \Delta^\nu A^j$, the second to last term in (3.22) tends to 0 by the dominated convergence theorem; of course, $\varepsilon_{n,m} = 5E\{(I^m)^{*2}\} + 5E\{(I^n)^{*2}\}$, which also tends to 0 as $n$, $m$ tend to $\infty$.

By elementary properties of the stochastic integral (cf. [2, 13]) and the Cauchy-Schwarz inequality, (3.22) becomes

$$(3.23)\quad \begin{aligned} &E\{[(Y^{(n)\alpha} - Y^{(m)\alpha})^*_{T-}]^2\} \\ &\le C_1 E\left\{\sum_i \int_0^T [\hat{F}_i^\alpha(\mathbf{Y}^{(n)})_s - \hat{F}_i^\alpha(\mathbf{Y}_s^{(m)})_s]^2\, d[M^i, M^i]_s\right\} \\ &+ C_2 E\left\{\sum_i (|A^i|_T + |V^i|_{T-}) \int_0^{T-} [\hat{F}_i^\alpha(\mathbf{Y}^{(n)})_s - \hat{F}_i^\alpha(\mathbf{Y}^{(m)})_s^*]^2 (|dA_s^i| + |dV_s^i|)\right\} \\ &+ C_3 E\left\{\sum_{i,j} |[M^i, M^j]|_T \int_0^T (\hat{G}_{ij}^\alpha(\mathbf{Y}^{(n)})_s - \hat{G}_{ij}^\alpha(\mathbf{Y}^{(m)})_s^*]^2\, |d[M^i, M^j]_s|\right\} + \varepsilon_{n,m}, \end{aligned}$$

where $\lim_{n,m\to\infty} \varepsilon_{n,m} = 0$. Note that by our hypotheses on the coefficients (cf. (2.8)),

$$(3.24)\quad \begin{aligned} &|\hat{F}_i^\alpha(\mathbf{Y}^{(n)})_s - \hat{F}_i^\alpha(\mathbf{Y}^{(m)})_s|^* \\ &\le K\{|t_\nu - t_\mu| + 2\|\mathbf{Y}_{t_\nu}^{(n)} - \mathbf{Y}_{t_\mu}^{(m)}\| + \|\mathbf{Z}_{t_\nu} - \mathbf{Z}_{t_\mu}\|\} \\ &< 2K\{\delta_m(\omega) + \|\mathbf{Y}_{t_\nu}^{(n)} - \mathbf{Y}_{t_\mu}^{(m)}\|\}, \end{aligned}$$

where $\delta_m = \max\{|t_\nu - t_\mu|, \|\mathbf{Z}_{t_\nu} - \mathbf{Z}_{t_\mu}\|\}$. Continuing with (3.24):

$$(3.25)\quad \begin{aligned} &|\hat{F}_i^\alpha(\mathbf{Y}^{(n)})_s - \hat{F}_i^\alpha(\mathbf{Y}^{(m)})_s|^* \\ &\le 2K\{\delta_m + \|\mathbf{Y}_{t_\nu}^{(n)} - \mathbf{Y}_{t_\nu}^{(m)}\| + \|\mathbf{Y}_{t_\nu}^{(m)} - \mathbf{Y}_{t_\mu}^{(m)}\|\} \\ &\le 2K\{\delta_m + \|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{s-}^* + C\textstyle\sum_i (|Z_{t_\nu}^i - Z_{t_\mu}^i| + |V_{t_\nu}^i - V_{t_\mu}^i|)\} \\ &\le K_1\{\Delta_m + \|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{s-}^*\}, \end{aligned}$$

where $\lim_{n,m\to\infty} E(\Delta_m^2) = 0$ by the continuity and right continuity of the paths of $Z^i$ and $V^i$. An analogous result holds for the $\hat{G}_{i,j}$.

Define an increasing process $L$ as follows:

$$(3.26)\quad L_t = t + \textstyle\sum_{i=1,r}([M^i, M^i]_t + |A^i|_t + |V^i|_t) + \sum_{i,j=1,r} |[M^i, M^j]|_t.$$

The process $L$ is right continuous and strictly increasing. Combining (3.23) with (3.25) and summing over $\alpha$ then yields

$$(3.27)\quad \begin{aligned} &E\{\|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{T-}^{*2}\} \\ &\le CE\left\{L_{T-}\int_0^{T-} \{\Delta_m^2 + \|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{s-}^{*2}\, dL_s\right\} + \varepsilon_{n,m}. \end{aligned}$$

Define $R_k = \inf\{s > 0; L_s > k\}$. To avoid the problem of large jumps, we stop the processes at $R_k -$; for $Z^i$ we could as well stop at $R_k$, since the paths are continuous. Thus, for a fixed $k$, we replace the processes $Z^i$ and $V^i$ as follows:

$$(3.28)\quad \begin{aligned} \tilde{Z}_t^i &= Z_{t\wedge R_k}^i \\ \tilde{V}_t^i &= V_{t\wedge R_k-}^i = V_{t\wedge R_k}^i - \Delta V_{R_k}^i 1_{\{t\ge R_k\}}, \end{aligned}$$

where $\Delta V_t = V_t - V_{t-}$ (the jump at $t$). Note then that $\tilde{L}_t = L_t^{R_k-} \le k$ a.s. Let $\tau_t = \inf\{s > 0: \tilde{L}_s > t\}$, the right continuous inverse of $\tilde{L}$. Then $\tau_t$ is a stopping time for each $t$ (cf., e.g., [1]), and (3.27) yields (for $t \le k$)

$$(3.29)\quad \begin{aligned} &E\{\|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{\tau_t-}^{*2}\} \\ &\qquad\le CkE\left\{\int_0^{\tau_t-} \Delta_m^2 + \|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{s-}^{*2}\, d\tilde{L}_s\right\} + \varepsilon_{n,m}, \end{aligned}$$

and absorbing the $Ck^2E(\Delta_m^2)$ into the $\varepsilon_{n,m}$, (3.29) becomes

$$(3.30)\quad \le CkE\left\{\int_0^{\tau_t-} \|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{s-}^{*2}\, d\tilde{L}_s\right\} + \varepsilon_{n,m}.$$

Let $\Gamma = [\![0, \tau_t[\![ = \{(s, \omega): 0 \le s < \tau_t(\omega)\}$. Then using Lebesgue's lemma, we have from (3.29) and (3.30)

$$(3.31)\quad \begin{aligned} E\{\|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{\tau_t-}^{*2}\} &\le KE\left\{\int_0^{\tilde{L}_\infty} 1_\Gamma(\tau_s)\|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{\tau_s-}^{*2}\, ds\right\} + \varepsilon_{n,m} \\ &\le KE\int_0^t \|\mathbf{Y}^{(n)} - \mathbf{Y}^{(m)}\|_{\tau_s-}^{*2}\, ds + \varepsilon_{n,m} \quad (t < k) \end{aligned}$$

and letting $\alpha(n, m, s) = \| \mathbf{Y}^{(n)} - \mathbf{Y}^{(m)} \|^{*2}_{T_{s-}}$, (3.31) simplifies to

$$E\{\alpha(n, m, t)\} \le K \int_0^t E\{\alpha(n, m, s)\}\, ds + \varepsilon_{n,m}.$$

Taking $\gamma_{n,m}(t) = E\{\alpha(n, m, t)\}$ and using the Bellman-Gronwall lemma (e.g., [5, page 393]), one obtains from (3.31)

$$\gamma_{n,m}(t) \le \varepsilon_{n,m} + K \int_0^t e^{k(t-s)} \varepsilon_{n,m}\, ds \le \varepsilon_{n,m}(1 + e^{Kt}). \tag{3.32}$$

Since $\lim_{n,m\to\infty} \varepsilon_{n,m} = 0$, we have that $(\mathbf{Y}^{(n)})^{R_{\bar{k}}}$ is Cauchy in $L^2$, uniformly in $s$. □

Two technical lemmas used in the proof of Lemma 3.40 follow.

LEMMA 3.33. *Let $(H^n)$ be a uniformly bounded sequence of processes in* raw *$\mathscr{R}^0$, and suppose for any $t$ $\lim_{n\to\infty}(H^n - H)^*_t = 0$, with convergence in probability, and $H \in \mathscr{R}^0$ (note that the limit process $H$ is assumed adapted). If $\Pi^n$ is a sequence of refining partitions of $[0, t]$ with $\lim_{n\to\infty}\mathrm{mesh}(\Pi^n) = 0$, then*

$$\lim_n \textstyle\sum_{t_\nu \in \Pi^n} H^n_{t_\nu}(Z_{t_{\nu+1}} - Z_{t_\nu}) = \displaystyle\int_0^t H_{s-}\, dZ_s$$

*uniformly in $t$ on compacts, in probability, for any semimartingale $Z$.*

PROOF. By changing to an equivalent probability if necessary, assume $Z \in \mathscr{H}^2$. Then

$$\begin{aligned}&\left(\sum H^n_{t_\nu} \Delta^\nu Z - \int H_s\, dZ_s\right)^*_t \\ &\qquad \le \left(\sum (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu Z\right)^* + \left(\sum H_{t_\nu}\Delta^\nu Z - \int H_s\, dZ_s\right)^*.\end{aligned} \tag{3.34}$$

The second term on the right side of (3.34) tends to 0 in probability as is well known (cf. [2, page 339]).

Let $Z = N + A$ be a decomposition of $Z$ such that $N$ is locally a square integrable martingale. Then

$$\left(\sum (H^n_{t_\nu} - H_{t_\beta})\Delta^\nu Z\right)^* \le \left(\sum (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu N\right)^* + \left(\sum (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu A\right)^*. \tag{3.35}$$

But

$$\left(\sum (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu A\right)^* \le \sum \varepsilon\, \Delta^\nu |A| + \int_{\{|H^n - H| > \varepsilon\}} 2K\, |dA_s|,$$

(where $K$ is a bound for $|H^n|$, $n \ge 1$)

$$\le \delta + 2K \int_{\{|H^n - H| > \varepsilon\}} |dA_s|.$$

Since $\lim_{n\to\infty} P(|H^n - H|^* > \varepsilon) = 0$, we have $\lim_{n\to\infty}(\sum (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu A)^* = 0$ in probability.

It remains to show that

$$\lim(\sum (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu N)^* = 0 \text{ in probability.} \tag{3.36}$$

Since $N$ is locally square integrable, and since we only need convergence in probability, without loss of generality we may assume that $N$ is stopped at an arbitrarily large stopping time, so that it is square integrable. Set

$$\Gamma^n_\nu = (H^n_{t_\nu} - H_{t_\nu}), \quad \Gamma^{n^*} = \sup_\nu |\Gamma^n_\nu|.$$

Then $\Gamma^{n^*}$ tends to 0 in probability and is bounded. Moreover,

$$\begin{aligned} E[(\sum_\nu (H^n_{t_\nu} - H_{t_\nu})\Delta^\nu N)^{*2}] &= E[(\sum_\nu \Gamma^n_\nu (\Delta^\nu N))^{*2}] \\ &\leq E[(\Gamma^{n^*} \sum_\nu |\Delta^\nu N|)^2] \\ &\leq E[(\Gamma^{n^*})^2\{\sum_\nu (\Delta^\nu N)^2 + \sum_{\nu\neq\mu} |\Delta^\nu N||\Delta^\mu N|\}], \end{aligned} \tag{3.37}$$

and since $\{\sum_\nu (\Delta^\nu N)^2 + \sum_{\nu\neq\mu} |\Delta^\nu N||\Delta^\mu N|\}$ is uniformly integrable as the mesh of the refining partitions tends to 0 (cf., e.g., [13, pages 355–356]), we deduce from (3.37) the convergence in probability of (3.36). □

LEMMA 3.38. *Let $(H^n)$ be uniformly bounded processes in* raw $\mathscr{R}^0$, *$H$ a process in (adapted) $\mathscr{R}^0$, such that $\lim_{n\to\infty}(H^n - H)^*_{t_0} = 0$, some $t_0 > 0$, with convergence in probability. Let $\Pi^n$ be a sequence of refining partitions of $[0, t_0]$ with $\lim_{n\to\infty}\text{mesh}(\Pi^n) = 0$. Let $Z$ and $Y$ be continuous semimartingales. Then*

$$\lim_{n\to\infty} \sum_{t_\nu\in\pi^n_t} H^n_{t_\nu}\Delta^\nu Z\Delta^\nu Y = \int_0^t H_s\, d[Z^c, Y^c]_s,$$

*uniformly on $[0, t_0]$, with convergence in probability.*

PROOF. By changing to an equivalent measure if necessary, assume $Z$ and $Y$ are in $\mathscr{H}^2$. Let $Z = M + A$ and $Y = N + B$, where $M$ and $N$ are the continuous martingale parts. It is simple to check that

$$\sum H^n_{t_\nu}\Delta^\nu M\Delta^\nu B, \quad \sum H^n_{t_\nu}\Delta^\nu A\Delta^\nu B, \quad \sum H^n_{t_\nu}\Delta^\nu A\Delta^\nu N$$

all tend to 0 a.s. as $\text{mesh}(\Pi^n)$ tends to 0. Consider the remaining term:

$$\sum H^n_{t_\nu}\Delta^\nu M\Delta^\nu N = \sum(H^n_{t_\nu} - H_{t_\nu})\Delta^\nu M\Delta^\nu N + \sum H_{t_\nu}\Delta^\nu M\Delta^\nu N. \tag{3.39}$$

The second term on the right in (3.39) tends in probability to

$$\int_0^t H_s\, d[M, N] = \int_0^t H_s\, d[Z^c, Y^c]_s$$

(cf. [2, pages 340, 344]).

An argument analogous to the one used to establish (3.36) shows that

$$E\{\sum_\nu |H^n_{t_\nu} - H_{t_\nu}||\Delta^\nu M||\Delta^\nu N|\} \leq E\{\Gamma^{n^*} \sum_\nu |\Delta^\nu M||\Delta^\nu N|\},$$

where $\sum_\nu |\Delta^\nu M||\Delta^\nu N|$ is uniformly integrable as the mesh of the refining partitions goes to 0. Thus the first term on the right side of (3.39) also tends to 0 in probability.

LEMMA 3.40. *Let $Z^i$, $1 \le i \le r$, be continuous $\mathscr{H}^4$ semimartingales, and let $V^i$, $1 \le i \le r$, be FV processes which are also $\mathscr{H}^4$. With the hypotheses and notation of Theorem* 3.4, *and with Hypothesis* 3.6 *holding, there exists a sequence of finite-valued stopping times $(T^k)$ increasing a.s. to $\infty$ such that $(\mathbf{Y}^{(n)})^{T_k-}$ converges in $\mathscr{R}^2$ to $\mathbf{Y}^{T_k-}$.*

PROOF. Lemma 3.19 assures us that $(\mathbf{Y}^{(n)})^{T^k-}$ is Cauchy in the Banach space $\mathscr{R}^2$. We must show that the limit is, indeed, $\mathbf{Y}^{T_k-}$. This, however, is a trivial consequence of Lemmas 3.7, 3.33, and 3.38. □

COROLLARY 3.41. *Under the additional Hypothesis* 3.6, *Theorem* 3.4 *holds.*

PROOF. Fix a $t$ and an $\varepsilon > 0$. Choose $k$ so large that $P(T^k \le t) < \varepsilon$. Then apply Lemma 3.40 and Lemma 3.5.

PROOF OF THEOREM 3.4. By Corollary 3.41 it remains only to remove Hypothesis 3.6. By Lemma 3.5 we may assume $Z^i$, $1 \le i \le r$, are continuous $\mathscr{H}^4$ semimartingales, and that $V^i$ $(1 \le i \le r)$ are $VF$ processes that are also $\mathscr{H}^4$. We now define a new operator $\beta$ as follows, defined for $X$ in the domain of $\Lambda$:

$$\beta X_t = \begin{cases} (\Lambda X)_t 1_{[\![0,R(X)[\![}(t) + (\Lambda X)_{R(X)-} 1_{[\![R(X),\infty[\![}(t) \\ R(X) = \inf\{s > 0\colon |\Lambda X|_s \ge \ell\}. \end{cases}$$

($R$ will be a stopping time only if $X$ is adapted; but this poses no problems as $\beta$ is well-defined in any case.)

Let $F_i^\alpha(X_{0-}, \beta X_{s-})$ be as is $F_i^\alpha(X_{s-})$, where $\beta$ replaces $\Lambda$. We must show equations of the form

$$X_t = x + \int_0^t F(X_{s-}, \beta X_{s-})\, dM_s \tag{3.42}$$

have solutions and that they are unique. Let $Y$ be the unique solution of

$$Y_t = x + \int_0^t F(Y_{s-}, \Lambda Y_{s-})\, dM_s,$$

which we know exists by Emery's theorem, since $\Lambda$ and hence $F$ are in Lip($K$). Define

$$T = \inf\{t > 0\colon |\Lambda Y|_t \ge \ell\}.$$

Then the operator $H$ defined by

$$H(Z)_s = F(Z_s, (\Lambda Z)_s^T)$$

is again in Lip($K$), and hence the equation $Z_t = x + \int_0^t H(Z)_{s-}\, dM_s$ also has a unique solution. Moreover, $Z^T = Y^T$, and hence $R(Z) = T = R(Y)$. Thus $Z$ is a solution of

$$Z_t = x + \int_0^t F(Z_{s-}, (\beta Z)_{s-})\, dM_s.$$

To show uniqueness, suppose $Z$ and $W$ are two solutions of equation (3.42). Let

$S = \min(R(Z), R(W))$. Then

$$Z_t^S = x + \int_0^{t\wedge S} F(Z_{s-}, (\beta Z)_{s-})\, dM_s$$

$$= x + \int_0^{t\wedge S} F(Z_{s-}, (\beta Z)_{s-}^S)\, dM_s$$

$$= x + \int_0^{t\wedge S} F(Z_{s-}^S, (\beta Z^S)_{s-})\, dM_s,$$

since $(\beta Z^S)_{s-} = (\beta Z)_{s-}$ on $[\![0, S]\!]$,where $S \le R(Z)$. Moreover, since $(\Lambda Z)_{s-} = (\beta Z)_{s-}$ on $[\![0, S]\!]$ when $S \le R(Z)$, this becomes:

$$= x + \int_0^{t\wedge S} F(Z_{s-}, (\Lambda Z)_{s-})\, dM_s.$$

Analogously, we have that

$$W_t^S = x + \int_0^{t\wedge S} F(W_{s-}, (\Lambda W)_{s-})\, dM_s.$$

Thus, both $W^S$ and $Z^S$ are solutions of

$$V_t = x + \int_0^t F(V_{s-}, (\Lambda V)_{s-})\, dM_s^S,$$

and by uniqueness we have $W^S = Z^S$. This implies $(\Lambda W)^S = (\Lambda Z)^S$, and hence $R(W) = R(Z)$, since $S$ is the minimum of $R(W)$ and $R(Z)$. Thus both $W^S$ and $Z^S$ are solutions of

$$U = x + \int_0^t F(U_{s-}, (\Lambda U)_{s-}^S)\, dM_s^S,$$

which has a unique solution. Hence $Z = W$ up to $R(W) = R(Z)$.

Since we have established existence and uniqueness for general equations of the form

$$X_t = x + \int_0^t F(X_{s-}, (\beta X)_{s-})\, dM_s$$

for $M$ a semimartingale, we can apply this result for the approximating differentials $M^{(n)i}$. (Taking the filtration to be $\mathscr{G}_t = \mathscr{F}_\infty$, for all $t$, each $M^{(n)i}$ is a semimartingale since it has paths of bounded variation on compacts.)

To complete the proof, set

$$Y_t = x + \int_0^t F(Y_{s-}(\Lambda Y)_{s-})\, dM_s \tag{3.43}$$

and let $R = R(Y)$. Then

$$Y^R = x + \int_0^{t\wedge R} F(Y_{s-}, (\Lambda Y)_{s-})\, dM_s$$
$$= x + \int_0^{t\wedge R} F(Y_{s-}, (\beta Y)_{s-})\, dM_s$$

since $(\Lambda Y)_{s-} = (\beta Y)_{s-}$ on $[\![0, R]\!]$. Let $U$ be the (unique, up to $R(U)$) solution of

$$U_t = x + \int_0^t F(U_{s-}, (\beta V)_{s-})\, dM_s. \tag{3.44}$$

Then $Y^R = U^R$, and so $(\Lambda Y)^R = (\Lambda U)^R$, and thus $R(Y) = R(U) = R$.

Analogously, $R(U^{(n)}) = R(Y^{(n)}) = R^n$, where $Y^{(n)}$, $U^{(n)}$ are solutions analogous to (3.43) and (3.44), respectively, with $M$ replaced by $M^{(n)}$.

Next implicitly stop $Z^i$ and $V^i$, $1 \le i \le d$, as in (3.40), at $T^k-$, so that for each $k$ the differentials are in $\mathscr{H}^4$. Then as in Lemma 3.19 we have that $U^{(n)}$ converges to $U$ in $\mathscr{R}^2$, using that $U^{(n)} = Y^{(n)}$ on $[\![0, R_n]\!]$. Therefore $\lim_{n\to\infty} R(U^{(n)}) = \lim_{n\to\infty} R(Y^{(n)}) = R$ in probability by (2.4)(ii). Therefore

$$P(\| Y^{(n)} - Y\|_R^* > \varepsilon) \le P(\| U^{(n)} - U\|_R^* > \varepsilon) + P(R^n < R).$$

tends to 0 as $n \to \infty$. Since $\varepsilon$ was arbitrary, and since $T^k$ increase to $\infty$ a.s., this completes the proof.

**4. Continuous case: $\mathscr{C}^1$ approximations.** In Section 3 we considered polygonal approximations $Z^{(n)i}$ of continuous semimartingales $Z^i$. In this section we consider processes $W^{(n)i}$ with paths that are $\mathscr{C}^1$ and which converge to the paths of $Z^i$. The processes $W^{(n)i}$ must be reasonably close to the polygonal approximations, and we make the following hypotheses, similar to those of Nakao and Yamato [15].

DEFINITION 4.1. Let $Z^1, \cdots, Z^r$ be continuous semimartingales. $(W^{(n)i})_{1\le i\le r}$ will be said to be a $\mathscr{C}^1$ approximation of $(Z^1, \cdots, Z^r)$ if the following four conditions hold for each $n \ge 1$, $1 \le i \le r$:

(4.1)(i) $W^{(n)i}$ have piecewise $\mathscr{C}^1$ paths, a.s.;

(4.1)(ii) there exists a sequence $\Pi^n$ of refining partitions with $\lim_{n\to\infty}\text{mesh}(\Pi^n) = 0$ such that

(a) $W_{t_\nu}^{(n)i} = Z_{t_\nu}^i$ for all $t_\nu \in \Pi^n$;

(b) $W_t^{(n)i}$ is $\mathscr{F}_{t_\nu}$-measurable, for all $t \le t_\nu$;

(c) For each $t$, $0 < t < \infty$, and $\Pi^n$ restricted to $[0, t]$

$$\sum_{i=1,r} \sup_{t_\nu < s < t_{\nu+1}} \left| \frac{\partial}{\partial s} W_s^{(n)i} - \frac{\Delta^\nu Z^i}{\Delta^\nu t} \right| \le K_t \frac{\| \Delta^\nu Z \|^2}{\Delta^\nu t}.$$

THEOREM 4.2. *Let $Z^i$, $1 \le i \le r$, be continuous semimartingales, $Z_0^i = 0$; let*

*$F_i^\alpha$ $(1 \le \alpha \le d)$ denote acceptable coefficients (cf. (2.9)); and let $W^{(n)i}$, $1 \le i \le r$, $n \ge 1$ be a $\mathscr{C}^1$ approximation of $Z^i$ as described in Definition 4.1. Let $V^i$, $1 \le i \le r$ be FV processes (cf. (2.1)), and let $\mathbf{X}_t^{(n)}$ denote the solution of* $(1 \le \alpha \le d)$:

$$X_t^{(n)\alpha} = x^\alpha + \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X}^{(n)}, \mathbf{W}^{(n)})_s \, d(W_s^{(n)i} + V_s^i). \tag{4.3}$$

*Let $G_{i,j}^\alpha$ be the associated coefficients (2.12), and let $X$ denote the solution of*

$$\begin{aligned} X_t^\alpha = x^\alpha &+ \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X}, \mathbf{Z})_s \, d(Z_s^i + V_s^i) \\ &+ \frac{1}{2} \sum_{i,j=1,r} \int_0^t G_{i,j}^\alpha(\mathbf{X}, \mathbf{Z})_s \, d[Z^{i,c}, Z^{j,c}]_s. \end{aligned} \tag{4.4}$$

*Then $\mathbf{X}^{(n)}$ converges uniformly on compacts, in probability, to $\mathbf{X}$.*

The proof of Theorem 4.2 begins with a lemma that lets us use our polygonal approximation result of Section 3. For simplicity, write $F_i^\alpha(\mathbf{X}^{(n)})$ for $F_i^\alpha(\mathbf{X}^{(n)}, \mathbf{Z}^{(n)})$, etc.

LEMMA 4.5. *Let $\mathbf{X}^{(n)}$ be as in Theorem 4.2, and $\mathbf{Y}^{(n)}$ be as in Theorem 3.4. If*

$$\begin{aligned} X_t^{(n)\alpha} - Y_t^{(n)\alpha} &= \sum_{t_\nu \in \Pi_t^n} \sum_{i=1,r} [F_i^\alpha(\mathbf{X}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}]\{\Delta^\nu Z^i + \Delta^\nu V^i\} \\ &\quad + \sum_{t_\nu \in \Pi_t^n} \tfrac{1}{2} \sum_{i,j=1,r} [G_{i,j}^\alpha(\mathbf{X}^{(n)})_{t_\nu} - G_{i,j}^\alpha(\mathbf{Y}^{(n)})_{t_\nu}]\Delta^\nu Z^i \Delta^\nu Z^j + I_n, \end{aligned}$$

*where $I_n$ tends to 0 in raw $\mathscr{R}^2$ $(1 \le \alpha \le d)$, then $\mathbf{X}^{(n)}$ converges uniformly on compacts, in probability, to $\mathbf{X}$.*

PROOF. Since

$$P(\| \mathbf{X}^{(n)} - \mathbf{X} \|_t^* > \varepsilon) \le P(\| \mathbf{X}^{(n)} - \mathbf{Y}^{(n)} \|_t^* > \varepsilon) + P(\| \mathbf{Y}^{(n)} - \mathbf{X} \|_t^* > \varepsilon),$$

and since

$$\lim_{n\to\infty} P(\| \mathbf{Y}^{(n)} - \mathbf{X} \|_t^* > \varepsilon) = 0$$

by Theorem 3.4 (note that $\mathbf{X} = \mathbf{Y}$ in our notation), it suffices to show

$$\lim_{n\to\infty} P(\| X^{(n)} - Y^{(n)} \|_t^* > \varepsilon) = 0.$$

By changing to an equivalent probability if necessary, assume without loss of generality that $Z^i$, $V^i$ are all $\mathscr{H}^4$ $(1 \le i \le r)$. Proceed as in the proof of Lemma 3.19 ((3.19)–(3.27)) to establish for any stopping time $T$:

$$E\{(X^{(n)\alpha} - Y^{(n)\alpha})_{T-}^{*2}\} \le CE\{L_{T-} \int_0^{T-} \{\Delta_n^2 + \| \mathbf{X}^{(n)} - \mathbf{Y}^{(n)} \|_s^{*2} \, dL_s\},$$

where $\lim_{n\to\infty} E(\Delta_n^2) = 0$ and $L_t$ is given in (3.26). The rest of the proof proceeds analogously to obtain that, for $t \le k$, $\| L^* \|_{L^\infty} < \infty$,

$$E\{\| \mathbf{X}^{(n)} - \mathbf{Y}^{(n)} \|_{\tau_t}^*\} \le \varepsilon_n(1 + e^{Kt}),$$

where $\varepsilon_n$ tends to 0. Since $\tau_k \equiv \infty$, we have $(\mathbf{X}^{(n)} - \mathbf{Y}^{(n)})_{R^k-}^*$ tends to 0 in probability for stopping times $R^k$ (defined in (3.28)) tending a.s. to $\infty$. This proves the theorem. □

LEMMA 4.6. *Under Hypothesis* 3.6, *Theorem* 4.2 *is true.*

PROOF. Recall that Hypothesis 3.6 assumes $|\Lambda H|_t \le K_t$, for $H$ in the domain of $\Lambda$, with $K_t$ increasing in raw $\mathscr{R}^0$.

With $\mathbf{Y}^{(n)}$ as in Theorem 3.4,

$$X_t^{(n)\alpha} - Y_t^{(n)\alpha} = \sum_{i=1,r}\Bigg[\int_0^t F_i^\alpha(\mathbf{Y}^{(n)})_s \{dW_s^{(n)i} - Z_s^{(n)i}\} + \int_0^t \{F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s\}\, dW_s^{(n)i} + \int_0^t \{F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s\}\, dV_s^i\Bigg]. \tag{4.7}$$

Consider the second term on the right side of (4.7). Integration by parts yields

$$\begin{aligned}\int_0^t F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s\, dW_s^{(n)i} &= \sum_{\nu\in\Pi_t^n}\{F_i^\alpha(\mathbf{X}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}\}\Delta^\nu Z^i \\ &\quad + \sum_{\nu\in\Pi_t^n}\int_{t_\nu}^{t_{\nu+1}} (W_{t_{\nu+1}}^{(n)i} - W_s^{(n)i})\, d[F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s].\end{aligned} \tag{4.8}$$

But

$$\begin{aligned} d[F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s] &= \sum_{j=1,r} G_i(\mathbf{X}^{(n)})_s dW_s^{(n)j} - G_{ij}^\alpha(\mathbf{Y}^{(n)})_s \frac{\Delta^\nu Z^j}{\Delta^\nu t} ds \\ &\quad + \left\{\frac{\partial f_i^\alpha}{\partial s}(s, \omega, \mathbf{X}^{(n)}, \cdots) - \frac{\partial f_i^\alpha}{\partial s}(s, \omega, \mathbf{Y}^{(n)}, \cdots)\right\} ds \\ &\quad + \frac{\partial f_i^\alpha}{\partial \lambda}(s, \cdots) d\Lambda(\mathbf{X}^{(n)})_s - \frac{\partial f_i^\alpha}{\partial \lambda}(s, \cdots) d\Lambda(\mathbf{Y}^{(n)})_s \end{aligned} \tag{4.9}$$

and integrating again by parts, when appropriate, changes (4.8) to

$$\begin{aligned}\int_0^t \{F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s\}\, dW_s^{(n)i} &= \sum_{\Pi_t^n}\{F_i^\alpha(\mathbf{X}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}\}\Delta^\nu Z^i \\ &\quad + \sum_{\Pi_t^n} \tfrac{1}{2} \sum_{i,j=1,r}\{G_{ij}^\alpha(\mathbf{X}^{(n)})_{t_\nu} - G_{ij}(\mathbf{Y}^{(n)})_{t_\nu}\}\Delta^\nu Z^i \Delta^\nu Z^j + I^n - J^n,\end{aligned} \tag{4.10}$$

where $I^n$ and $J^n$ go to 0 in raw $\mathscr{R}^2$, since (3.6) holds, provided $Z^i$, $V^i$ are in $\mathscr{H}^4$, $1 \le i \le r$, by arguments analogous to those used in Lemma 3.7. (Note that without loss of generality we may assume once again that $Z^i$, $V^i$ are in $\mathscr{H}^4$ by the arguments presented in Lemma 3.5.)

The third term in (4.7) is easily disposed of:

$$\int_0^t F_i^\alpha(\mathbf{X}^{(n)})_s - F_i^\alpha(\mathbf{Y}^{(n)})_s \, dV_s^i$$

$$(4.11) \qquad = \sum_{\Pi_t^n} [F_i^\alpha(\mathbf{X}^{(n)})_{t_\nu} - F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}]\Delta^\nu V^i$$

$$+ \sum_{\Pi_t^n} \int_{t_\nu}^{t_{\nu+1}} (V_{t_{\nu+1}}^i - V_s^i) \, d[F(\mathbf{X}^{(n)}) - F(\mathbf{Y}^{(n)})],$$

where the second term on the right side of (4.11) tends to 0 by arguments analogous to those establishing that $I^n$ and $J^n$ tend to 0.

Finally, we are left with the first term on the right side of (4.7). Integrating by parts yields

$$\int_0^t F_i^\alpha(\mathbf{Y}^{(n)})_s \, d\{W_s^{(n)i} - Z_s^{(n)i}\}$$

$$= \sum_{\Pi_t^n} F_i^\alpha(\mathbf{Y}^{(n)})_{t_\nu}(\Delta^\nu Z^i - \Delta^\nu Z^i)$$

$$+ \sum_{\Pi_t^n} \int_{t_\nu}^{t_{\nu+1}} (W_{t_{\nu+1}}^{(n)i} - W_s^{(n)i}) - (Z_{t_{\nu+1}}^{(n)i} - Z_s^{(n)i}) \, dF_i^\alpha(\mathbf{Y}^{(n)})_s$$

$$(4.12) \qquad = 0 + \sum_{\Pi_t^n} \int_{t_\nu}^{t_{\nu+1}} \left[\frac{\partial W^{(n)i}}{\partial t}(c(s)) - \frac{\Delta^\nu Z}{\Delta^\nu t}\right](t_{\nu+1} - s) \, dF_i^\alpha(\mathbf{Y}^{(n)})_s$$

$$\leq K_t \sum_{\Pi_t^n} \frac{\| \Delta^\nu Z \|^2}{\Delta^\nu t} \int_{t_\nu}^{t_{\nu+1}} (b - s)\left[\frac{\partial f_i^\alpha}{\partial t}(s, \cdots) + \frac{\partial f_i^\alpha}{\partial \lambda}(s, \cdots) + \sum_{j=1,r} G_{ij}^\alpha(\mathbf{Y}^{(n)})_s \frac{\Delta^\nu Z^j}{\Delta^\nu t}\right] ds$$

where $c(s)$ is a value in $[t_\nu, t_{\nu+1}]$,

$$(4.13) \qquad \leq K \sum_{\Pi_t^n} (\| \Delta^n Z \|^2 \Delta^\nu Q^i + \| \Delta^\nu Z \|^2 (\textstyle\sum_{j=1,r} | \Delta^\nu Z^j |))$$

where $Q^i$ is an FV process, and where the mean value theorem gives the second equality. The right side of (4.13) is dominated by

$$K(\sup_\nu | \Delta^\nu Q | + \textstyle\sum_j | \Delta^\nu Z^j |) \sum_{\Pi_t^n} \| \Delta^\nu Z \| \leq C(\omega)(\sup_{\nu \in \Pi_t^n}[| \Delta^\nu Q | + \sum_j | \Delta^\nu Z^j |]),$$

which tends to 0 a.s. and in $L^2$.

The conditions of Lemma 4.5 are now satisfied, implying Theorem 4.2, under Hypothesis 3.6. □

PROOF OF THEOREM 4.2. In view of Lemma 4.6, it remains only to remove Hypothesis 3.6. But this can be done exactly as in the polygonal case, and we refer the reader to the conclusion of the proof of Theorem 3.4 in Section 3 (which follows Corollary 3.41).

**5. Right continuous case.** Consider now equations of the form ($1 \leq \alpha \leq d$):

$$X_t^\alpha = x^\alpha + \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X})_s \, dM_s^i$$

where $M^i$ are arbitrary right continuous semimartingales. Every such $M$ can be decomposed: $M_t = N_t^c + N_t^d + A_t$, where $N^c$ is a continuous local martingale, $A$ is an FV process, and $N^d$ is a "purely discontinuous" local martingale (cf., e.g., [20]). Both $N^c$ and $N^d$ can have, in general, paths of infinite variation on compacts; thus it is of interest to approximate their paths with processes with smooth paths and find the limits of the approximating solutions. Unlike the case for continuous martingales, however, one can approximate $N^d$ by a sequence of *local martingales with paths of finite variation* (e.g., Example 5.2 below).

DEFINITION 5.1. For a (purely discontinuous) local martingale $N$, a sequence $N^k$ will be called an approximating martingale sequence of $N$ if

(i) $N^k$ is a local martingale with paths of finite variation on compacts ($k \geq 1$);
(ii) $N^k$ converges to $N$ locally in $\mathscr{H}^1$; that is, there exist stopping times $T^\ell$ increasing to $\infty$ a.s. such that $\lim_{k\to\infty} E\{[N - N^k, N - N^k]_{T^\ell}^{1/2}\} = 0$, ($\ell \geq 1$);
(iii) there exists $D \in \mathscr{R}^0$ such that $[N^k, N^k] \leq D$, for each $k \geq 1$.

COMMENTS. For a local martingale $N$, the $\mathscr{H}^1$ norm, $\| N \|_{\mathscr{H}^1}$, is defined to be $E\{[N, N]_\infty^{1/2}\}$. Every local martingale is locally in $\mathscr{H}^1$. Moreover one can check that $\| N \|_{\mathscr{H}^1} \leq E\{\int_0^\infty |dN_s|\}$, and also that $\| N \|_{\mathscr{H}^1} = \sup_\ell \| N^{T_\ell} \|_{\mathscr{H}^1}$, for any sequence of stopping times $T_\ell$ increasing to $\infty$ a.s.

We write $[M]$ for $[M, M]$ when there is no ambiguity.

EXAMPLE 5.2. One way to obtain such a sequence is to take

$$A_t^k = \sum_{s \leq t} \Delta N_s 1_{\{k > |\Delta N_s| > \varepsilon_k\}}, \quad N_t^k = A_t^k - \tilde{A}_t^k.$$

where $\tilde{A}_t^k$ is the compensation of $A$, and where $\varepsilon_k$ is a sequence decreasing to 0.

Any semimartingale $M$ can be written $M = Z + N + V$, where $Z$ is a continuous semimartingale, $N$ is a purely discontinuous local martingale, and $V$ is an FV process. Such a decomposition is not unique, and in particular the choice of the continuous semimartingale part $Z$ is, in general, arbitrary. Here is our main result of this section.

THEOREM 5.3. *Let $M^i$, $1 \leq i \leq r$, be right continuous semimartingales, and let $M^i = Z^i + N^i + V^i$ be any decomposition of $M^i$ such that $Z^i$ is a continuous semimartingale, $N^i$ is a purely discontinuous local martingale, and $V^i$ is an FV process. Let $W^{(n)i}$ be $\mathscr{C}^1$ approximations of $Z^i$* (cf. Definition 4.1), *and let $N^{(n)i}$ be an approximating martingale sequence of $N^i$. Suppose $F_i(1 \leq \alpha \leq d)$ denote* *acceptable coefficients* (cf. (2.9)), *and let* $\mathbf{X}^{(n,k)}$ *denote the solution of* $(1 \le \alpha \le d)$:

$$(5.4) \qquad X_t^{(n,k)\alpha} = x^\alpha + \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X}^{(n,k)}, \mathbf{W}^{(n)})_s \, d(W^{(n)i} + N_s^{(k)i} + V_s^i).$$

*Let* $G_{1,j}^\alpha$ *be the associated coefficients* (2.12) *and let* $X$ *denote the solution of* $(1 \le \alpha \le d)$:

$$(5.5) \quad X_t^\alpha = x^\alpha + \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X}, \mathbf{Z})_s \, dM_s^i + \sum_{i,j=1,r} \int_0^t G_{i,j}^\alpha(\mathbf{X}, \mathbf{Z})_s \, d[M^{ic}, M^{jc}]_s.$$

*Then for any t and for any* $\delta$, $\varepsilon > 0$, *there exists a* $K$ *and a function* $n(k)$ *such that* $k > K$ *and* $n > n(k)$ *implies*

$$P(\| X^{(n,k)} - X \|_t^* > \delta) < \varepsilon.$$

Most of the work involved in proving Theorem 5.3 is contained in the proof of Theorem 3.4. Nevertheless, we will need the following lemma. Again, for simplicity write $F_i^\alpha(\mathbf{X}^{(n,k)})_s$ for $F_i^\alpha(\mathbf{X}^{(n,k)}, \mathbf{W}^{(n)})$, etc.

LEMMA 5.6. *Let* $M^i$ *be semimartingales* $(1 \le i \le r)$, $M^i = Y^i + N^i$, *where each* $N^i$ *is a purely discontinuous local martingale. Let* $N^{(k)i}$ *be an approximating martingale sequence of* $N^i$. *Let* $F_i$ $(1 \le \alpha \le d)$ *be in* Lip$(K)$ *and bounded. Let* $\mathbf{X}^{(k)}$ *and* $\mathbf{X}$ *be solutions, respectively, of* $(1 \le \alpha \le d)$:

$$X_t^{(k)} = x^\alpha + \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X}^{(k)})_s \, d(Y_s^i + N_s^{(k)i})$$

$$X_t^\alpha = x^\alpha + \sum_{i=1,r} \int_0^t F_i^\alpha(\mathbf{X})_s \, dM_s^i.$$

*Then* $\lim_{k\to\infty} \mathbf{X}^{(k)} = \mathbf{X}$, *in probability, uniformly on compacts.*

PROOF. For simplicity of notation assume $r = d = 1$ (the extension to general finite $r$, $d$ is easy). Let $T^\ell$ be stopping times increasing to $\infty$ a.s. such that $\lim_{k\to\infty} E\{[N - N^{(k)}]_{T^\ell}^{1/2}\} = 0$. Without loss of generality we assume $Y$, $N^{(k)}$, and $M$ are all stopped at $T^\ell$, for a fixed $\ell$. By changing to an equivalent probability $Q$, if necessary, we may further assume that $N$ and $Y$ are in $\mathscr{H}^2(Q)$ on compact time intervals and that the process $D$ in Definition 5.1 is in $L^1(dQ)$. Further, one can choose $Q$ so that the density $dQ/dP$ is bounded (cf. [2, page 271]).

Under $Q$, the processes $N$, $N^k$ need not be local martingales, but they are special semimartingales. Let

$$(5.7) \qquad N = \tilde{N} + \tilde{A}, \quad N^k = \tilde{N}^k + \tilde{A}^k$$

be their canonical decompositions under $Q$. Let $J_t = E\{dQ/dP \mid \mathscr{F}_t\}$ and we take the right continuous version of this bounded martingale. The martingale $J$ is nonnegative, and letting $R^p = \inf\{t: J_t < 1/p\}$, we have $R^p$ increases to $\infty$ a.s. as $p$ increases to $\infty$.

Fix $t_0$ and let $\varepsilon > 0$ be given. Choose $p$ sufficiently large that $P(R^p < t_0) < \varepsilon$.

The processes $\tilde{A}$ and $\tilde{A}^k$ of (5.7) are known to be given by

$$\tilde{A}_t = \int_0^t \frac{1}{J_{s-}}\, d\langle N, J\rangle_s$$

$$\tilde{A}_t^k = \int_0^t \frac{1}{J_{s-}}\, d\langle N^k, J\rangle_s \text{ (cf. [2, page 259]);}$$

For $p$ fixed let $R = R^p$. On $[\![0, R]\!]$ we have $|1/J_{s-}| \leq p$, and therefore

$$\begin{aligned} E_Q\{|\tilde{A} - \tilde{A}^k|_R\} &\leq E_Q\left\{\int_0^R \frac{1}{J_{s-}} |d\langle N - N^k, J\rangle_s|\right\} \\ &\leq E_P\left\{J \int_0^R \frac{1}{J_{s-}} |d\langle N - N^k, J\rangle_s|\right\} \\ &\leq E_P\left\{\int_0^R J_{s-} \frac{1}{J_{s-}} |d\langle N - N^k, J\rangle_s|\right\} \\ &= E_P\left\{\int_0^R |d\langle N - N^k, J\rangle_s|\right\} \\ &= E_P\left\{\int_0^R H_s\, d\langle N - N^k, J\rangle_s\right\} \end{aligned}$$

(where $H$ is predictable and takes on only the values $\pm 1$)

$$\begin{aligned} &= E_P\left\{\int_0^R H_s\, d[N - N^k, J]_s\right\} \\ &\leq E_P\left\{\int_0^R |d[N - N^k, J]_s|\right\} \\ &\leq C E_P\{[N - N^k]_R^{1/2}\}, \end{aligned}$$

by Fefferman's inequality and using that $J$ is bounded (cf. [2, page 295]). Summarizing the above, we have that

$$E_Q\{|\tilde{A} - \tilde{A}^k|_R\} \leq C E_P\{[N - N^k]_R^{1/2}\}, \tag{5.8}$$

and thus

$$\lim_{k\to\infty} E_Q\{|\tilde{A} - \tilde{A}^k|_R\} = 0. \tag{5.9}$$

Since $F$ is bounded, this then implies that if

$$\delta_k(R) = E_Q\left\{\sup_{t\leq R}\left(\int_0^t F X_s^k\, d(\tilde{A}^k - \tilde{A})_s\right)^2\right\}, \tag{5.10}$$

then

$$\lim_{k\to\infty}\delta_k(R) = 0. \tag{5.11}$$

Next, observe that, for any stopping time $T$,

$$\begin{aligned} E_Q\left\{\sup_{t\le T}\left(\int_0^t FX_s^k\, d(\tilde N^k - \tilde N)_s\right)^2\right\} &\le 4E_Q\left\{\int_0^T (FX_s^k)^2\, d[\tilde N^k - \tilde N, \tilde N^k - \tilde N]_s\right\} \\ &\le 4CE_Q\{[\tilde N^k - \tilde N, \tilde N^k - \tilde N]_T\} \\ &\le 16CE_Q\{[N^k - N, N^k - N]_T\}, \end{aligned}$$

since $F$ is bounded and where the last inequality follows since $\tilde N^k - \tilde N$ is the local martingale in the canonical decomposition of $N^k - N$ (cf. [2, page 264]). Thus if we set

$$\gamma_k(T) = E_Q\left\{\sup_{t\le T}\left(\int_0^t FX_s^k\, d(\tilde N^k - \tilde N)_s\right)^2\right\}, \tag{5.12}$$

we have from (5.11) and by $[N - N^k, N - N^k]$ tending to 0 in probability and being dominated by $2([N, N] + D) \in L^1(dQ)$, that

$$\lim_{k\to\infty}\gamma_k(T) = 0, \quad \text{for any stopping time } T. \tag{5.13}$$

Finally, we recall the following fundamental result of Métivier and Pellaumail [12]: for $Z$ a semimartingale, there exists an increasing process $B$ that "controls" $Z$ in the following sense:

$$E\left\{\sup_{t<T}\left(\int_0^t H_s\, dZ_s\right)^2\right\} \le E\left\{B_{T-}\int_0^{T-} H_s^2\, dB_s\right\} \tag{5.14}$$

for any locally bounded predictable process $H$ and any stopping time $T$. Let $L$ denote a process controlling the semimartingale $M$ of the hypothesis. Then we have, for any stopping time $T \le R$,

$$\begin{aligned} E\{\sup_{t<T}(X_t - X_t^k)^2\} &\le E\left\{\sup_{t<T}\left(\int_0^t FX_s^k\, dM_s^k - \int_0^t FX_s\, dM_s\right)^2\right\} \\ &\le 2E\left\{\sup_{t<T}\left(\int_0^t (FX_s^k - FX_s)\, dM_s\right)^2\right\} \\ &\quad + 2E\left\{\sup_{t<T}\left(\int_0^t FX_s^k\, d(M_s^k - M_s)\right)^2\right\} \\ &\le 2E\left\{\sup_{t<T}\left(\int_0^t (FX_s^k - FX_s)\, dM_s\right)^2\right\} \\ &\quad + C[\delta_k(R) + \gamma_k(R)] \end{aligned} \tag{5.15}$$

where $C$ is a constant, $\lim_{k\to\infty}C[\delta_k(R) + \gamma_k(R)] = 0$, from (5.10), (5.11), (5.12), and (5.13). Let $\alpha_k = C[\delta_k(R) + \gamma_k(R)]$. Then (5.15) becomes

$$\begin{aligned} E\{\sup_{t<T}(X_t - X_t^k)^2\} &\le 2E\left\{\sup_{t<T}\left(\int_0^t (FX_s^k - FX_s)\, dM_s\right)^2\right\} + \alpha_k \\ &\le 2E\left\{L_{T-} \int_0^{T-} (FX_s^k - FX_s)^2\, dL_s\right\} + \alpha_k \\ &\le 2KE\left\{L_{T-} \int_0^{T-} [(X^k - X)_{s-}^*]^2\, dL_s\right\} + \alpha_k \end{aligned} \tag{5.16}$$

using (5.14) and the Lipschitz property of $F$.

Define $\tau_t = \inf\{s: L_s > t\}$, and let $T = \min(\tau_t, R)$. Then for $t \le t_0$, we have from (5.16)

$$\begin{aligned} E\{\sup_{s<T}(X_s - X_s^k)^2\} &\le 2KE\left\{t \int_0^{T-} (X^k - X)_{s-}^{*2}\, dL_s\right\} + \alpha_k \\ &\le 2KtE\left\{\int_0^{T-} 1_{[\![0,T[\![}(s)(X^k - X)_{s-}^{*2}\, dL_s\right\} + \alpha_k \\ &\le 2KtE\left\{\int_0^{L_\infty} 1_{[\![0,R\wedge\tau_t[\![}(\tau_s)(X^k - X)_{\tau_{s-}}^{*2}\, ds\right\} + \alpha_k \\ &\le 2Kt_0E\left\{\int_0^t (X^k - X)_{R\wedge\tau_{s-}}^{*2}\, ds\right\} + \alpha_k, \end{aligned} \tag{5.17}$$

where we have used Lebesgue's lemma ([1, page 91]). Letting

$$\beta(k, t) \le E\{\sup_{s<R\wedge\tau_t} |X_s - X_s^k|^2\},$$

we have from (5.17) that

$$\beta(k, t) \le 2Kt_0 \int_0^t \beta(k, s)\, ds + \alpha_k,$$

and hence, by the Bellman-Gronwall lemma ([5, page 393]),

$$\beta(k, t) \le \alpha_k + 2Kt_0 \int_0^t e^{c(t-s)}\alpha_k\, ds$$

which tends to 0 as $k \to \infty$ since $\lim_{k\to\infty}\alpha_k = 0$. Thus $(X - X^k)^*$ tends to 0 in $L^2(dQ)$ on $[\![0, R[\![$, and hence in probability $(Q)$; since $Q$ and $P$ are equivalent, we have convergence in probability $(dP)$. Since $P(R < t_0) < \varepsilon$, $\varepsilon$ arbitrary, we thus have $(X - X^k)^*$ tends to 0 in probability on $[0, t_0]$. But $t_0$ was arbitrary, and the proof is complete. □

PROOF OF THEOREM 5.3. Let $\mathbf{X}(n, k)$ be as given in (5.4), and let $\mathbf{X}$ be as given in (5.5). Under the same hypotheses, let $\mathbf{Y}^{(k)}$ be the solution of $(1 \le \alpha \le d)$:

$$\mathbf{Y}_t^{(k)} = x^\alpha + \textstyle\sum_{i=1,r} \displaystyle\int_0^t F_i^\alpha(\mathbf{Y}^{(k)})_s \, d(Z_s^i + N_s^{(k)i} + V_s^i)$$

$$+ \tfrac{1}{2} \textstyle\sum_{i,j=1,r} \displaystyle\int_0^t G_{i,j}^\alpha(\mathbf{Y}^{(k)})_s \, d[M^{ic}, M^{jc}]_s.$$

Given $\delta, \varepsilon > 0$, for each $k$, Theorem 3.4 assures the existence of an $m_k$ such that if $n \ge m_k$, then

$$P(\| \mathbf{X}^{(n,k)} - \mathbf{Y}^{(k)} \|_t^* > \delta) < \varepsilon/2. \tag{5.18}$$

Analogously, Lemma 5.6 ensures the existence of a $K$ such that, for $k \ge K$,

$$P(\| \mathbf{Y}^{(k)} - \mathbf{X} \|_t^* > \delta) < \varepsilon/2. \tag{5.19}$$

Since

$$P(\| \mathbf{X}^{(n,k)} - \mathbf{X} \|_t^* > \delta) \le P(\| \mathbf{X}^{(n,k)} - \mathbf{Y}^{(k)} \|_t^* > \delta) + P(\| \mathbf{Y}^{(k)} - \mathbf{X} \|_t^* > \delta),$$

(5.18) and (5.19) yield the result. □

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MATHEMATICS DEPARTMENT
PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907