The Annals of Probability
1985, Vol. 13, No. 3, 676–692

THE I-FUNCTION FOR DIFFUSION PROCESSES WITH BOUNDARIES

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The Donsker-Varadhan I-Function which measures the asymptotic rate of decay of the distribution of the occupation measure of a Markov process is evaluated for diffusion processes with boundaries.

1. Introduction. In this paper the Donsker-Varadhan I-function is evaluated explicitly for diffusion processes with boundaries. Stroock and Varadhan [8] have shown that there exists a unique strong Feller diffusion process with state space A, a compact region of R^n with C^2 boundary, and whose generator $(L, \mathscr{D})$ is an extension of $(L, \tilde{\mathscr{D}})$ where $L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$ and $\tilde{\mathscr{D}} = \{u \in C^2(A): J \cdot \nabla u = 0 \text{ on } \partial A\}$. Here a is a strictly positive $n \times n$ matrix with entries $a_{ij} \in C^1(A)$, b is an n-vector with terms $b_i \in C(A)$ and J is an n-vector with terms $J_i \in C^1(\partial A)$ and with $|J \cdot n| \geq \gamma > 0$, where n is the outward unit normal to the region A. The work of Stroock and Varadhan actually allows for more general coefficients. For our purposes, we need the above smoothness conditions.

Let $p(t, x, dy)$ be the transition probability of a Feller process with state space A, a complete separable metric space, and let $\omega = x(\cdot)$, $\omega \in \Omega$, be a realization of the process. For each x, $p(t, x, dy)$ induces a probability measure P_x on Ω. Consider $L_t(\omega, B) = \int_0^t \chi_{(B)}(x(s))\, ds$. Then $(1/t)L_t(\omega, B)$ is the proportion of time up to t that a particular path ω spends in the set B. Thus $(1/t)L_t(\omega, \cdot) \in \mathscr{P}(A)$, the set of probability measures on A; it is the occupation measure for the process. We endow $\mathscr{P}(A)$ with the weak topology. For $\mu \in \mathscr{P}(A)$, define

$$I(\mu) = -\inf_{u \in \mathscr{D}^+} \int_A \frac{Lu}{u}\, d\mu \quad \text{where} \quad \mathscr{D}^+ = \mathscr{D} \cap \{u: u \geq c > 0\}.$$

Note that $I(\mu)$ is lower semicontinuous under weak convergence on $\mathscr{P}(A)$. Under suitable transitivity and recurrence conditions, Donsker and Varadhan [1, 2, 3] have proven that for $x \in A$ and open sets $G \subset \mathscr{P}(A)$

$$\liminf_{t\to\infty} (1/t) \log P_x((1/t)L_t(\omega, \cdot) \in G) \geq -\inf_{\mu \in G} I(\mu), \tag{1.1}$$

and for closed sets $C \subset \mathscr{P}(A)$,

$$\limsup_{t\to\infty} (1/t) \log P_x((1/t)L_t(\omega, \cdot) \in C) \leq -\inf_{\mu \in C} I(\mu). \tag{1.2}$$

Received April 1983; revised January 1985.
AMS 1980 *subject classifications*. Primary 60J60; secondary 60F10.
Key words and phrases. Diffusion processes with boundaries, large deviations.

[1] Work done while the author was at the University of California, Los Angeles.

Thus for large t, if U_μ is a small neighborhood of μ, then

$$e^{-t(I(\mu)+\varepsilon)} \le P_x((1/t)L_t(\omega, \cdot) \in U_\mu) \le e^{-t(I(\mu)-\varepsilon)}.$$

The transitivity and recurrence conditions are easily satisfied by the diffusion processes we have described above.

The first difficulty that arises in computing the I-function is that the domain $\mathscr{D}^+$ is not known explicitly. In [7], it was shown in a general context that the infimum could be taken over a nice dense subdomain. In particular, for the diffusion processes above,

$$(1.3)\qquad -\inf_{u\in\mathscr{D}^+} \int_A \frac{Lu}{u}\,d\mu = -\inf_{u\in\tilde{\mathscr{D}}^+} \int_A \frac{Lu}{u}\,d\mu$$

where $\tilde{\mathscr{D}}^+ = \tilde{\mathscr{D}} \cap \{u\colon u \ge c > 0\}$. Thus, it is the right-hand side of (1.3) that we will evaluate.

Let the region A be given by $\theta(x_1, x_2, \cdots, x_n) \ge 0$, $\theta \in C^2(R^n)$ and $|\nabla\theta| \ne 0$ on ∂A. We may write $J = ((J \cdot an)/|an|^2)an + J_1$ with $J_1 \perp an$. Since $|J \cdot n| \ge \gamma > 0$ and since a is strictly elliptic, the boundary condition $\nabla u \cdot J = 0$ is equivalent to the boundary condition $-\nabla u \cdot T_0 + \nabla u \cdot T = 0$ where $T_0(x) = a(x)n(x)$ and $T(x)$ is a vector field on ∂A—that is, $T(x)$ is a vector in the tangent space to ∂A at $x \in \partial A$. Since $n = (1/|\nabla\varphi|)(-\theta_{x_1}, \cdots, -\theta_{x_n}) \in C^1(\partial A)$, and since $J \in C^1(\partial A)$, $T(x)$ is a C^1-vector field on ∂A.

In Section 2, we prove

THEOREM 1.4. *Let $\mu \in \mathscr{P}(A)$ with density $\varphi \in C^1(A)$ and set $g = \varphi^{1/2}$. If $\int_A (|\nabla\varphi|^2/\varphi)\,dx = \infty$, or equivalently, $\int_A |\nabla g|^2\,dx = \infty$, then $I(\mu) = \infty$. If $\int_A (|\nabla\varphi|^2/\varphi)\,dx < \infty$, or equivalently, $\int_A |\nabla g|^2\,dx < \infty$, then*

$$(1.5)\qquad \begin{aligned} I(\mu) &= \frac{1}{2}\int_A \left(\frac{\nabla g}{g} - a^{-1}b\right)a\left(\frac{\nabla g}{g} - a^{-1}b\right)g^2\,dx - \frac{1}{2}\int_{\partial A}\left(\frac{\nabla g}{g}\cdot T\right)g^2\,d\sigma \\ &\quad - \inf_{h\in C^2(A)}\left[\frac{1}{2}\int (\nabla h - a^{-1}b)a(\nabla h - a^{-1}b)g^2\,dx - \frac{1}{2}\int_{\partial A}(\nabla h \cdot T)g^2\,d\sigma\right]. \end{aligned}$$

In fact there exist positive constants c_1, c_2, c_3 and c_4 depending on a, b, T and A but not on φ such that

$$(1.6)\qquad c_1\int \frac{|\nabla\varphi|^2}{\varphi}\,dx - c_2 \le I(\mu) \le c_3\int\frac{|\nabla\varphi|^2}{\varphi}\,dx + c_4.$$

Furthermore, $I(\mu)$ can be written as

$$(1.7)\qquad \begin{aligned} I(\mu) &= \int_A \frac{\nabla\varphi a\nabla\varphi}{8\varphi}\,dx - \int_A \frac{b\cdot\nabla\varphi}{2}\,dx \\ &\quad - \int_{\partial A}\frac{\nabla\varphi\cdot T}{4}\,d\sigma + \frac{1}{2}\int_A (\nabla h a \nabla h)\varphi\,dx \end{aligned}$$

where $h \in W_1^2(A, d\mu)$ is the unique weak solution in $W_1^2(A, d\mu)$ (up to a constant) *to the variational equation*

$$(1.8)\quad \int_A (\nabla h a \nabla q)\varphi\, dx - \int_A (\nabla q \cdot b)\varphi\, dx - \frac{1}{2}\int_{\partial A} (\nabla q \cdot T)\varphi\, d\sigma = 0$$

for all $q \in C^1(A)$.

If $b \in C^1(A)$ *and* $\varphi > 0$, *then h also is the unique solution (up to a constant) of the elliptic equation*

$$(1.9)\quad \begin{aligned} &\text{i) } \nabla \cdot [\varphi(b - a\nabla h)] = 0 \quad \text{on} \quad A \\ &\text{ii) } 2\varphi(b - a\nabla h) \cdot n = \nabla \cdot (\varphi T) \quad \text{on} \quad A. \end{aligned}$$

($d\sigma$ is the volume element on ∂A and $W_1^2(A, d\mu) \subset L^2(A, d\mu)$ consists of functions with one generalized $L^2(A, d\mu)$ derivative. More explicitly, we define $W_1^2(A, d\mu)$ as the closure of $C^1(A)$ under the norm $\|\ \|_1$ defined by $\|u\|_1 = (\int_A u^2\, d\mu + \int_A |\nabla u|^2\, d\mu)^{1/2}$. For Lebesgue measure, dx, we will write $W_1^2(A) \equiv W_1^2(A, dx)$.)

Theorem 1.4 can be slightly strengthened with no additional work. At the end of Section 2 we prove

COROLLARY 1.10. *Let* $\mu \in \mathscr{P}(A)$ *with density* $\varphi \in W_1^2(A)$. *Then* $I(\mu)$ *is finite or infinite according as* $\int_A (|\nabla\varphi|^2/\varphi)\, dx$ *is finite or infinite (equivalently,* $g = \varphi^{1/2}$ *does or does not belong to* $W_1^2(A)$). *In the finite case,* (1.6) *is still valid and* (1.5), (1.7), *and* (1.8) *hold with the term*

$$\frac{1}{4}\int_{\partial A} (\nabla\varphi \cdot T)\, d\sigma \quad \left(= \frac{1}{2}\int_{\partial A} \left(\frac{\nabla g}{g} \cdot T\right) g^2\, d\sigma\right)$$

replaced by

$$\frac{1}{4}\int_A \left(\textstyle\sum_{i,j=1}^n (c_{ij})_{x_j}\varphi_{x_i}\right) dx. \quad \left(= \frac{1}{2}\int_A \left(\textstyle\sum_{i,j=1}^n (c_{ij})_{x_j}(g_{x_i}/g) g^2\, dx\right)\right).$$

Here $C = \{c_{ij}\}$ *is the skew symmetric matrix discussed in Lemma* 2.6.

Several remarks are in order.

REMARK 1. Suppose there exists a reference measure β with respect to which the semigroup is selfadjoint. Then there exists a corresponding selfadjoint generator $(\hat{L}, \hat{\mathscr{D}})$ on $L_2(A, \beta(dx))$. Donsker and Varadhan [2] have shown that for $\mu \in \mathscr{P}(A)$,

$$(1.11)\quad \begin{aligned} I(\mu) &= \|(-\hat{L})^{1/2} f^{1/2}\|_2^2 \\ &\quad \text{if} \quad \mu \text{ has a density } d\mu/d\beta = f \quad \text{with} \quad f^{1/2} \in \hat{\mathscr{D}}_{1/2} \\ &= \infty, \quad \text{otherwise.} \end{aligned}$$

Here $\hat{\mathscr{D}}_{1/2}$ is the domain of the selfadjoint operator $(-\hat{L})^{1/2}$. In our case, taking β to be Lebesgue measure, we encounter selfadjointness if $T \equiv 0$ and $b \equiv 0$, in which case we obtain from Theorem 1.4, $I(\mu) = \int_A (\nabla g a \nabla g/2)\, dx$. Since $\hat{\mathscr{D}}_{1/2} = \{h: \int_A (\nabla h a \nabla h)\, dx < \infty\}$, this is in accordance with (1.11).

REMARK 2. If $b = a\nabla\psi$ for some $\psi \in C^1(A)$, and $T \equiv 0$, then the variational part of Theorem 1.4 vanishes and we are left with $I(\mu) = \frac{1}{2} \int_A ((\nabla g/g) - a^{-1}b)a((\nabla g/g) - a^{-1}b)g^2\, dx$. In this case, the semigroup is selfadjoint with respect to the measure with density $e^{2\psi}$ and one can check readily that the calculation agrees with (1.11).

REMARK 3. In [4], Donsker and Varadhan obtained a formula for $I(\mu)$ for diffusions on R^n with C^∞ coefficients in the case where μ has a density $\varphi \in C_0^\infty(R^n)$. Their formula is identical to (1.5) if we set $T \equiv 0$ and $A = R^n$.

The theorem and corollary show that under the a priori assumption that μ has a density $\varphi \in W_1^2(A)$, $I(\mu)$ is finite or infinite according as $\int_A (|\nabla\varphi|^2/\varphi)\, dx < \infty$ or $\int_A (|\nabla\varphi|^2/\varphi)\, dx = \infty$. In Section 3, we briefly discuss the case in which μ has no density in $W_1^2(A)$. Our theorem and corollary and Remark 1 strongly suggest that $I(\mu) = \infty$ for all such measures. However, this is difficult to prove—we cannot integrate by parts so it is difficult to get a handle on $\int_A (Lu/u)\, d\mu$. In fact, we have shown elsewhere that $I(\mu) = \infty$ for most such measures [6].

2. Proof of Theorem 1.4. To simplify notation, we will sometimes write $-T_0 \cdot \nabla g + T \cdot \nabla g = 0$ as $J \cdot \nabla g = 0$. For $u \in \hat{\mathscr{D}}^+$, let $u = e^g$. Then

$$(2.1) \quad I(\mu) = -\inf_{\{g \in C^2(A): J\cdot\nabla g = 0 \text{ on } \partial A\}} \int_A \left[\frac{1}{2}\nabla \cdot a\nabla g + b \cdot \nabla g + \frac{1}{2}\nabla g a \nabla g\right]\varphi\, dx.$$

We would like to write $g = \frac{1}{2}\log\varphi - h$ for some $h \in C^2(A)$. But if $\varphi \notin C^2(A)$ or if φ is not strictly positive, this is not possible. If we extend φ continuously to a function on all of R^n with compact support and mollify with the smooth kernel $G_\varepsilon(x) = (2\pi\varepsilon)^{-n/2}e^{-|x|^2/2\varepsilon}$, then $\varphi_\varepsilon = G_\varepsilon * \varphi$ is smooth and strictly positive on A and as ε approaches zero $\varphi_\varepsilon \to \varphi$ and $\nabla_{\varphi_\varepsilon} \to \nabla_\varphi$ uniformly on A since $\varphi \in C^1(A)$. In order to apply the dominated convergence theorem later on, we need to consider $\varphi_\varepsilon + \delta$ rather than φ_ε. After we let $\varepsilon \to 0$, we will let $\delta \to 0$. Thus we write $g = \frac{1}{2}\log(\varphi_\varepsilon + \delta) - h$ for $h \in C^2(A)$. Since g satisfies $-\nabla g \cdot T_0 + \nabla g \cdot T = 0$, h satisfies

$$-\nabla h \cdot T_0 + \nabla h \cdot T = (\nabla\varphi_\varepsilon \cdot T)/(2(\varphi_\varepsilon + \delta)) - (\nabla\varphi_\varepsilon \cdot T_0)/(2(\varphi_\varepsilon + \delta)) \equiv \gamma.$$

Substituting $g = \frac{1}{2}\log(\varphi_\varepsilon + \delta) - h$ into (2.1) gives

$$I(\mu) = -\inf_{\{h \in C^2(A): J\cdot\nabla h = \gamma \text{ on } \partial A\}}$$

$$(2.2) \quad \cdot\left[\int_A \left(-\frac{1}{2}\nabla \cdot a\nabla h - b \cdot \nabla h + \frac{1}{2}\nabla h a \nabla h - \frac{1}{2}\frac{\nabla h a \nabla\varphi_\varepsilon}{\varphi_\varepsilon + \delta}\right)\varphi\, dx\right]$$

$$- \int_A \frac{\nabla\varphi_\varepsilon a \nabla\varphi_\varepsilon}{8(\varphi_\varepsilon + \delta)^2}\varphi\, dx - \frac{1}{4}\int_A \varphi\nabla \cdot \left(\frac{a\nabla\varphi_\varepsilon}{\varphi_\varepsilon + \delta}\right) dx - \int_A \frac{b \cdot \nabla\varphi_\varepsilon}{2(\varphi_\varepsilon + \delta)}\varphi\, dx.$$

By the divergence theorem, the boundary condition on h, and the fact that $a(x)$ is hermitian,

$$\int_A \frac{1}{2}\nabla\cdot(a\nabla h)\varphi\,dx = \frac{1}{2}\int_{\partial A}(\varphi\nabla h\cdot T)\,d\sigma + \frac{1}{4}\int_{\partial A}\left(\frac{\nabla\varphi_\varepsilon\cdot T_0}{\varphi_\varepsilon+\delta}\varphi\right)d\sigma - \frac{1}{4}\int_{\partial A}\left(\frac{\nabla\varphi_\varepsilon\cdot T}{\varphi_\varepsilon+\delta}\varphi\right)d\sigma - \frac{1}{2}\int_A(\nabla\varphi a\nabla h)\,dx.$$

Using this in (2.2), we obtain

$$\begin{aligned} I(\mu) = &-\int_A \frac{\nabla\varphi_\varepsilon a\nabla\varphi_\varepsilon}{8(\varphi_\varepsilon+\delta)^2}\varphi\,dx - \frac{1}{4}\int_A \varphi\nabla\cdot\left(\frac{a\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}\right)dx \\ &-\int_A \frac{b\cdot\nabla\varphi_\varepsilon}{2(\varphi_\varepsilon+\delta)}\varphi\,dx + \int_{\partial A}\frac{\nabla\varphi_\varepsilon\cdot T_0}{4(\varphi_\varepsilon+\delta)}\varphi\,d\sigma - \int_{\partial A}\frac{\nabla\varphi_\varepsilon\cdot T}{4(\varphi_\varepsilon+\delta)}\varphi\,d\sigma \\ &-\inf_{\{h\in C^2(A):J\cdot\nabla h=\gamma \text{ on } \partial A\}}\left[\int_A\left(\left(\frac{1}{2}\nabla ha\nabla h\right)\varphi - \frac{1}{2}\frac{\nabla ha\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}\varphi + \frac{1}{2}\nabla\varphi a\nabla h \right.\right. \\ &\left.\left. -(b\cdot\nabla h)\varphi\right)dx - \int_{\partial A}\frac{(\nabla h\cdot T)}{2}\varphi\,d\sigma\right]. \end{aligned} \tag{2.3}$$

Consider now just the variational term. For $\varepsilon, \delta > 0$, let

$$\begin{aligned} \psi_{\varepsilon,\delta}(h) \equiv &\left[\int_A\left(\frac{1}{2}(\nabla ha\nabla h)\varphi - \frac{1}{2}\frac{\nabla ha\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}\varphi\right.\right. \\ &\left.\left.+\frac{1}{2}\nabla\varphi a\nabla h - (b\cdot\nabla h)\varphi\right)dx - \int_{\partial A}\frac{(\nabla h\cdot T)\varphi}{2}\,d\sigma\right]. \end{aligned} \tag{2.4}$$

An important step in proving our theorem is

LEMMA 2.5.

$$\inf_{\{h\in C^2(A):J\cdot\nabla h=\gamma \text{ on } \partial A\}}\psi_{\varepsilon,\delta}(h) = \inf_{h\in C^2(A)}\psi_{\varepsilon,\delta}(h).$$

To prove the lemma, we will need the estimate $\psi_{\varepsilon,\delta}(h) \le M\int_A|\nabla h|^2\,dx$. To show this, we first must show that the integrals on ∂A may be converted to integrals on A involving no derivatives higher than the first order. This can be done using the divergence theorem.

LEMMA 2.6. *Let $f, g \in C^1(A)$ and let T be a C^1-vector field on ∂A. Then there exists a skew-symmetric matrix $C(x)$ with $c_{ij}(x)\in C^1(A)$, and for $x\in\partial A$, $T = Cn$ where n is the outward unit normal on ∂A. Furthermore,*

$$\int_{\partial A}(f\nabla g\cdot T)\,d\sigma = \int_A\left(\textstyle\sum_{i,j=1}^n (fc_{ij})_{x_j}g_{x_i}\right)dx. \tag{2.7}$$

Thus, in particular $\int_{\partial A}(f\nabla g\cdot T)\,d\sigma$ can be expressed as an integral over A involving f, g, ∇f, and ∇g but no derivatives of higher order.

PROOF. Let Q be the constant matrix with entries $q_{21} = -q_{12} = 1$ and all other entries zero. Let $S(x)$ be the $n \times n$ matrix whose first two columns are, respectively, $n(x)$ and $T(x)$, and the rest of whose columns are identically zero. Then $C(x) \equiv S(x)QS^T(x)$ is skew symmetric, satisfies $C(x)n(x) = T(x)$, and has entries $c_{ij}(x)$ in $C^1(\partial A)$ since the components of $n(x)$ and $T(x)$ are in $C^1(\partial A)$. Now extend $C(x)$ to the interior so that it is still skew symmetric and continuously differentiable. This proves the first part of the lemma. The second part is just the divergence theorem. First assume $g \in C^2(A)$. We have

$$\begin{aligned}\int_{\partial A} (f\nabla g \cdot T)\, d\sigma &= \int_{\partial A} (f\nabla gC) \cdot n\, d\sigma = \int_A \nabla \cdot (f\nabla gC)\, dx \\ &= \int_A \left(\textstyle\sum_{i,j=1}^n (fc_{ij})_{x_j} g_{x_i}\right) dx + \int_A \left(\textstyle\sum_{i,j=1}^n fc_{ij} g_{x_i x_j}\right) dx \\ &= \int_A \left(\textstyle\sum_{i,j=1}^n (fc_{ij})_{x_j} g_{x_i}\right) dx,\end{aligned}$$

since $c_{ij} = -c_{ji}$. Thus if $g \in C^2(A)$, we have

$$\int_{\partial A} (f\nabla g \cdot T)\, d\sigma = \int_A \left(\textstyle\sum_{i,j=1}^n (fc_{ij})_{x_j} g_{x_i}\right) dx.$$

It is clear then that in fact this must hold for $g \in C^1(A)$. This completes the proof of Lemma 2.6.

We use Lemma 2.6 to prove Lemma 2.5.

PROOF OF LEMMA 2.5. Apply Lemma 2.6 to the boundary term in $\psi_{\varepsilon,\delta}(h)$. We obtain

$$\frac{1}{2}\int_{\partial A} (\varphi\nabla h \cdot T)\, d\sigma = \frac{1}{2}\int_A \left(\textstyle\sum_{i,j=1}^n (\varphi c_{ij})_{x_j} h_{x_i}\right) dx.$$

Hence from (2.4) we may write

$$\begin{aligned}\psi_{\varepsilon,\delta}(h) &= \int_A \frac{(\nabla h a \nabla h)}{2}\varphi\, dx - \int_A \nabla h \cdot \left[\frac{1}{2}\frac{a\nabla\varphi_\varepsilon}{\varphi_\varepsilon + \delta}\varphi - \frac{1}{2}a\nabla\varphi + \varphi b\right] dx \\ &\quad - \int_A \frac{1}{2}\left(\textstyle\sum_{i=1}^n \sum_{j=1}^n (\varphi c_{ij})_{x_j} h_{x_i}\right) dx.\end{aligned} \tag{2.8}$$

In particular,

$$|\psi_{\varepsilon,\delta}(h)| \le M \int_A |\nabla h|^2\, dx \quad \text{for some constant} \quad M,$$

by the Schwarz inequality and the fact that φ is bounded. Pick any $q \in C^2(A)$. Define $\nu(x) \equiv -\nabla q \cdot T_0 + \nabla q \cdot T$. To prove our lemma, it suffices to show that one can pick $r(x) \in C^2(A)$ with $-\nabla r \cdot T_0 + \nabla r \cdot T = -\nu(x) + \gamma(x)$ on ∂A and $\int_A |\nabla r|^2\, dx$ arbitrarily small. For then $h \equiv q + r$ satisfies the boundary condition and $|\psi_{\varepsilon,\delta}(h) - \psi_{\varepsilon,\delta}(q)|$ is as small as we like. Let $-\nu(x) + \gamma(x) \equiv \zeta(x)$. Note that $\zeta \in C^1(\partial A)$. Extend ζ to all of R^n in such a manner that $\zeta \in C^1(R^n)$ and is bounded. It suffices to show that for any $\delta > 0$, we can pick an $r_\delta(x) \in C^2(A)$ satisfying $-\nabla r_\delta \cdot T = \zeta$ on ∂A wit $r_\delta = 0$ except in a δ-neighborhood of ∂A and with $|\nabla r_\delta| \le D$ for some constant D independent of δ.

The idea behind our method is as follows. We would like to obtain a smooth one-to-one map which takes each point x in a neighborhood of $\partial A \cap A$ to a pair (y, t) with $y \in \partial A$ and $t \in R^+$ such that $y + t(-T_0 + T)(y) = x$. Let $\rho = \sup_{x\in\partial A}|(-T_0 + T)(x)|$ and let $\delta_1 = \delta/\rho$. We would like to define $r_\delta(x) = (\delta_1 - t)^3/3\delta_1^2)\zeta(y)$ if $t \le \delta_1$ and $r_\delta(x) = 0$ if $t > \delta_1$. Then $r_\delta(x)$ is supported in the δ-neighborhood of ∂A and for $y \in \partial A$, $\nabla r_\delta(x) \cdot (-T_0 + T)(x)|_{x=y} = (\partial/\partial t)r_\delta(x)|_{x=y} = \zeta(y)$. In actuality the function will be a little more complicated than this because $\zeta(y)$ is only C^1 and $r_\delta(x)$ must be C^2; we will use a mollification procedure. The one-to-one mapping that we wish to obtain may be achieved locally using the implicit and inverse function theorems. With the help of a partition of unity, this is enough. Unfortunately, we will need a somewhat cumbersome notation to prove all this rigorously.

Let x_i denote the ith coordinate of $x \in R^n$. Let $x^i = (x_1, \cdots, \check{x}_i, \cdots, x_n) \in R^{n-1}$ and let $x_z^i = (x_1, \cdots, x_{i-1}, z, x_{i+1}, \cdots, x_n) \in R^n$. For $y \in \partial A$, pick $i_y \in \{1, 2, \cdots, n\}$ such that $\theta_{x_{i_y}}(y) \ne 0$ and consider the map $M_y\colon R^n \to R^n$ defined by $M_y x = x^{i_y}_{\theta(x)}$. Since $\theta_{x_{i_y}}(y) \ne 0$, the inverse function theorem guarantees the existence of a neighborhood U_y of y and a C^2-function $\varphi_y\colon \{x^{i_y}\colon x \in U_y\} \to R^1$ satisfying $\theta(x^{i_y}_{\varphi_y(x^{i_y})}) = 0$ for $x \in U_y$. Let $V_y = \{x^{i_y}\colon x \in U_y\}$. Define the map $S_y\colon V_y \times R \to R^n$ by

$$S_y(x^{i_y}, t) = x^{i_y}_{\varphi(x^{i_y})} + t(-T_0 + T)(x^{i_y}_{\varphi(x^{i_y})}).$$

One can check that the determinant of the Jacobian of the map S_y evaluated at $(y^{i_y}, 0)$ is $-\nabla\theta(y) \cdot (-T_0 + T)(y)$ which is nonzero by hypothesis. Thus by the inverse function theorem, S_y^{-1} exists on an open neighborhood W_y of y. Thus through S_y^{-1}, every point $x \in W_y$ is specified uniquely by a boundary point $z \in \partial A$ and a real number t such that $x = z + t(-T_0 + T)(z)$. Note that $t \equiv t_y(x) = (S_y^{-1}(x))_n$ for $x \in W_y$. Let

$$Q_y(x) = ((S_y^{-1}(x))_1, \cdots, (S_y^{-1}(x))_{i_y-1}, \varphi_y((S_y^{-1}(x))^n), (S_y^{-1}(x))_{i_y}, \cdots, (S_y^{-1}(x))_{n-1})$$

for $x \in W_y$. $Q_y(x)$ is just the point $z \in \partial A$ mentioned above. That is,

$$x = Q_y(x) + t_y(x)(-T_0 + T)(Q_y(x)). \tag{2.9}$$

From this, it is clear that for functions $g(x)$ and $h(x)$ of the form $g(x) = f(Q_y(x))$ and $h(x) = F(t_y(x))$,

$$\nabla g(x) \cdot (-T_0 + T)(x)|_{x=z} = 0 \quad \text{and}$$

$$\nabla h(x) \cdot (-T_0 + T)(x)|_{x=z} = (F')(t_y(z)) = F'(0)$$

for $z \in \partial A \cap W_y$.

Pick $\{y_j\}_{j=1}^m \subset \partial A$ such that $\cup_{j=1}^m W_{y_j}$ covers ∂A and let $\{\psi_j\}_{j=1}^m$ be a partition of unity for ∂A subordinate to the cover $\{W_{y_j}\}_{j=1}^m$. That is, each ψ_j is smooth and vanishes off W_{y_j} and $\sum_{j=1}^{m} \psi_j(x) = 1$ for all $x \in \partial A$. To simplify notation, we will call Q_{y_i}, W_{y_i} and t_{y_i} by Q_i, W_i and t_i. Let $\Gamma(x) = e^{-|x|^2/2}/(2\pi)^{n/2}$, $x \in R^n$ and let

$$P_i(x, t) = \frac{1}{t^{n-1}} \int_{R^n} \zeta(y)\psi_i(y)\Gamma\left(\frac{x-y}{t}\right) dy, \quad t > 0.$$

Note that $\lim_{t\to 0} P_i(x, t) = 0$ and $\lim_{t\to 0}(1/t)P_i(x, t) = \zeta(x)\psi_i(x)$. Thus define $P_i(x, 0) = 0$. Set

$$\begin{aligned} r_\delta^i(x, t) &= ((\delta - t)^3/\delta^3)P_i(x, t), \quad 0 \le t \le \delta \\ &= 0, \qquad\qquad\qquad\qquad\quad t > \delta. \end{aligned}$$

Since $r_\delta^i(x, 0) = 0$, we have

$$(\partial/\partial t)r_\delta^i(x, t)\,|_{t=0} = \lim_{t\to 0} r_\delta^i(x, t)/t = \zeta(x)\psi_i(x).$$

We will show that $|\nabla r_\delta^i(x, t)|$ is bounded independent of δ and that $r_\delta^i(x, t) \in C^2(R^n \times R^+)$. Assuming this for the moment, we can now construct the function $r_\delta(x)$ with the desired properties. Pick $\delta_0 > 0$ such that $[0, \delta_0] \subset \bigcap_{j=1}^{n}\{t_j(x)\colon x \in W_j\}$. Let $\rho = \sup_{x\in\partial A}|(-T_0 + T)(x)|$ and let $\delta_1 = \delta/\rho$. For $\delta_1 < \delta_0$ and $x \in A$, define $r_\delta(x) = \sum_{i=1}^{m} r_{\delta_1}^i(Q_i(x), t_i(x))$. Thus $r_\delta(y) \in C^2(A)$, $|\nabla r_\delta(x)|$ is bounded independent of δ, $r_\delta(x)$ vanishes outside the δ-neighborhood of ∂A, and using (2.9) and the remark which follows it, we see that for $y \in \partial A$,

$$\begin{aligned} \nabla r_\delta(t) \cdot (-T_0 + T)(x)\,|_{x=y} &= \textstyle\sum_{i=1}^{m} (\partial/\partial t)\, r_{\delta_1}^i(Q_i(y), t)\,|_{t=0} \\ &= \textstyle\sum_{i=1}^{m} \zeta(Q_i(y))\psi_i(Q_i(y)) \\ &= \textstyle\sum_{i=1}^{m} \zeta(y)\psi_i(y) = \zeta(y). \end{aligned}$$

We now show that $|\nabla r_\delta^i(x, t)|$is bounded independent of δ. Since $r_\delta^i(x, t) \equiv 0$ for $t > \delta$ and since $(\partial/\partial t)r_\delta^i(x, t)\,|_{t=0} = \zeta(x)\psi_i(x)$ independent of δ, we may restrict to $0 < t \le \delta$ when considering $(\partial/\partial t)r_\delta^i(x, t)$. We have

$$\frac{\partial}{\partial t} r_\delta^i(x, t) = \frac{(\delta - t)^3}{\delta^3}\frac{\partial}{\partial t}P_i(x, t) - \frac{3(\delta - t)^2}{\delta^3}P_i(x, t)$$

From above, we have $P_i(x, 0) = 0$ and $\lim_{t\to\infty}(1/t)P_i(x, t) = \varphi(x)\psi_i(x)$. This is enough to conclude that $\lim\sup_{\delta\to 0}\sup_{0\le t\le\delta}(\partial/\partial t)r_\delta^i(x, t) < \infty$. For the gradient in x, we have $\nabla_x r_\delta^i(x, t) = ((\delta - t)^3/3)\nabla_x P_i(x, t)$. In the course of proving that $r_\delta^i(x, t) \in C^2(R^n \times R^+)$, we will prove that $P_i(x, t) \in C^2(R^n \times R^+)$. Hence $\nabla_x P(x, t)$ causes no trouble as $t \to 0$, and it is clear that

$$\lim\sup_{\delta\to 0}\sup_{0\le t\le\delta}((\delta - t)^3/\delta^3)\nabla_x P_i(x, t) < \infty.$$

Thus we have shown that $|\nabla r_\delta^i(x, t)|$ is bounded independent of δ.

Now we show that $r_\delta^i(x, t) \in C^2(R^n \times R^+)$. We must show that $P_i(x, t) \in C^2(R^n \times R^+)$. Of course $P_i(x, t)$ is C^∞ for $t > 0$ so it is only at $t = 0$ that we must be careful. Changing variables, we have

$$P_i(x, t) = \frac{1}{t^{n-1}} \int_{R^n} \zeta\psi_i(y)\Gamma\left(\frac{x-y}{t}\right) dy = t \int_{R^n} \zeta\psi_i(x - tu)\Gamma(u)\, du.$$

Thus

$$\frac{\partial}{\partial t}P_i(x, t) = \int_{R^n} \zeta\psi(x - tu)\Gamma(u)\, du - t\int_{R^n} (\nabla(\zeta\psi_i))(x - tu)\cdot u\Gamma(u)\, du.$$

Changing variables in the second term on the right-hand side above, we obtain

$$\frac{\partial}{\partial t}P_i(x, t) = \int_{R^n} \zeta\psi(x - tu)\Gamma(u)\, du - \frac{1}{t^n}\int_{R^n} (\nabla(\zeta\psi_i))(v)\cdot(x - v)\Gamma\left(\frac{x - v}{t}\right) dv.$$

Thus

$$\begin{aligned}\frac{\partial^2}{\partial t^2} P_i(x, t) = &- \int_{R^n} (\nabla(\zeta\psi_i))(x - tu)\cdot u\Gamma(u)\, du \\ &+ \frac{n}{t^{n+1}}\int_{R^n} (\nabla(\zeta\psi_i))(v)\cdot(x - v)\Gamma\left(\frac{x - v}{t}\right) dv \\ &+ \frac{1}{t^{n+2}}\int_{R^n}\left[(\nabla(\zeta\psi_i))(v)\cdot(x - v)\right]\left[(\nabla\Gamma)\left(\frac{x - v}{t}\right)\cdot(x - v)\right] dv \\ = &- \int_{R^n} (\nabla(\zeta\psi_i))(x - tu)\cdot u\Gamma(u)\, du \\ &+ n\int_{R^n} (\nabla(\zeta\psi_i))(x - tu)\cdot u\Gamma(u)\, du \\ &+ \int_{R^n} [(\nabla(\zeta\psi_i))(x - tw)\cdot w][\nabla\Gamma(w)\cdot w]\, dw.\end{aligned}$$

Since $\zeta \in C^1(R^n)$, it is clear that $(\partial^2/\partial t^2)P_i(x, t)$ is continuous for all $t \geq 0$ and one can show similarly that the same is true of the other mixed partial derivatives. Thus $P_i(x, t) \in C^2(R^n \times R^+)$. This concludes the proof of Lemma 2.5.

At this point, we can show that $I(u) = \infty$ if $\int_A (|\nabla\varphi|^2/\varphi)\, dx = \infty$. Since $\psi_{\varepsilon,\delta}(h) = 0$ if h is a constant, we have $\inf_{h\in C^2(A)}\psi_{\varepsilon,\delta}(h) \leq 0$. Thus, using Lemma 2.5, applying the divergence theorem to $\frac{1}{4}\int_A \varphi\nabla\cdot(a\nabla\varphi_\varepsilon/(\varphi_\varepsilon + \delta))\, dx$, and letting $\varepsilon \to 0$, we obtain from (2.3),

$$I(u) \geq \int_A \frac{\nabla\varphi a\nabla\varphi}{8(\varphi + \delta)}\, dx - \int_A \frac{\varphi b\cdot\nabla\varphi}{2(\varphi + \delta)}\, dx - \frac{1}{4}\int_{\partial A} \frac{\varphi(\nabla\varphi\cdot T)}{\varphi + \delta}\, d\sigma.$$

Since a is strictly positive, letting $\delta \to 0$ gives us $I(u) = \infty$.

From here on in the proof, we assume $\int_A (|\nabla\varphi|^2/\varphi)\, dx < \infty$.

We want to let $\varepsilon \to 0$. Notice that only one integral in $\psi_{\varepsilon,\delta}$ depends on ε, and as $\varepsilon \to 0$, that integral approaches $\frac{1}{2} \int_A (\nabla h a \nabla \varphi/(\varphi + \delta))\varphi \, dx$. Let

$$\psi_{0,\delta}(h) \equiv \lim_{\varepsilon\to 0}\psi_{\varepsilon,\delta}(h)$$

$$= \int_A \frac{(\nabla h a \nabla h)}{2} \varphi \, dx - \int_A (b \cdot \nabla h)\varphi \, dx + \frac{1}{2}\int_A (\nabla h a \nabla \varphi) \, dx - \frac{1}{2}\int_A \frac{\nabla h a \nabla \varphi}{\varphi + \delta} \varphi \, dx - \int_{\partial A} \frac{\nabla h \cdot T}{2} \varphi \, d\sigma. \tag{2.10}$$

LEMMA 2.11.

$$\lim_{\varepsilon\to 0}\inf_{h\in C^2(A)}\psi_{\varepsilon,\delta}(h) = \inf_{h\in C^2(A)}\lim_{\varepsilon\to 0}\psi_{\varepsilon,\delta}(h) = \inf_{h\in C^2(A)}\psi_{0,\delta}(h).$$

PROOF. From (2.8), we may write for $\lambda > 0$,

$$\psi_{\varepsilon,\delta}(h) = \int_A \left[\frac{\nabla h a \nabla h}{2}\varphi - \frac{\nabla h a \nabla \varphi_\varepsilon}{2(\varphi_\varepsilon + \delta)}\varphi + \frac{\nabla h a \nabla \varphi}{2} - \varphi \nabla h \cdot b - \frac{1}{2}\sum_{i=1}^n \left(\sum_{j=1}^n (\varphi c_{ij})_{x_j}\right) h_{x_i}\right] dx$$

$$\geq \int_A \frac{(\nabla h a \nabla h)}{2}\varphi \, dx - \lambda \int_A \frac{(\nabla h a \nabla h)}{2}\varphi \, dx - \frac{1}{\lambda}\int_A \frac{(\nabla \varphi_\varepsilon a \nabla \varphi_\varepsilon)}{8(\varphi_\varepsilon + \delta)^2}\varphi \, dx$$

$$- \lambda \int_A \frac{(\nabla h a \nabla h)}{2}\varphi \, dx - \frac{1}{\lambda}\int_A \frac{\nabla \varphi a \nabla \varphi}{8\varphi} \, dx - \lambda \int_A \frac{\varphi |\nabla h|^2}{2} \, dx$$

$$- \frac{1}{\lambda}\int_A \frac{\varphi |b|^2}{2} \, dx - \lambda \int_A \frac{|\nabla h|^2}{2}\varphi \, dx$$

$$- \frac{1}{\lambda}\int_A \sum_{i=1}^n \left|\sum_{j=1}^n (\varphi c_{ij})x_j\right|^2/8\varphi \, dx$$

where in four places we have used the inequality

$$|AB| \leq \tfrac{1}{2}\lambda A^2 + (B^2/2\lambda)/2.$$

By the bounded and the dominated convergence theorems,

$$\lim_{\delta\to\infty}\lim_{\varepsilon\to 0}\int_A \frac{\nabla \varphi_\varepsilon a \nabla \varphi_\varepsilon}{8(\varphi_\varepsilon + \delta)^2}\varphi \, dx = \int_A \frac{\nabla \varphi a \nabla \varphi}{8\varphi} \, dx.$$

By hypothesis, $\int_A (\nabla \varphi a \nabla \varphi/8\varphi) \, dx < \infty$. Since $a(x)$ is strictly positive, by picking λ small enough we have from the above calculation,

$$\psi_{\varepsilon,\delta}(h) \geq c_1 \int_A |\nabla h|^2 \varphi \, dx - c_2 \tag{2.12}$$

for $\varepsilon \geq 0$ and $\delta > 0$ and also for $\varepsilon = \delta = 0$, where c_1 and c_2 are positive constants independent of h, ε and δ. Hence there exists a constant N independent of ε and δ such that

$$\inf_{h\in C^2(A)}\psi_{\varepsilon,\delta}(h) = \inf_{\{h\in C^2(A):\int_A|\nabla h|^2\varphi\,dx\le N\}}\psi_{\varepsilon,\delta}(h), \quad \text{for } \varepsilon \text{ and } \delta$$

in the same set as above.

Thus to prove the lemma, we need only show that the one term in $\psi_{\varepsilon,\delta}(h)$ which depends on ε, $\frac{1}{2}\int_A (\nabla h a \nabla\varphi_\varepsilon/\varphi_\varepsilon + \delta)\,\varphi\,dx$, converges to

$$\frac{1}{2}\int_A \frac{\nabla h a \nabla\varphi}{\varphi+\delta}\,\varphi\,dx$$

uniformly over $\{h\in C^2(A): \int_A |\nabla h|^2\varphi\,dx \le N\}$. But, by the Schwarz inequality, for such an h,

$$\left|\frac{1}{2}\int_A \frac{\nabla h a\nabla\varphi_\varepsilon}{(\varphi_\varepsilon+\delta)}\varphi\,dx - \frac{1}{2}\int_A\frac{(\nabla h a\nabla\varphi)}{\varphi+\delta}\varphi\,dx\right|^2$$
$$= \left|\frac{1}{2}\int_A\left[\nabla h a\left(\frac{\varphi\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}-\frac{\varphi\nabla\varphi}{\varphi+\delta}\right)\right]dx\right|^2$$
$$\le \frac{1}{2}\int_A(\nabla h a\nabla h)\varphi\,dx\cdot\int_A\left(\frac{\varphi^{1/2}\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}-\frac{\varphi^{1/2}\nabla\varphi}{\varphi+\delta}\right)a\left(\frac{\varphi^{1/2}\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}-\frac{\varphi^{1/2}\nabla\varphi}{\varphi+\delta}\right)dx$$
$$\le\frac{1}{2}\,\|a\|^2 N\int_A\left|\frac{\varphi^{1/2}\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}-\frac{\varphi^{1/2}\nabla\varphi}{\varphi+\delta}\right|^2\to 0 \quad\text{as}\quad \varepsilon\to 0,$$

by the bounded convergence theorem. This proves Lemma 2.11.

Taking the limit as $\varepsilon\to 0$ in (2.3) and using Lemmas 2.5 and 2.11, we obtain

$$I(\mu) = \lim_{\varepsilon\to 0}\left[-\int_A\frac{\nabla\varphi_\varepsilon a\nabla\varphi_\varepsilon}{8(\varphi_\varepsilon+\delta)^2}\varphi\,dx - \frac{1}{4}\int_A\varphi\nabla\cdot\left(\frac{a\nabla\varphi_\varepsilon}{\varphi_\varepsilon+\delta}\right)dx\right.$$
$$\left.-\int_A\frac{b\cdot\nabla\varphi_\varepsilon}{2(\varphi_\varepsilon+\delta)}\varphi\,dx+\int_{\partial A}\frac{(\nabla\varphi_\varepsilon\cdot T_0)}{4(\varphi_\varepsilon+\delta)}\varphi\,d\sigma-\int_{\partial A}\frac{\nabla\varphi_\varepsilon\cdot T}{4(\varphi_\varepsilon+\delta)}\varphi\,d\sigma\right] \tag{2.13}$$
$$-\inf_{h\in C^2(A)}\psi_{0,\delta}(h).$$

Applying the divergence theorem to the term

$$\frac{1}{4}\int_A\varphi\nabla\cdot\left(a\frac{\nabla\varphi_\varepsilon}{\varphi_{\varepsilon+\delta}}\right)dx,$$

then using this in (2.13) and letting $\varepsilon\to 0$, gives

$$I(\mu) = \int_A\frac{\nabla\varphi a\nabla\varphi}{8(\varphi+\delta)}dx - \int_A\frac{\varphi b\cdot\nabla\varphi}{2(\varphi+\delta)}dx - \frac{1}{4}\int_A\frac{\varphi(\nabla\varphi\cdot T)}{\varphi+\delta}d\sigma$$
$$-\inf_{h\in C^2(A)}\psi_{0,\delta}(h)$$

Let

$$\psi_{0,0}(h) \equiv \lim_{\delta\to 0}\psi_{0,\delta}(h)$$
$$= \frac{1}{2}\int_A (\nabla h a \nabla h)\varphi\, dx - \int_A (b\cdot\nabla h)\varphi\, dx - \int_{\partial A} \frac{\nabla h\cdot T}{2}\varphi\, d\sigma.$$

Since (2.12) holds for $\psi_{0,\delta}(h)$, the same argument used to prove Lemma 2.11 goes through here to show that

$$\lim_{\delta\to 0}\inf_{h\in C^2(A)}\psi_{0,\delta}(h) = \inf_{h\in C^2(A)}\lim_{\delta\to 0}\psi_{0,\delta}(h) = \inf_{h\in C^2(A)}\psi_{0,0}(h).$$

Thus, letting $\delta\to 0$ in (2.14), and adding and subtracting $\frac{1}{2}\int_A (ba^{-1}b)\varphi\, dx$, we obtain

$$I(\mu) = \int_A \frac{\nabla\varphi a\nabla\varphi}{8\varphi}\, dx + \int_A \frac{1}{2}(ba^{-1}b)\varphi\, dx$$
$$(2.14)\qquad - \int_A \frac{(b\cdot\nabla\varphi)}{2}\, dx - \frac{1}{4}\int_{\partial A}(\nabla\varphi\cdot T)\, d\sigma$$
$$- \inf_{h\in C^2(A)}\left[\frac{1}{2}\int_A (\nabla h - a^{-1}b)a(\nabla h - a^{-1}b)\varphi\, dx - \int_{\partial A}\frac{\nabla h\cdot T}{2}\varphi\, d\sigma\right].$$

Replacing φ by g^2 and collecting terms gives us (1.5).

We now show that (1.6) holds. We have for $\lambda > 0$,

$$(2.15)\qquad \left|\int_A \frac{b\cdot\nabla\varphi}{2}\, dx\right| \le \lambda\int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx + \frac{1}{16\lambda}\int_A |b|^2\varphi\, dx$$
$$\le \lambda\int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx + c$$

for c independent of φ. By (2.7),

$$\int_{\partial A}(\nabla\varphi\cdot T)\, d\sigma = \int_A \left(\textstyle\sum_{i=1}^n \left(\sum_{j=1}^n (c_{ij})_{x_j}\right)\varphi_{x_i}\right) dx.$$

Hence

$$(2.16)\qquad \frac{1}{4}\int_{\partial A}(\nabla\varphi\cdot T)\, d\sigma \le \lambda\int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx + \hat{c}$$

with $\hat{c}$ independent of φ. Also since $\psi_{0,0}(0) = 0$,

$$(2.17)\qquad \inf_{h\in C^2(A)}\psi_{0,0}(h) \le 0.$$

Thus, using the strict positivity of a and picking λ small enough, (2.14)–(2.17) give

$$(2.18)\qquad I(\mu) \ge c_1\int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx - c_2$$

for positive c_1, c_2 independent of φ. To obtain

$$I(\mu) \le c_3 \int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx + c_4 \tag{2.19}$$

for positive constants c_3, c_4 independent of φ, we need only show that $\inf_{h\in C^2(A)}\psi_0(h) \ge -k_1 - k_2 \int_A |\nabla\varphi|^2/\varphi\, dx$ for positive k_1, k_2 independent of φ. We have for $\lambda > 0$,

$$\int_A (b \cdot \nabla h)\varphi\, dx \le \lambda \int_A |\nabla h|^2 \varphi\, dx + c \tag{2.20}$$

for c independent of φ. Also

$$\begin{aligned} \left| \int_{\partial A} \frac{(\nabla h \cdot T)\varphi}{2}\, d\sigma \right| &= \left| \frac{1}{2} \int_A \textstyle\sum_{i=1}^n \sum_{j=1}^n (\varphi c_{ij})_{x_j} h_{x_i}\, dx \right| \\ &\le \lambda \int_A |\nabla h|^2 \varphi\, dx + \frac{1}{16\lambda} \int_A \textstyle\sum_{i=1}^n \left(\sum_{j=1}^n (\varphi c_{ij})_{x_j}/\varphi^{1/2}\right)^2 dx \\ &\le \lambda \int_A |\nabla h|^2 \varphi\, dx + \hat{c} \int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx + \hat{c} \end{aligned} \tag{2.21}$$

for $\hat{c} > 0$ independent of φ. Hence (2.20), (2.21) and strict positivity give for small enough λ,

$$\psi_{0,0}(h) \ge k_3 \int_A |\nabla h|^2 \varphi\, dx - k_2 \int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx - k_1 \tag{2.22}$$

for positive k_1, k_2, k_3 independent of φ. Thus,

$$\inf_{h\in C^2(A)} \psi_{0,0}(h) \ge -k_1 - k_2 \int_A \frac{|\nabla\varphi|^2}{\varphi}\, dx,$$

and (2.19) holds. Combining (2.18) and (2.19) gives (1.6).

We now show that (1.7) and (1.8) hold. From (2.14), we need only show that

$$\inf_{q\in C^2(A)} \psi_{0,0}(q) = -\int_A \frac{(\nabla h a \nabla h)}{2} \varphi\, dx \tag{2.23}$$

where h satisfies (1.8).

From (2.7), we have for $q \in C^2(A)$,

$$\begin{aligned} \psi_{0,0}(q) &= \int_A \left(\frac{1}{2} (\nabla q a \nabla q)\varphi - (b \cdot \nabla q)\varphi \right) dx - \int_{\partial A} \frac{\nabla q \cdot T}{2} \varphi\, d\sigma \\ &= \int_A \left(\frac{1}{2} (\nabla q a \nabla q)\varphi - (b \cdot \nabla q)\varphi \right) dx - \int_A \frac{1}{2} \textstyle\sum_{i=1}^n \left(\sum_{j=1}^n (\varphi c_{ij})_{x_j}\right) q_{x_i}\, dx. \end{aligned}$$

For $q \in W_1^2(A, d\mu)$, define

$$\tilde{\psi}_{0,0}(q) = \int_A \frac{1}{2} (\nabla q a \nabla q)\varphi\, dx - \int_A (b \cdot \nabla q)\varphi\, dx - \int_A \frac{1}{2} \sum_{i=1}^n \left(\sum_{j=1}^n (\varphi c_{ij})_{x_j}\right) q_{x_i}\, dx. \tag{2.24}$$

Thus $\tilde{\psi}_{0,0}(q) = \psi_{0,0}(q)$ for $q \in C^2(A)$.

From (2.22),

$$\tilde{\psi}_{0,0}(q) \ge k_1 \int_A |\nabla q|^2 \varphi\, dx - k_2 \quad \text{for constants} \quad k_1, k_2 > 0. \tag{2.25}$$

Now pick a minimizing sequence $q_n \in C^2(A)$ so that

$$\inf_{q\in C^2(A)} \psi_{0,0}(q) = \inf_{q \in C^2(A)} \tilde{\psi}_{0,0}(q) = \lim_{n\to\infty} \tilde{\psi}_{0,0}(q_n).$$

Since $\tilde{\psi}_{0,0}(q)$ depends on ∇q but not on q, we may fix a point $x_0 \in A$ and require $q_n(x_0) = 0$ for all n. By (2.25), there exists a c with $\int_A |\nabla q_n|^2 \varphi\, dx \le c$, for all n. Hence $\{q_n\}$ is weakly relatively compact in $W_1^2(A, d\mu)$. Pick a subsequence $\{q_{n'}\}$ with $q_{n'} \to_w h_0 \in W_1^2(A, d\mu)$. From (2.24) and the fact that $\int_A |\nabla\varphi|^2/\varphi\, dx < \infty$, we see that $\tilde{\psi}_{0,0}$ is lower semicontinuous with respect to weak convergence in $W_1^2(A, d\mu)$. This, together with the fact that $\{q_{n'}\}$ is a minimizing sequence, gives us

$$\inf_{q\in C^2(A)} \tilde{\psi}_{0,0}(q) \ge \tilde{\psi}_{0,0}(h_0).$$

But $C^2(A)$ is dense in $W_1^2(A, d\mu)$ and since $\int_A |\nabla\varphi|^2/\varphi < \infty$, $\tilde{\psi}_{0,0}$ is continuous in the $W_1^2(A, du)$ topology. Thus we have

$$\inf_{q\in C^2(A)} \tilde{\psi}_{0,0}(q) = \inf_{q \in W_1^2(A,\, d\mu)} \tilde{\psi}_{0,0}(q).$$

Hence, in fact

$$\tilde{\psi}_{0,0}(h_0) = \inf_{q\in W_1^2(A,d\mu)} \tilde{\psi}_{0,0}(q) = \inf_{q\in C^2(A)} \tilde{\psi}_{0,0}(q) = \inf_{q \in C^2(A)} \psi_{0,0}(q). \tag{2.26}$$

Then using the calculus of variations—that is, solving

$$d/d\varepsilon(\tilde{\psi}_{0,0}(h_0 + \varepsilon q))\,|_{\varepsilon=0} = 0, \quad \text{for all} \quad q \in W_1^2(A, d\mu),$$

we find that the function h_0 satisfies the weak variational equation,

$$\int_A ((\nabla h_0 a \nabla q)\varphi - (b \cdot \nabla q)\varphi)\, dx - \int_A \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\varphi c_{ij})_{x_j} q_{x_i}\, dx = 0, \tag{2.27}$$

$$\text{for all} \quad q \in W_1^2(A, d\mu).$$

Since $C^1(A)$ is dense in $W_1^2(A, d\mu)$ and since the variational equation as a function of q is continuous in the $W_1^2(A, d\mu)$ topology, it suffices to consider (2.27) for $q \in C^1(A)$.

Converting the second integral to a boundary integral, we have

$$\int_A ((\nabla h_0 a \nabla q)\varphi - (b \cdot \nabla q)\varphi)\, dx - \frac{1}{2} \int_{\partial A} (\nabla q \cdot T)\varphi\, d\sigma = 0, \tag{2.28}$$

for all $q \in C^1(A)$. This is (1.8).

Consider (2.27) for $q = h_0$. Using this with (2.24) and (2.26), we see that $\inf_{q \in C^2(A)} \psi_{0,0}(q) = -\int_A ((\nabla h_0 a \nabla h_0)/2)\varphi\, dx$. This gives us (1.7).

We now assume $b \in C^1(A)$ and $\varphi > 0$. We will show that (1.9) holds. Say (2.28) holds for some $h \in C^2(A)$ in place of h_0. Then, integrating by parts, we obtain

$$\int_A q[-\nabla \cdot (\varphi(a\nabla h - b))]\, dx$$

$$(2.29) \qquad + \int_{\partial A} q\left[\varphi a \nabla h \cdot n - \varphi b \cdot n + \frac{1}{2}\nabla \cdot (\varphi T)\right] d\sigma = 0,$$

$$\text{for all} \quad q \in C^1(A).$$

(In integrating by parts, we have used the fact that the divergence theorem gives

$$\int_{\partial A} (\varphi \nabla q \cdot T + q\nabla \cdot (\varphi T))\, d\sigma = \int_{\partial A} \nabla \cdot (\varphi q T)\, d\sigma = \int_{\partial(\partial A)} \varphi q T \cdot n = 0$$

since $\partial(\partial A)$ is empty.) So (2.29) implies that h satisfies the elliptic boundary value equation (1.9). It is easy to see that, conversely, any solution of (1.9) is also a solution of the weak variational equation (2.28). Equation (1.9) has a C^2 solution [5, page 122]. Call it h. If we consider $k \equiv h - h_0$ in (2.28), we obtain

$$(2.30) \qquad \int_A (\nabla k a \nabla q)\varphi\, dx = 0 \quad \text{for all} \quad q \in C^1(A).$$

But then in fact, (2.30) is true for all $q \in W_1^2(A, d\mu)$. Pick $q = k$. Then (2.30) becomes $\int_A (\nabla k a \nabla k)\varphi\, dx = 0$. Since φ is positive and a is strictly positive, we have, up to a constant, $k \equiv 0$ a.e. or $h = h_0$ a.e. Thus h_0 may in fact be identified with a C^2 function h which satisfies (1.9). This analysis also shows the uniqueness of the solution to (1.8) and concludes the proof of Theorem 1.4.

We now turn to Corollary 1.10. Since it is still true that $\varphi_\varepsilon \to_{L^2} \varphi$ and $\nabla\varphi_\varepsilon \to_{L^2} \nabla\varphi$, all we need do is replace the term $\int_{\partial A} ((\nabla\varphi_\varepsilon \cdot T)/4(\varphi_\varepsilon + \delta))\varphi\, d\sigma$ in (2.13) by

$$\frac{1}{4}\int_A \sum_{i,j=1}^n \left(\frac{\varphi}{\varphi_\varepsilon + \delta} c_{ij}\right)_{x_j} (\varphi_\varepsilon)_{x_i}\, dx$$

and proceed as before.

3. Measures without nice densities. An important application of the I-function theory is to the evaluation of asymptotic limits of functional integrals. This evaluation usually involves the infimum of the I-function over all probability measures supported in a certain region. Thus, we are interested in a representation for the I-function for all measures. The representation in Section 2 is for a measure μ with a density $\varphi \in W_1^2(A)$. We relied heavily on integration by parts, and it seems clear that this is the only way to obtain much of an explicit representation for the I-function. However, if μ does not posses a density $\varphi \in W_1^2(A)$, we cannot generally integrate by parts. Mollifying μ to give μ_ε with a smooth density φ_ε does not help because the I-function is only lower semicontinuous. Thus $I(\mu_\varepsilon) \to \infty$ does not allow us to conclude that $I(\mu) = \infty$. If L has constant coefficients and there is no boundary, that is $A = R^n$, then one can convolve with a translation invariant kernel and use the convexity of the I-function along with lower semicontinuity to obtain $I(\mu) = \lim_{\varepsilon\to 0} I(\mu_\varepsilon)$. For bounded regions, this is not possible.

However, from Section 2, we see that under the a priori assumption that $\varphi \in W_1^2(A)$, then $I(\mu)$ is finite if and only if $\int_A (|\nabla\varphi|^2/\varphi)\,dx < \infty$, that is, if and only if $g = \varphi^{1/2} \in W_1^2(A)$. In fact, (1.6) holds. Furthermore, in the case that $a^{-1}b = \nabla\psi$ and $T \equiv 0$, the semigroup is selfadjoint on $L_2(A, d\nu)$ with $d\nu/dx = e^{2\psi}$, and (1.11) holds. For any measure μ with $d\mu/dx = \varphi$, considering $d\mu/d\nu = \varphi e^{-2\psi}$ and using (1.11), we see that

$$I(\mu) = \int_A \frac{\nabla\varphi a \nabla\varphi}{8\varphi}\,dx + \frac{1}{2}\int_A (ba^{-1}b)\varphi\,dx - \int_A \frac{b\cdot\nabla\varphi}{2}\,dx$$

if the right-hand side is finite and equals infinity otherwise. Since a is strictly elliptic, we see that in the selfadjoint case

$$\text{(3.1)} \quad I(\mu) < \infty \quad \text{if and only if } \mu \text{ has a density } \varphi \text{ with } \quad \varphi^{1/2} \in W_1^2(A).$$

From the above, it appears likely that (3.1) is also true in the nonselfadjoint case. In fact, in [6] we proved the following theorem which places no a priori restrictions on the density of μ.

THEOREM 3.2. *(a) If $T \equiv 0$, then $I(\mu) < \infty$ if and only if μ has a density φ with $\varphi^{1/2} \in W_1^2(A)$.*

(b) If $T \not\equiv 0$, let $V = \{D: D$ is open, $D \subset A$ with $\bar{D} \cap \partial A = \emptyset$. For any $D \in V$, consider $\mu \in \mathscr{P}(A)$ restricted to D as a measure $\mu \in \mathscr{M}(D)$, the space of finite measures on D. Define

$$I_0^D(\mu) = \int_D \frac{|\nabla\varphi|^2}{\varphi}\,dx \quad \text{if } \mu \in \mathscr{M}(D) \text{ has a density } \varphi \text{ with } \quad \varphi^{1/2} \in W_1^2(D)$$

$$= \infty, \qquad \text{otherwise.}$$

If there exists a $D \in V$ for which $I_0^D(\mu) = \infty$, then $I(\mu) = \infty$.

Thus, if $T \not\equiv 0$, we are left to account for the case in which the measure μ has a density $\varphi \notin W_1^2(A)$ with $\varphi^{1/2} \in W_1^2(D)$ for all $D \in V$. That is, the only singularities in φ occur on ∂A. We conjecture that $I(\mu) = \infty$ for such measures, and thus, that (3.1) holds.

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