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SPECIAL INVITED PAPER

LIMIT THEOREMS FOR LARGE DEVIATIONS AND REACTION-DIFFUSION EQUATIONS

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Dedicated to the 60th birthday of Professor E. B. Dynkin

The equation $u_t = u_{xx} + u(1 - u)$ is the simplest reaction-diffusion equation. Introduction of a small parameter allows construction of geometric optics approximations for the solutions of such equations; these solutions are approximated by step-functions with the values 0 and 1. The region where the solution is close to 1 propagates according to the Huygens principle for the corresponding velocity field $v(x, e)$ which is calculated via the equation. New effects may emerge, such as stops and jumps of the wave front.

The Feynman–Kac formula implies that the solutions of certain Cauchy problems obey some integral equations in the space of trajectories of the corresponding Markov processes.

Examination of this equation requires the study of Laplace-type asymptotics for functional integrals. These asymptotics are defined by large deviations for the corresponding family of processes and are expressed through action functionals.

1. Introduction. By now it has become usual to use probabilistic methods when studying one or another problem for second-order elliptic and parabolic differential operators. As it is known (see Dynkin, 1965), a Markov process (X_t, P_x) is associated with every such operator. The solutions of basic boundary value problems for the linear differential equation containing this operator may be represented as the mean value of the corresponding functional of the trajectories of the process (X_t, P_x) (as a functional integral). Direct probabilistic methods developed in recent decades for examining Markov processes enable one to study these functionals and thus to study the solutions of boundary value problems. This approach has allowed one to obtain new results on the spectral properties of operators and to examine boundary value problems for degenerate equations, problems with small parameters in higher derivatives and a number of other problems (Kac, 1951; Simon, 1979; Freidlin and Wentzell, 1984).

Methods employed in the research of linear equations frequently throw light on related nonlinear problems. Such is the case with probabilistic methods. Besides, one should mention that some classes of nonlinear equations are directly connected with specific probabilistic objects, such as controlled diffusion processes (Krylov, 1977) and branching processes with diffusion (Skorohod, 1964; Ikeda and Watanabe, 1970; McKean, 1975).

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This paper considers some asymptotic problems for reaction-diffusion equations. Such equations arise in many problems in physics, chemical kinetics and biology (for references, see Smoller, 1983). In this field there are fairly many interesting results and far more unsolved problems.

By a reaction-diffusion (R-D) equation, we usually mean an equation or a system of differential equations of the form:

$$\frac{\partial u_k(t,x)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{r}\frac{\partial}{\partial x^i}\left(a_k^{ij}(x)\frac{\partial u_k}{\partial x^j}\right) + \sum_{i=1}^{r} b_k^i(x)\frac{\partial u_k}{\partial x^i} + f_k(x, u_1, \cdots, u_n) \tag{1}$$

$$= L_k u_k + f_k(x, u_1, \cdots, u_n), \quad t > 0, \quad x \in R^r; \quad k = 1, 2, \cdots, n.$$

The coefficients are assumed to be sufficiently smooth, and the quadratic forms $\sum_{i,j=1}^{r} a_k^{ij}(x)\lambda_i\lambda_j$, $k = 1, \cdots, n$, to be positive semidefinite. Sometimes the coefficients of the operators L_k are allowed to depend on u_k.

The system of ordinary differential equations

$$\dot{u} = du/dt = f(x, u), \quad u = (u_1, \cdots, u_n), \quad f = (f_1, \cdots, f_n), \tag{2}$$

is called the local system corresponding to (1). The point $x \in R^r$ in (2) plays the role of a parameter.

Equations of type (1) describe the evolution of the concentrations $u_1(t, x)$, $\cdots$, $u_n(t, x)$ of particles of n different types. This evolution is caused by the diffusion of the particles, by their drift (the diffusion and drift coefficients are, generally, different for different particles) and by the transmutation of certain particles into others. In the case of no diffusion and no drift, this evolution is described by system (2).

Equations of the form (1), in particular, describe the propagation of a disturbance. Since the R-D equations are widely used in the models describing the propagation of nerve impulses, the term "propagation of excitation" is often used. There is also another model describing the propagation of disturbances—the phenomenological one. According to this model, every point can be in one of two states: either excited or not excited. If at any time t_0 a point is excited, then it will remain excited for all $t > t_0$. Excitation propagates according to Huygens principle: in the space R^r, one defines the velocity field $v(x, e)$, $x \in R^r$, $e \in R^r$, $|e| \doteq 1$. If at time $s \geq 0$ the excited region is $G_s \subset R^r$, then by time $t > s$ the excited region will be

$$G_t = \left\{x \in R^r: \inf_{\varphi:\varphi_0=x,\varphi_1\in G_s} \int_0^1 \frac{|\dot{\varphi}_s|\, ds}{v(\varphi_s, \dot{\varphi}_s|\dot{\varphi}_s|^{-1})} \leq t - s\right\}.$$

Questions arise about the connection between these two ways of describing the propagation of a disturbance and about the evaluation of the velocity field $v(x, e)$ using equation (1).

We will consider an asymptotic problem which in a number of cases enables one to go over from the diffusion-kinetic description via system (1) to the phenomenological model. We will usually consider the case of one equation

($n = 1$). The simplest version of the R-D equation

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in R^1, \tag{3}$$

has been investigated since the 1930s (Fisher, 1937; Kolmogorov, Petrovskii, and Piskunov, 1937). One of the most interesting properties of equation (3) is the existence of solutions of this equation of the propagating wave type $u = q(x - \alpha t)$, and the convergence of the solutions of the Cauchy problem for equation (3) to one of these wave solutions as $t \to \infty$. Of course, certain assumptions should be made concerning the nonlinear term and initial conditions.

We denote by $\mathscr{F}_1$ the class of continuously differentiable functions $f(u)$, $u \in R^1$, such that $f(0) = f(1) = 0$, $f(u) > 0$ for $0 < u < 1$, $f(u) < 0$ for $u \notin [0, 1]$ and $\sup_{0<u<1} u^{-1}f(u) = f'(0)$.

We denote by $\mathscr{F}_2$ the class of continuously differentiable $f(u)$ such that $f(0) = f(\mu) = f(1) = 0$ for some $\mu \in (0, 1)$, $f(u) < 0$ for $u \in (0, \mu) \cup (1, \infty)$ and $f(u) > 0$ for all other $u \in R^1$. Moreover, if the function $f(u)$ satisfies the condition $\int_0^1 f(u)\, du > 0$, we shall then write $f \in \mathscr{F}_2^+$.

Sometimes we will face nonlinear terms f not belonging to $\mathscr{F}_1 \cup \mathscr{F}_2$. These cases will be discussed briefly.

Consider Cauchy's problem for equation (3) with the initial condition

$$u(0, x) = \chi^-(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0. \end{cases} \tag{4}$$

It is possible to prove (see Aronson and Weinberger, 1975, 1978) that the solution of the Cauchy problem (3), (4) for $f \in \mathscr{F}_1 \cup \mathscr{F}_2$ converges to the wave solution $q(x - \alpha^* t)$ as $t \to \infty$. If $f \in \mathscr{F}_1$, then $\alpha^* = \sqrt{2f'(0)}$. In the case when $f \in \mathscr{F}_2$, there is no such simple formula for α^*. However, one can prove that $\alpha^* = \alpha^*[f]$ is a continuous monotone functional of f. The wave $q(\xi)$, $\xi \in R^1$, can be interpreted as the solution of the boundary value problem

$$\tfrac{1}{2} q''(\xi) + \alpha^* q'(\xi) + f(q(\xi)) = 0, \quad -\infty < \xi < \infty, \quad q(-\infty) = 1, \quad q(\infty) = 0. \tag{5}$$

The function $q(\xi)$ decreases monotonically from 1 to 0 and can be defined in a unique way up to a shift of the argument. In the case when $f \in \mathscr{F}_2$, problem (5) can be solved for a unique $\alpha^* \in R^1$.

Therefore, for large t, the solution of problem (3), (4) is close to $q(x - \alpha^* t)$ and is thus characterized by the shape of the wave $q(\xi)$ and by the velocity α^*. If we want to go over to the phenomenological description, then we must separate the problems of the evaluation of the velocity and of the shape so that only the velocity will remain in the initial approximation. The shape of the wave appears under more detailed approximation.

To carry out this scheme, consider the following function

$$u^\varepsilon(t, x) = u(t/\varepsilon, x/\varepsilon)$$

where $u(t, x)$ is the solution of problem (3), (4). The above implies that $u^\varepsilon(t, x) = u(t/\varepsilon, x/\varepsilon) \approx q((x - \alpha^* t)/\varepsilon)$ for $t \gg 1$. Since $q(\xi)$ decreases monotonically from

1 to 0, we conclude that $q((x - \alpha^*t)/\varepsilon) \to \chi^-(x - \alpha^*t)$ as $\varepsilon \downarrow 0$. Therefore, one can expect that $u^\varepsilon(t, x) \to \chi^-(x - \alpha^*t)$ as $\varepsilon \downarrow 0$. The function $u^\varepsilon(t, x)$ satisfies the differential equation

$$\frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon}f(u^\varepsilon), \quad u^\varepsilon(0, x) = \chi^-(x). \tag{6}$$

Therefore, the solution of problem (6) must converge to $\chi^-(x - \alpha^*t)$ as $\varepsilon \downarrow 0$. Thus, when studying problem (6) we obtain only the velocity α^* in the principal term $u^\varepsilon(t, x)$ as $\varepsilon \downarrow 0$. The wave $q(\xi)$ appears only under a more accurate examination of the behavior of $u^\varepsilon(t, x)$ as $\varepsilon \downarrow 0$.

Note that for equation (3) and in general for equations in R^r which are homogeneous in space the introduction of a small parameter is in fact equivalent to the study of $\lim_{t\to\infty} u(t, at)$, $a \in R^r$. Such a problem was studied in (Aronson and Weinberger, 1975, 1978). As these authors have shown, on the unit sphere in R^r a function $\alpha(e)$, $e \in R^r$, $|e| = 1$, is defined such that the above limit is equal to 1 for $|a| < \alpha(a/|a|)$ and to 0 for $|a| > \alpha(a/|a|)$. It is natural to interpret the value $\alpha(e)$ as the velocity of propagation of the disturbance in the direction e. However, the introduction of the small parameter provides more opportunities for generalization of the problem.

The above reasoning leads us to the following statement of the problem in the general case.

Consider the Cauchy problem

$$\begin{aligned} \frac{\partial u^\varepsilon(t, x)}{\partial t} &= \frac{\varepsilon}{2}\sum_{i,j=1}^r \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial u^\varepsilon}{\partial x^j}\right) + \sum_{i=1}^r b^i(x)\frac{\partial u^\varepsilon}{\partial x^i} + \frac{1}{\varepsilon}f(x, u^\varepsilon) \\ &= L^\varepsilon u^\varepsilon + \frac{1}{\varepsilon}f(x, u^\varepsilon), \quad t > 0, \quad x \in R^r, \quad u^\varepsilon(0, x) = g(x) \ge 0. \end{aligned} \tag{7}$$

We will not strive for the greatest possible generality and we suppose that $f(x, u)$ belongs to $\mathscr{F}_1$ for every $x \in R^r$ or $f(x, u)$ belongs to $\mathscr{F}_2$ for every $x \in R^r$. We will consider these cases separately, although it is also not difficult to examine the case where $f(x, \cdot) \in \mathscr{F}_1$ for some $x \in R^1$ and $f(x, \cdot) \in \mathscr{F}_2$ for other x. The initial function $g(x)$ is assumed to be bounded nonnegative and continuous everywhere with the possible exception of a finite number of manifolds of lower dimensions on which it has simple discontinuities.

Under minor additional restrictions we will show that the solution of problem (7) tends, as $\varepsilon \downarrow 0$, to a function $u^0(t, x)$ which only takes the two values 0 and 1. Denote by Γ_t, $t > 0$, the set in R^r on which this limit is 1: $\Gamma_t = \{x \in R^r: \lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1\}$. It is clear that the evolution of the function $u^0(t, x) = \lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x)$ as t grows reduces to the evolution of the set Γ_t.

In some respects it is simpler to describe the evolution of the set Γ_t in the case of $f \in \mathscr{F}_2$. If $f(x, \cdot) \in \mathscr{F}_2$ for $x \in R^r$ and there is no drift in equation (7), then the set Γ_t changes in accordance with the Huygens principle for an appropriate velocity field $v(x, e)$; $\Gamma_0 = \{x \in R^r, g(x) > \mu(x)\}$, where $\mu(x) \in (0, 1)$, $f(x, \mu(x)) = 0$. The field $v(x, e)$ is defined by the diffusion coefficients and nonlinear term, the dependence being of local nature; the velocity at a point x in a direction e is defined by the coefficients $a^{ij}(x)$ and $f(x, u)$ with "frozen" x. For $f \in \mathscr{F}_2^+$ the problem in fact reduces to the one-dimensional equation with the coefficients independent of x.

If $f \in \mathscr{F}_2$ and $\int_0^1 f(x, u)\, du < 0$ then according to similar rules the unexcited region expands as time grows. The case where $f \in \mathscr{F}_2$ is considered in Section 3.

Section 2 studies problem (7) with nonlinear term from class $\mathscr{F}_1$. In general, for $f(x, \cdot) \in \mathscr{F}_1$, $x \in R^r$, effects arise which are not covered by the phenomenological theory. However, if $f(x, \cdot) \in \mathscr{F}_1$ and $(\partial f(x, u)/\partial u)|_{u=0} = c = \text{const}$, then the propagation of disturbances allows phenomenological description. In this case, disturbances propagate in accordance with the Huygens principle along the velocity field $v_0(x, e) = \sqrt{2c}(\sum_{i,j=1}^r a_{ij}(x)e^ie^j)^{-1/2}$, $(a_{ij}(x)) = (a^{ij}(x))^{-1}$, if one puts $\Gamma_0 = \operatorname{supp} g = \{x \in R^r\colon g(x) > 0\}$. This assertion is contained in Example 1. Now let $f(x, \cdot) \in \mathscr{F}_1$ and $c(x) = (\partial f(x, u)/\partial u)|_{u=0}$ be nonconstant. We will introduce the velocity field $v(x, e) = \sqrt{2c(x)}(\sum a_{ij}(x)e^ie^j)^{-1/2}$. The disturbance would propagate along this velocity field if, just as in the case of $f \in \mathscr{F}_2$, the "frozen" coefficients principle were valid for $f \in \mathscr{F}_1$. Let us denote by G_t the region which will be excited by time t with the disturbance propagating according to the Huygens principle along this field $v(x, e)$, if at the initial moment the region $G_0 = \Gamma_0 = \operatorname{supp} g$ was excited. Example 1 and comparison theorems imply that the set Γ_t contains G_t (Lemma 1). It turns out that without additional restrictions the set Γ_t is larger than G_t. This follows from Example 2. The example considers one-dimensional problem (7) with $a^{11}(x) \equiv 1$, $f(x, u) = c(x)u(1 - u)$, where $c(x) = 1$ for $x \leq 0$ and $c(x) = 1 + x$ for $x > 0$. Let the support of the initial function be $G_0^a = \{x \in R^1, x < a\}$, where $a \geq 0$. In this example $v(x, \pm 1) = \sqrt{2(1 + x)}$, $x > 0$, and, say for $a = 0$, $G_t = \{x \in R^1\colon x < t^2/2 + t\sqrt{2}\}$. As it follows from Example 2, for $a = 0$ the region Γ_t has the form: $\Gamma_t = \{x \in R^1\colon x < t^2/2 + t\sqrt{2 + t^2/3}\}$. Therefore, for every $t > 0$, Γ_t is larger than G_t. Example 2 also implies that in the problem under consideration a universal velocity field $v(x, e)$ does not exist at all. These fields turned out to be different for different initial conditions (different a).

Theorem 2 gives sufficient conditions concerning the equation and initial function for the sets Γ_t and G_t to coincide. It follows from Example 4, that in the one-dimensional case with $a^{11}(x) \equiv 1$ it is sufficient that the function $c(x)$ be nonincreasing as the point x moves away from the support G_0 of the initial function. In particular, if as in Example 2, $c(x) = 1 + x$ for $x > 0$ and $c(x) = 1$ for $x \leq 0$ and $G_0 = \{x \in R^1\colon x > 1\}$, then the wave front (that is the boundary of the excited region) will move from the point $x = 1$ to the point $x = 0$ with velocity $\tilde{v}(x, -1) = \sqrt{2(1 + x)}$. Therefore, one can conclude from Examples 2 and 4 that in the direction of the decrease of $c(x)$ the wave front moves more slowly than in the direction of the increase of $c(x)$.

While Lemma 1 gives a lower bound for the set Γ_t and Theorem 2 provides conditions under which Γ_t coincides with this lower bound, Theorem 1 gives an upper bound for Γ_t. This theorem describes the set which always belongs to the complement of Γ_t and gives a condition (condition (N)) under which the closure of this set coincides with the closure of $R^r \setminus \Gamma_t$. Therefore condition (N) ensures the quickest propagation of disturbances. In this case not only the velocity of disturbance propagation may be larger than $v(x, e)$, and may become infinite, but the wave front (the boundary of the set Γ_t) may have jumps. Such a situation is described in Example 3. This example in fact implies that for $f \in \mathscr{F}_1$ sufficiently quick growth of the function $c(x)$ leads to jumps of the wave front. As we shall see later, all these effects are connected with the asymptotics of the probabilities of large deviations for diffusion processes with small diffusion.

The introduction of a small parameter enables one to give a number of useful generalizations of the problem on propagation of disturbances. Some of these generalizations are discussed in Section 4. This section deals with initial boundary value problems for equation (7), problems with nonlinear boundary conditions, equations with periodic and random coefficients, the effect of drift on wave propagation and a number of other questions.

Section 5 is devoted to systems of R-D equations. An integral equation in the space of trajectories of the corresponding Markov process is written for the solution of the system. The law of propagation of the disturbance is calculated for the systems analog of the Kolmogorov-Petrovskii-Piskunov equation (class $\mathscr{F}_1$).

The final section, 6, concerns some ways of introducing a small parameter in R-D equations which differ from those considered in the previous sections. Boundary value problems for R-D equations are studied for the case when only some of the diffusion coefficients are small, when the drift is of the same order as the nonlinear term. A system of R-D equations is considered in which only one equation contains a small parameter.

In order to analyze problem (7), we will consider the Markov diffusion process (X_t^ε, P_x) in R^r governed by the operator

$$L^\varepsilon = \frac{\varepsilon}{2} \sum_{i,j=1}^r \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial xj}\right) + \sum_{i=1}^r b^i(x)\frac{\partial}{\partial x^i}$$

(see Dynkin, 1965). The trajectories of this process can be constructed with the help of the stochastic differential equation

$$\dot{X}_t^\varepsilon = \sqrt{\varepsilon}\sigma(X_t^\varepsilon)\dot{W}_t + \tilde{b}(X_t^\varepsilon), \quad X_0^\varepsilon = x. \tag{8}$$

Here W_t is a Wiener process in R^r, $\sigma(x)$ is a matrix such that $\sigma(x)\sigma^*(x) = (a^{ij}(x))$; $\tilde{b}(x) = (\tilde{b}^1(x), \cdots, \tilde{b}^r(x))$, $\tilde{b}^k(x) = b^k(x) + (\varepsilon/2)\sum_{i=1}^r (\partial a^{ik}(x)/\partial x^i)$.

Let $c(x, u) = u^{-1}f(x, u)$. The Feynman-Kac formula implies that the solution of problem (7) obeys the following equation

$$u^\varepsilon(t, x) = E_x g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_s^\varepsilon, u(t-s, X_s^\varepsilon))\, ds\right\}, \tag{9}$$

where X_t^ε is the solution of equation (8).

Equation (9) is quite suitable for examining the function $u^\varepsilon(t, x)$. Under minor additional assumptions concerning the nonlinear terms $f_k(x, u_1, \cdots, u_n)$, one can write similar equations for the solution of the system (1) as well (see Section 5). If the drift coefficients depend on unknown functions, then also in this case, using the Cameron-Martin-Girsanov formula for the density of measures in the space of trajectories, it is possible to write an equation similar to (9) for the solution of the Cauchy problem. Below we shall use equation (9) for analysing u^ε as $\varepsilon \downarrow 0$. Similar integral equations in the space of trajectories can also be utilized in other problems connected with quasilinear equations (see, e.g. Freidlin, 1967, 1968).

To examine the behavior of the solution of equation (9) as $\varepsilon \downarrow 0$, we shall need asymptotic formulae for expressions of the form

$$E_x g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_s^\varepsilon)\,ds\right\}, \quad \varepsilon \downarrow 0, \tag{10}$$

where $c(x)$ is a continuous function which is bounded from above. It is clear that, uniformly in the interval $[0, t]$, the trajectories X_s^ε tend in probability to the solution of the ordinary (nonstochastic) differential equation which can be obtained from (8) by setting $\varepsilon = 0$. If the exponent in (10) did not contain the factor ε^{-1}, then the limit of this expression as $\varepsilon \downarrow 0$ would be equal to $g(X_t^0)\exp\{\int_0^t c(X_s^0)\,ds\}$. Deviations of order 1 of the trajectories X_s^ε from X_s^0 have probabilities of order $\exp\{-\text{const} \times \varepsilon^{-1}\}$, so the asymptotics of expression (10) as $\varepsilon \downarrow 0$ are defined by the deviations of X_s^ε from X_s^0 of order 1.

In order to describe these deviations, we shall introduce an action functional.

By the action functional for the family of processes (X_t^ε, P_x) as $\varepsilon \downarrow 0$, we mean the functional $S_{0t}(\varphi)$, $\varphi \in C_{0t}$, which is defined by the equality:

$$S_{0t}(\varphi) = \frac{1}{2}\int_0^t \sum_{i,j=1}^r a_{ij}(\varphi_s)(\dot\varphi_s^i - b^i(\varphi_s))(\dot\varphi_s^j - b^j(\varphi_s))\,ds, \quad (a_{ij}(x)) = (a^{ij}(x))^{-1},$$

for the absolutely continuous functions φ: $[0, t] \to R^r$. For other $\varphi \in C_{0t}$, we put $S_{0t}(\varphi) = +\infty$.

The functional $S_{0t}(\varphi)$ is lower semicontinuous. With its help, it is possible to describe the logarithmic asymptotics for the probabilities of large deviations. If $\mathscr{A} \subset \{\varphi \in C_{0t}, \varphi_0 = x\}$, then

$$\begin{aligned} &-\inf\{S_{0t}(\varphi)\colon \varphi \in (\mathscr{A})\} \\ &\le \liminf_{\varepsilon\downarrow 0}\varepsilon \ln P_x\{X_\cdot^\varepsilon \in \mathscr{A}\} \le \limsup_{\varepsilon\downarrow 0}\varepsilon \ln P_x\{X_\cdot^\varepsilon \in \mathscr{A}\} \\ &\le -\inf\{S_{0t}(\varphi)\colon \varphi \in [\mathscr{A}]\}, \end{aligned} \tag{11}$$

where (A) denotes the interior of the set A, and $[A]$ denotes the closure of A.

For the expression (10), we have the following asymptotic formula of Laplace-type:

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\varepsilon \ln E_x g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_s^\varepsilon)\,ds\right\} \\ &= \sup\left\{\int_0^t c(\varphi_s)\,ds - S_{0t}(\varphi)\colon \varphi_0 = x, \varphi_t \in \operatorname{supp} g\right\} \end{aligned} \tag{12}$$

where $g(x) \ge 0$, $\operatorname{supp} g = \{x \in R^r\colon g(x) > 0\}$.

The proof of formulae (11) and (12) and other properties of the action functional may be found in (Freidlin and Wentzell, 1984).

2. Wave front propagation for equations with nonlinear term of $\mathcal{F}_1$ class. This section considers the limiting behavior as $\varepsilon \downarrow 0$ of the solution of problem (7) for the nonlinear terms $f(x, u)$ such that $f(x, \cdot) \in \mathcal{F}_1$ for every $x \in R^r$. For the sake of brevity we shall assume that equation (7) has no drift, i.e. $b^i(x) \equiv 0$, $i = 1, \cdots, r$. The case when the drift is not zero is discussed briefly in Section 4.

Noting that the initial function is nonnegative, we can conclude, for example from equality (9), that $u^\varepsilon(t, x) \geq 0$ for $t \geq 0$, $x \in R^r$. Let $c(x) = c(x, 0) = \lim_{u \downarrow 0} u^{-1} f(x, u)$. For every $x \in R^r$ the condition $f(x, \cdot) \in \mathcal{F}_1$ implies that

$$c(x, u) = u^{-1} f(x, u) \leq c(x).$$

Taking into account this bound, we obtain from (9) and (12)

$$\begin{aligned} 0 \leq u^\varepsilon(t, x) &\leq E_x g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon} \int_0^t c(X_s^\varepsilon)\, ds\right\} \\ &\asymp \exp\left\{\frac{1}{\varepsilon}\left[\sup\left\{\int_0^t c(\varphi_s)\, ds - S_{0t}(\varphi) \colon \varphi \in C_{0t}, \varphi_0 = x, \varphi_t \in [G_0]\right\}\right]\right\}, \quad \varepsilon \downarrow 0, \end{aligned} \tag{13}$$

where the sign "$\asymp$" denotes logarithmic equivalence and G_0 is the support of the initial function $g(x)$, $[G_0]$ being its closure. We always assume that the closure of the interior of G_0 coincides with $[G_0]$.

Let

$$V(t, x) = V_g(t, x) = \sup\left\{\int_0^t \left(c(\varphi_s) - \frac{1}{2} \textstyle\sum_{i,j=1}^r a_{ij}(\varphi_s)\dot{\varphi}_s^i \dot{\varphi}_s^j\right) ds \colon \varphi_0 = x,\ \varphi_t \in [G_0]\right\}.$$

The functional under the supremum sign is upper semicontinuous. Therefore, the upper bound in (13) is attained at some function $\hat{\varphi} \in C_{0t}$, the function $V(t, x)$ being continuous.

From (13) it follows that $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x) = 0$ on the set $\{(t, x) \colon t > 0, x \in R^r, V(t, x) < 0\}$. This convergence is uniform on every compactum lying in the region $\{(t, x) \colon t > 0, x \in R^r, V(t, x) < 0\}$. If we succeed in proving that $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x) = 1$ for $V(t, x) > 0$, then the manifold $\sum_t = \{x \in R^r \colon V(t, x) = 0\}$ can be considered as the position of the wave front (i.e., the boundary between the excited and nonexcited regions) at the time t. (Note that in "reasonable" cases, $\sum_t$ is in fact either a manifold of dimension $(r - 1)$ or the union of manifolds of dimension $(r - 1)$ and less). Without supplementary assumptions, $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x)$ is not necessarily equal to 1 on the set $\{(t, x) \colon V(t, x) > 0\}$. Now we will introduce an assumption which ensures the validity of this assertion. Then we will examine the case when this assumption cannot be fulfilled.

Let $Q_- = \{(t, x) \colon V(t, x) < 0\}$. We shall say that condition (N) is fulfilled, if the following relation

$$\begin{aligned} V(t, x) = \sup\Big\{\int_0^t c(\varphi_s)\, ds - S_{0t}(\varphi) \colon \\ \varphi \in C_{0t}, \varphi_0 = x, \varphi_t \in [G_0], (t - s, \varphi_s) \in Q_- \text{ for } 0 < s < t\Big\} \end{aligned}$$

holds for any $t > 0$ and $x \in \sum_t$.

Condition (N) means that $V(t, x)$ is the supremum of the functional $\int_0^t c(\varphi_s)\,ds - S_{0t}(\varphi)$ on the set of functions lying in the region, where by virtue of the above, $u^\varepsilon(s, y)$ is close to zero. In this region $c(x, u^\varepsilon) \approx c(x)$, and inequality (13) is not too rough. This enables us to bound $u^\varepsilon(t, x)$ from below and show that $u^\varepsilon(t, x) \to 1$ as $\varepsilon \downarrow 0$ in the region $\{(t, x): t > 0, x \in R^r, V(t, x) > 0\}$.

First of all, we note that if condition (N) holds, then for any $\delta > 0$, $T > 0$ one can find ε_0 such that for $x \in \sum_t$, $0 < t < T$

$$u^\varepsilon(t, x) > \exp\{-\delta/\varepsilon\}, \tag{14}$$

provided $\varepsilon < \varepsilon_0$. To see this, first note that (13) implies that $u^\varepsilon(t, x) \to 0$ as $\varepsilon \downarrow 0$ in the domain Q_-. Suppose that $\hat{\varphi}_s$, $0 \le s \le t$, is a function such that $\hat{\varphi}_0 = x \in \sum_t$, $\hat{\varphi}_t \in (G_0)$, $(t - s, \hat{\varphi}_s) \in Q_-$ for $0 < s < t_1 < t$ for some $t_1 \in (0, t)$and $-\delta/2 < \int_0^{t_1} c(\hat{\varphi}_s)\,ds - S_{0t}(\hat{\varphi})$. Such a $\hat{\varphi}$ exists on account of the fact that $x \in \sum_t$ and condition (N) is valid. Denote $\mathscr{E}_h = \{\varphi \in C_{0t}: \varphi_0 = x, \sup_{0 \le s \le t} |\varphi_s - \hat{\varphi}_s| < h\}$ and let χ_h designate the indicator of the set $\mathscr{E}_h \subset C_{0t}$. From (9), noting that the initial function is nonnegative and the functional $\int_0^t c(\varphi_s)\,ds$ is continuous, we conclude that for some $c_1 > 0$,

$$\begin{aligned} u^\varepsilon(t, x) &= E_x g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon} \int_0^t c(X_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon))\,ds\right\} \\ &\ge E_x \chi_h g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon} \int_0^t c(X_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon))\,ds\right\} \\ &\ge c_1 \exp\left\{\frac{1}{\varepsilon}\left(\int_0^t c(\hat{\varphi}_s)\,ds - \frac{\delta}{4}\right)\right\} P_x\{\sup_{0 \le s \le t} |X_s^\varepsilon - \hat{\varphi}_s| < h\}, \end{aligned} \tag{15}$$

provided h is small enough. Furthermore, the properties of the action functional imply that for any $\delta > 0$:

$$P_x\{\sup_{0 \le s \le t} |X_s^\varepsilon - \hat{\varphi}_s| < h\} > \exp\{-1/\varepsilon[S_{0t}(\hat{\varphi}) + \delta/8]\}, \tag{16}$$

provided ε is small enough. Finally, the definition of $\hat{\varphi}$ together with (15) and (16) results in inequality (14).

Now we will show that if condition (N) is fulfilled and

$$(t, x) \in Q_+ = \{(t, x): t > 0, x \in R^r, V(t, x) > 0\},$$

then

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x) = 1.$$

First of all, we notice that the condition $f(x, u) < 0$ for $u > 1$ implies the relation: $\limsup_{\varepsilon \downarrow 0} u^\varepsilon(t, x) \le 1$. Therefore, it is sufficient to show that $\liminf_{\varepsilon \downarrow 0} u^\varepsilon(t, x) \ge 1$ for $(t, x) \in Q_+$. Suppose the opposite were true: $(t, x) \in Q_+$ and for some $h > 0$ one can find arbitrarily small $\varepsilon > 0$ such that $u^\varepsilon(t, x) < 1 - h$. Denote by λ the distance from the point (t, x) to the set $\{(t, x): t \ge 0, x \in R^r, V(t, x) \le 0\}$. We introduce the domains

$$D_1 = \{(t, y): u^\varepsilon(t, y) < 1 - h\},$$

$$D_2 = \{(t, y): V(t, y) > 0\},$$

$$D_3 = \{(t, y): t > 0, |y - x| < \lambda/2\}, \quad D = D_1 \cap D_2 \cap D_3.$$

Suppose that $\tau = \tau^\varepsilon$ is the first exit time of the "heat" process $(t - s, X_s^\varepsilon)$ from the domain D: $\tau = \inf\{s: (t - s, X_s^\varepsilon) \notin D\}$. Since τ is a Markov time and X_t^ε is a strong Markov process, from (9) we deduce the relation

$$u^\varepsilon(t, x) = E_x u^\varepsilon(t - \tau, X_\tau^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^\tau c(X_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon))\, ds\right\}. \tag{17}$$

If $(t - \tau, X_\tau^\varepsilon) \in \partial D_1$, then $u^\varepsilon(t - \tau, X_\tau^\varepsilon) = 1 - h$. Next, from the properties of the action functional it is easy to obtain

$$P_x\{(t - \tau, X_\tau^\varepsilon) \in \partial D_3\} \asymp \exp\{-c_1/\varepsilon\}, \quad \varepsilon \downarrow 0$$

for some $c_1 > 0$. Finally, if $(t - \tau, X_\tau^\varepsilon) \in \partial D_2$, then observing that $\lambda > 0$, we conclude that $\tau > \lambda/2 > 0$. From this, taking into account that $c(y, u) > c_0 > 0$ for $|x - y| < \lambda/2$, $u < 1 - h$, we deduce that on the set $\mathscr{A}_\varepsilon = \{(t - \tau, X_\tau^\varepsilon) \in \partial D_2\}$ the exponential in (17) is bounded from below by $\exp\{\lambda c_0/2\varepsilon\}$.

Relying on (14) for $\delta < \frac{1}{4}\lambda c_0$, we conclude that on the set $\mathscr{A}_\varepsilon$ the value under the expectation sign in (17) is larger or equal to $\exp\{\lambda c_0/4\varepsilon\}$ for ε small enough. Gathering the above bounds, we conclude from (17) that $\liminf_{\varepsilon\downarrow 0} u^\varepsilon(t, x) \geq 1 - h$. The obtained contradiction shows that $\liminf_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$, and thus $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$ for $(t, x) \in Q_+$.

So, we have proved the following result (Freidlin, 1979).

THEOREM 1. *Suppose that* $f(x, \cdot) \in \mathscr{F}_1$ *for* $x \in R^r$ *and let condition* (N) *be fulfilled. Then, for the solution* $u^\varepsilon(t, x)$ *of problem* (7) *the following relation holds*:

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = \begin{cases} 1, & \text{if } V(t, x) > 0 \\ 0, & \text{if } V(t, x) < 0. \end{cases}$$

This convergence is uniform on every compactum lying in the regions $\{(t, x): t > 0, x \in R^r, V(t, x) > 0\}$ *and* $\{(t, x): t > 0, x \in R^r, V(t, x) < 0\}$, *respectively.*

Therefore, the equation

$$V(t, x) = 0 \tag{18}$$

defines the wave front which divides the regions where $u^\varepsilon(t, x)$ is close to 0 and is close to 1 for small ε. From equation (18), one can find $t^* = t^*(x)$ which is the time necessary for the disturbance to reach the point x. As will be seen below, the disturbance (i.e., the region where $u^\varepsilon(t, x)$ is close to 1 for small ε) can be propagated in a noncontinuous way. In this case one cannot manage with the phenomenological description in the form in which it is given in Section 1. On the other hand, under appropriate additional conditions, the level set

$\{V(t, x) = 0\}$ has sufficiently simple structure so that the behavior of $u^\varepsilon(t, x)$ for small ε may be described with the help of the corresponding velocity field and the Huygens principle.

EXAMPLE 1. Suppose that the function $f(x, u)$ does not depend on $x \in R^r$: $f(x, u) = f(u) \in \mathscr{F}_1$. In this case the expression for $V(t, x)$ is as follows:

$$(19)\qquad V(t, x) = ct - \inf\left\{\frac{1}{2}\int_0^t \textstyle\sum_{i,j=1}^r a_{ij}(\varphi_s)\dot\varphi_s^i\dot\varphi_s^j\, ds: \varphi_0 = x,\ \varphi_t \in [G_0]\right\},$$

where $c = (\partial f(u)/\partial u)\,|_{u=0}$. Denote by $d(x, y)$ the Riemann metric in R^r corresponding to the metric tensor $ds^2 = \sum_{i,j=1}^r a_{ij}(x)\, dx^i\, dx^j$. Simple calculation shows that the lower bound on the right-hand side of (19) is equal to $(1/2t)d^2(x, G_0)$and thus

$$V(t, x) = ct - \frac{d^2(x, G_0)}{2t}.$$

It is not difficult to verify that in this case condition (N) holds. Therefore, at a time $t > 0$ the wave front is defined by the equality

$$d(x, G_0) = t\sqrt{2c}.$$

This implies that in the case $f(x, u) = f(u) \in \mathscr{F}_1$, the excited region expands according to Huygens principle, the corresponding velocity field being homogeneous and isotropic in the metrics $d(x, y)$ and equal to $\sqrt{2c}$. In the original Euclidean metric, this field has the form

$$v(x, e) = \sqrt{2f'(0)}\left(\textstyle\sum_{i,j=1}^r a_{ij}(x)e^ie^j\right)^{-1/2},$$

where $e = (e^1, \cdots, e^r)$.

Now we will show that if the function $f(x, u) = u \cdot c(x, u)$ depends on x in an essential way, then some new effects can appear and generally the propagation of disturbances can no longer be described with the help of the Huygens principle.

EXAMPLE 2. Let $x \in R^1$, $a^{11}(x) \equiv 1$, $f(x, u) = c(x) \cdot u(1 - u)$, where $c(x) = 1$ for $x < 0$ and $c(x) = 1 + x$ for $x > 0$. Suppose that the initial function has the support $G_0 = G_0^a = \{x \in R^1, x < a\}$, $a \geq 0$. In this case

$$V(t, x) = V_a(t, x) = \sup\left\{\int_0^t \left[1 + \varphi_s - \frac{1}{2}\dot\varphi_s^2\right] ds: \varphi_0 = x,\ \varphi_t = a\right\}.$$

The Euler equation for such a functional has the form: $\ddot\varphi = -1$. Taking into account the boundary conditions we can therefore find the extremal on which the supremum is attained

$$\hat\varphi_s = -(s^2/2) + (t/2 + (a - x)/t)s + x.$$

For $V_a(t, x)$ we obtain the expression

$$V_a(t, x) = \int_0^1 \left[1 + \hat\varphi_s - \frac{1}{2}\dot{\hat\varphi}_s^2\right] ds = \frac{t^3}{24} + t\left(1 + \frac{a + x}{2}\right) - \frac{(a - x)^2}{2t}.$$

By equating $V_a(t, x)$ to zero, we find the expression for the front position $X_a(t)$

at time t:

$$X_a(t) = t^2/2 + a + \sqrt{t^4/3 + 2t^2(1 + a)}.$$

Note that since the function $X_a(t)$ is convex and the extremals are concave, condition(N) is fulfilled. Therefore, in this case Theorem 1 can be applied.

For every $a \geq 0$ the function $X_a(t)$ is strictly increasing, and at each time t its derivative can be represented in the form of a function of $X_a(t)$. It may seem that in this case the front propagation also admits phenomenological description with the help of an appropriate velocity field. However, the velocity field here turns out to depend on the initial condition. To see this it is sufficient to evaluate $\tilde{a} = X_0(1)$, $X_{\tilde{a}}(1)$ and $X_0(2)$:

$$X_0(1) = \tilde{a} = 1/2 + \sqrt{7/3}, \quad X_0(2) = 2 + \sqrt{40/3} \approx 5.6,$$
$$X_{\tilde{a}}(1) = 1 + \sqrt{7/3} + \sqrt{10/3 + \sqrt{28/3}} \approx 5.1.$$

If the velocity field did not depend on the initial conditions, then the equality $X_0(2) = X_{\tilde{a}}(1)$ would be valid. In our case $X_0(2) > X_{\tilde{a}}(1)$, and thus the velocity field is not of such a universal nature as in the phenomenological theory.

In the next example, we will first choose a discontinuous function for $c(x, u)$. Note that this is done solely for the sake of simplification of the computations. In the final part of the example, we will see that "new sources" also arise in the case where $c(x, u)$ is continuous, provided $c(x, u)$ increases sufficiently quickly in some finite interval.

EXAMPLE 3. Consider the one-dimensional problem with $a^{11}(x) \equiv 1$, $c(x, 0) = c_1 > 0$ for $x < h$ and $c(x) = c_2 > 2c_1$ for $x \geq h > 0$. As the initial function $g(x)$, we take the step function $\chi^-(x)$ which is the indicator of the set $\{x \leq 0\}$. Inside of each of the half-lines $\{x < h\}$ and $\{x > h\}$, Euler's equation for the functional $R_{0t} = \int_0^t c(\varphi_s)\, ds - S_{0t}(\varphi)$ has the form $\ddot{\varphi} = 0$. Therefore, the extremals of the functional $R_{0t}(\varphi)$ are line segments or unions of line segments with vertices on the line $x = h$. This reasoning permits us to calculate

$$V(t, x) = \sup\{R_{0t}(\varphi)\colon \varphi_0 = x, \varphi_t = 0\}$$

and the law of the wave front propagation which is defined by the equation $V(t, x) = 0$, and also to check the validity of condition (N).

It turns out that up to time $T_0 = (h/c_2)\sqrt{2(c_2 - c_1)}$ the disturbance propagates from the point $0 \in R^1$ to the right with constant velocity $\sqrt{2c_1}$ and by time T_0 reaches the point $T_0\sqrt{2c_1} < h$. On the other hand, simple calculation shows that $V(T_0, h) = 0$ and $V(t, h) > 0$ for $t > T_0$. This means that at time T_0 a new source appears at the point $x = h$. From this source a wave propagates in both directions: to the left with velocity $\sqrt{2c_1}$ and to the right with a velocity which is at first larger than $\sqrt{2c_2}$, but for large t tends to $\sqrt{2c_2}$. The waves from the point 0 and from the point 1 will meet by time $T_1 = (2\sqrt{2c_1})^{-1}(h - T_0\sqrt{2c_1})$.

Therefore, at the time T_0 a jump of the wave front occurs and the excited region has two connected components for $t \in (T_0, T_1)$.

Now let $c(x, 0) = c(x)$ be a smooth monotone function which is equal to c_1 for $x < \bar{x} = 1/2[h + T_0\sqrt{2c_1}] + \delta$, where $\delta > 0$ is small enough, and is equal to c_2 for

$x > h$. Using comparison theorems one can show that for such a function $c(x)$ the disturbance reaches the region $\{x > \bar{x} + \delta\}$ before it reaches the point $\bar{x}$. Thus here we also have a jump of the wave front.

Therefore, in the case when $c(x) = f'_u(x, 0)$, $f \in \mathscr{F}_1$, is nonconstant, the propagation of the region occupied by the excitation cannot, generally speaking, be described via Huygens principle.

Now we will discuss condition (N) and consider possible ways of weakening this condition. For this, we shall need a result which gives a lower bound for the region occupied by the disturbance.

Let us introduce the velocity field

$$v(x, e) = \sqrt{2c(x)}\left(\textstyle\sum_{i,j=1}^{r} a_{ij}(x)e^i e^j\right)^{-1/2}, \quad x \in R^r, \quad |e| = 1.$$

We put

$$\tau_{G_0}(x) = \inf\left\{\int_0^t \frac{|\dot{\varphi}_s|\, ds}{v(\varphi_s, \dot{\varphi}_s|\dot{\varphi}_s|^{-1})} : \varphi \in C_{0t},\ \varphi_0 = x,\ \varphi_t \in G_0\right\}.$$

LEMMA 1. *Suppose that $f(x, \cdot) \in \mathscr{F}_1$, $x \in R^r$, and let the function $c(x, u)$ be bounded from above and continuously differentiable. Then $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$ for $x \in G_t = \{y \in R^r: \tau_{G_0}(y) < t\}$.*

PROOF. The proof of this lemma is simple and we will only outline it. Given two sufficiently smooth functions $f_1(x, u)$, $f_2(x, u)$, $x \in R^r$, $u \in R^1$, suppose that $f_1(x, u) \le f_2(x, u)$. Denote by $u_1^\varepsilon(t, x)$ and $u_2^\varepsilon(t, x)$ the solutions of problem (7) with $f = f_1$ and $f = f_2$, correspondingly. It is not difficult to deduce from(9) that $u_1^\varepsilon(t, x) \le u_2^\varepsilon(t, x)$ for $t > 0$, $x \in R^r$. The same monotonicity holds when considering the solutions of the first boundary problem for equation (7). Next, consider the Riemannian metric $ds^2 = 2c(x) \sum_{i,j=1}^{r} a_{ij}(x)\, dx^i\, dx^j$. In this metric the domain G_t, $t \ge 0$, expands according to the Huygens principle with velocity one. Let Γ be the minimal geodesic connecting the point x and $[G_0]$. For a given $\delta > 0$, we will decompose the curve Γ into a finite number of small segments $\gamma_1, \cdots, \gamma_m$ so that each such segment γ_k can be covered by a neighborhood inside of which it is possible to choose $c_k(u)$ so that the following relations hold:

$$c(x, u) \ge c_k(u), \quad |c(x, u) - c_k(u)| < \delta.$$

The propagation of the disturbance for $c(x, u) \equiv c(u)$ is described by Theorem 1 (see Example 1). By virtue of the comparison principle given above, the disturbance in a piecewise-constant medium will reach the point x not earlier than in the original medium. On the other hand, by choosing δ sufficiently small, one can ensure that the time of propagation from G_0 to x will be arbitrarily close to $\tau_{G_0}(x)$. The assertion of the lemma follows.

Example 3 shows why such "local" arguments do not give the corresponding upper bound for $u^\varepsilon(t, x)$.

We put $H = \{(t, x): t > 0, x \in G_t\}$, where G_t is the region in R^r defined in the formulation of Lemma 1 and $[H]$ is the closure of the set H in $[0, \infty) \times R^r$.

THEOREM 2. *Suppose that $f(x, \cdot) \in \mathscr{F}_1$ for $x \in R^r$, and let $c(x, u) = u^{-1}f(x, u)$ be continuously differentiable and bounded from above. Set*

$$\tilde{V}(t, x) = \sup\left\{\int_0^s c(\varphi_\sigma)\, d\sigma - S_{0s}(\varphi): \varphi_0 = x,\ (s, \varphi_s) \in [H],\right. \tag{19}$$
$$\left.(s - \sigma, \varphi_\sigma) \notin [H] \text{ for } \sigma \in [0, s),\ s \le t\right\}.$$

Suppose that $\tilde{V}(t, x) < 0$ for $(t, x) \notin [H]$. Then $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$ for $(t, x) \in H$ and $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 0$ for $(t, x) \notin [H]$.

PROOF. The equality $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$ for $(t, x) \in H$ was established in Lemma 1. To prove that $u^\varepsilon(t, x) \to 0$ on the set $\{(t, x) \notin [H]\}$, we shall consider the stopping time $\tau^\varepsilon = t \wedge \inf\{s: (t - s, X_s^\varepsilon) \in [H]\}$. Taking into account that the process X_s^ε is a strong Markov process, we have

$$u^\varepsilon(t, x) = E_x u^\varepsilon(t - \tau^\varepsilon, X_{\tau^\varepsilon}^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^{\tau^\varepsilon} c(X_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon))\, ds\right\}$$
$$\le E_x u^\varepsilon(t - \tau^\varepsilon, X_{\tau^\varepsilon}^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^{\tau^\varepsilon} c(X_s^\varepsilon)\, ds\right\}.$$

From this and using a generalization of (11) one can deduce that $\lim u^\varepsilon(t, x) = 0$, provided $(t, x) \notin [H]$.

EXAMPLE 4. Suppose that $x \in R^1$, $a^{11}(x) \equiv 1$, $f(x, \cdot) \in \mathscr{F}_1$, $g(x) = \chi^-(x)$. The function $c(x) = (\partial f(x, u)/\partial u)\,|_{u=0}$ is assumed to be monotonically decreasing. We will define the function $\psi(s)$, $s \ge 0$, as a solution of the differential equation

$$\dot{\psi}_s = \sqrt{2c(\psi_s)}, \quad \psi_0 = 0. \tag{20}$$

This function increases monotonically; its derivative is positive and decreases as s increases. It is readily checked that in this case the set $H = \{(t, x): t > 0, \tau_{G_0}(x) < t\}$ is $\{(t, x): t > 0, x < \psi_t\}$. Simple calculations show that outside the closure of this set the corresponding function $\tilde{V}(t, x)$ is negative. Therefore by Theorem 2,

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = \begin{cases} 1, & \text{for } x < \psi_t \\ 0, & \text{for } x > \psi_t. \end{cases}$$

In this case, condition (N) is not fulfilled.

In this example, one can describe the propagation of disturbances from the region $\{x < 0\}$ to the right with the help of phenomenological theory with the velocity field $v(x, 1) = \sqrt{2c(x)}$. One should, however, keep in mind that the front velocity at a point x depends on which direction the wave moves. If at the initial time the region $\{x > 10\}$ were excited, then the front would propagate to the left (for the same function $c(x)$) with a different velocity and, in particular, in this case the wave front may have jumps as in Example 3. Note that the front propagation in the direction where $c(x)$ decreases (for instance, from the point 0 to the point 10) is slower than in the direction of increasing $c(x)$ (from $x = 10$ to $x = 0$).

3. Geometric optics approximation for the nonlinearities of $\mathcal{F}_2$ type. Now suppose that $f(x, \cdot) \in \mathcal{F}_2$ for every $x \in R^r$, and let $c(x, u) = u^{-1}f(x, u)$ be a sufficiently smooth function which is bounded on sets of the form $\{(x, u): x \in R^r, |u| \leq M\}$, $M > 0$, with $c(x, 0) = c(x, \mu(x)) = c(x, 1) = 0$. For the sake of brevity, we will assume that there is no drift and that the initial function obeys the condition $0 \leq g(x) \leq 1$. The last condition is not difficult to weaken by noting that $f(x, u) > 0$ for $u < 0$ and $f(x, u) < 0$ for $u > 1$.

We put

$$G_+ = \{x \in R^r: g(x) > \mu(x)\}, \quad G_- = \{x \in R^r: g(x) < \mu(x)\},$$
$$G_\mu = \{x \in R^r: g(x) = \mu(x)\}.$$

If either G_+ or G_- is empty, then the problem reduces to the case considered in the previous section; we will therefore assume that G_+ and G_- are not empty. We will also suppose that the closure of either set coincides with the closure of its interior. Sometimes we will also make the following assumption:

$$\partial G_+ = \partial G_- = G_\mu \tag{21}$$

Suppose that $x \in G_+$ and let $\delta_1 = \rho(x, G_\mu)$ be the Euclidean distance from x to G_μ. We choose $\delta_2 > 0$ small enough so that $g(y) - \mu(y) \geq \lambda > 0$ for $|y - x| < \delta_2$ and put $\delta = \delta_1 \wedge \delta_2$. From the definition of the action functional, it follows that

$$\begin{aligned} &P_x\{\sup_{0\leq t\leq h}|X_t^\varepsilon - x| \geq \delta/2\} \\ &\quad \asymp \exp\{-(1/\varepsilon)\inf\{S_{0h}(\varphi): \varphi \in C_{0h}, \varphi_0 = x, \sup_{0\leq s\leq h}|\varphi_s - x| \geq \delta/2\}\} \end{aligned} \tag{22}$$

for any $h > 0$. It is easy to verify that the lower bound in the exponent in (22) is positive and tends to infinity as $h \downarrow 0$. In particular, one can pick an $h_1 > 0$ such that for $h \in (0, h_1]$ this lower bound is larger than $\sup_{x\in R^r,|u|\leq 1}|c(x, u)|$. This implies that, for $t < h_1$, the contribution to the right-hand side of (9) of the trajectories X_s^ε leaving the $\delta/2$-neighborhood of the point x in the time $[0, h_1]$ tends to zero as $\varepsilon \downarrow 0$.

Let

$$\bar{g} = \min\{g(y): |x - y| \leq \delta/2\},$$
$$\bar{c} = \min\{c(y, u): |y - x| \leq \delta/2, \mu(y) + \lambda \leq u \leq 1 - \varkappa\},$$

where $\varkappa$ is an arbitrary small positive number. Relying on the above reasoning, we obtain from equation (9) the bound

$$u^\varepsilon(h, x) \geq \tfrac{1}{2}\bar{g}\exp\{\bar{c}h/\varepsilon\}, \quad 0 < h < h_1,$$

which holds for $u^\varepsilon < 1 - \varkappa$. From this it follows that $\liminf_{\varepsilon\downarrow 0} u^\varepsilon(t, x) \geq 1$. On the other hand, note that since $g(x) \leq 1$ and $f(x, u)$ is negative for $u > 1$, we can deduce that the function $u^\varepsilon(t, x)$ cannot exceed 1. This implies that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(h, x) = 1$, $x \in (G_+)$, $h \in (0, h_1)$. Moreover, it follows from the above argument that for any compactum $K \subset (G_+)$ the convergence of $u^\varepsilon(t, x)$ to the limit is uniform in $x \in K$ and $t \in [\delta, h_1]$, for any $\delta > 0$. In the same way, one can show that for $x \in (G_-)$, $\lim_{\varepsilon\downarrow 0} u^\varepsilon(h, x) = 0$ for $h \in (0, h_1)$ and appropriately small h_1.

Therefore, the following result holds.

LEMMA 2. *For each compactum $K \subset (G_+)$ ($K \subset (G_-)$) there exists an $h_1 > 0$ such that* $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$ ($\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 0$) *uniformly in $x \in K$ and $0 < \delta < t < h_1$ for any fixed $\delta > 0$.*

Suppose now that condition (21) is fulfilled. Then G_μ is a "thin" set with no interior. Lemma 2 implies that for small t, outside a small narrow neighborhood of the set G_μ the function $u^\varepsilon(t, x)$ tends (as $\varepsilon \downarrow 0$) to the function which is equal to 1 on G_+ and to 0 on G_-. It turns out that for any $t > 0$, the function $u^\varepsilon(t, x)$ converges (as $\varepsilon \downarrow 0$) to the step function taking values 0 and 1. With the growth of t, the only evolution is that of the manifold dividing the regions where $u^\varepsilon(t, x)$ is close to 0 and 1. This manifold can be interpreted as a wave front. The evolution of the wave front describes the change of $u^\varepsilon(t, x)$ with time for $\varepsilon \ll 1$. Let us see how one can describe this evolution.

For $f(x, \cdot) \in \mathscr{F}_2$, the function $c(x, u)$ attains its maximum somewhere between $\mu(x)$ and 1. The main contribution to the mathematical expectation on the right-hand side of (9) will therefore be given by the trajectories staying far from 0 and 1, that is, by the trajectories which are near the wave front for every t. Therefore, even to compute the logarithmic asymptotics of the right-hand side in (9), it is necessary to know the behavior of $u^\varepsilon(t, x)$ in the transient area. This is even more the case if one wishes to compute more precise asymptotics. We recall that in the case of $f(x, \cdot) \in \mathscr{F}_1$, the main contribution in (9) was given by those trajectories passing before the front—i.e., in the region where the solution is close to zero. Therefore, in the case of $f(x, \cdot) \in \mathscr{F}_2$, roughly speaking, one cannot completely separate the problem of velocity computation and that of describing the wave shape.

However, there are circumstances which simplify the problem in the case $f(x, \cdot) \in \mathscr{F}_2$, $x \in R^r$. First of all the fact that $f(x, u)$ is negative for $0 < u < \mu(x)$ leads us to the conclusion that the small values of $u^\varepsilon(t, x)$ fade rapidly and cannot generate new sources as was the case for $f(x, \cdot) \in \mathscr{F}_1$. This property enables one to localize the problem, provided, of course, that the diffusion coefficients and the nonlinear term are sufficiently smooth. Near every point $x \in R^r$ the movement of the wave front is defined by the diffusion coefficients and the nonlinear term which are nearly constant in the vicinity of this point. The problem, in fact, becomes one-dimensional because near each point $x \in R^r$ the disturbance propagates in the direction of the normal to the wave front.

Another important feature is that the function $v^\varepsilon(t, x) = 1 - u^\varepsilon(t, x)$ satisfies an equation like (7) with the nonlinear term $\tilde{f} = -f(x, 1 - v)$, which also belongs to class $\mathscr{F}_2$. This property allows one to confine the proof to the bounds of $u^\varepsilon(t, x)$, for example, from below as in Lemma 1. The upper bound can be obtained as the lower bound for the function $v^\varepsilon = 1 - u^\varepsilon$. Note that class $\mathscr{F}_1$ does not have this property.

We also recall the main facts concerning the equation which is homogeneous in space with nonlinearity of $\mathscr{F}_2$ type. It is possible to prove (Fife and McLeod, 1977) that the solution of equation (3) with the initial condition $u(0, x) = \chi^-(x)$ converges to the solution of the running wave type $q(x - \alpha^* t)$ as $t \to \infty$. The function $q(\xi)$ defining the wave shape is the solution of the boundary problem

$$(23)\quad \tfrac{1}{2}q''(\xi) + \alpha^* q'(\xi) + f(q) = 0, \quad -\infty < \xi < \infty, \quad q(-\infty) = 1, \quad q(\infty) = 0.$$

If $f \in \mathscr{F}_2$, then problem (23) can be solved for a unique $\alpha^* = \alpha^*[f]$. The constant α^* is the wave velocity. In the case $f \in \mathscr{F}_2$, generally speaking, one cannot write down such a simple formula for the velocity as for $f \in \mathscr{F}_1$, but it is possible to prove that the functional $\alpha^*[f]$ is continuous and monotone. For the velocity $\alpha^*[f]$, one can give the following formula (Volpert, 1983):

$$\alpha^*[f] = (1/\sqrt{2})\sup_{\rho \in Z_{01}} \inf_{0<s<1}[\rho'(s) + f(s)/\rho(s)],$$

where Z_{01} is the totality of continuously differentiable functions $\rho(s)$, $0 \le s \le 1$, such that $\rho(0) = \rho(1) = 0$, $\rho(s) > 0$, for $s \in (0, 1)$, $\rho'(0) > 0$, $\rho'(1) < 0$. Notice that for $f \in \mathscr{F}_2$ the wave velocity $\alpha^* = \alpha^*[f]$ can be either positive (the region occupied by the disturbance propagates) or negative (the disturbance dies out). The sign of $\alpha^*[f]$ coinsides with that of $\int_0^1 f(u)\,du$. Also note that, if $q(\xi)$ is the solution of problem (23), then $\tilde{q}(\xi) = 1 - q(\xi)$ is the solution of problem (23) where instead of f we take the function $\tilde{f} = -f(1-u)$ and $-\alpha^*$ is taken rather than α^*. This implies that $\alpha^*[f] = -\alpha^*[\tilde{f}]$. This equality allows the construction of sufficiently precise lower and upper bounds for $u(t, x)$.

We introduce the velocity field in R^r:

$$(24)\quad \begin{aligned} v(x, e) &= \alpha^*[f(x, \cdot)](\textstyle\sum_{i,j=1}^r a_{ij}(x)e^i e^j)^{-1/2}, \\ (a_{ij}(x)) &= (a^{ij}(x))^{-1}, \quad e = (e^1, \cdots, e^r), \quad \textstyle\sum_1^r (e^i)^2 = 1. \end{aligned}$$

Denote by $T_g(x)$ the time necessary for the disturbance propagating in accordance with the velocity field $v(x, e)$ to reach the point x provided that at the initial moment it occupied a region G_+:

$$T_g(x) = \inf\left\{\int_0^1 \frac{|\dot\varphi_s|\,ds}{v(\varphi_s, \dot\varphi_s|\dot\varphi_s|^{-1})} : \varphi \in C_{01}, \varphi_0 = x, \varphi_1 \in G_+\right\}.$$

THEOREM 3. *Suppose that $f(x, \cdot) \in \mathscr{F}_2^+$ for $x \in R^r$, that condition* (21) *is fulfilled, and assume that the function $c(x, u)$ is continuously differentiable with $\sup_{x\in R^r, 0\le u\le 1} c(x, u) < \infty$. Suppose that the coefficients $a^{ij}(x)$ have bounded second-order derivatives. Then*

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = \begin{cases} 1, & \text{if } T_g(x) < t \\ 0, & \text{if } T_g(x) > t. \end{cases}$$

Using such arguments as locality, this result was given in (Freidlin and Sivak, 1979). Then the proof was developed in detail in (Gärtner, 1980, 1984). This proof was based on a generalization of Lemma 2, on the detailed analysis of the functional $\alpha^*[f]$ and on the above remarks. The scheme of this proof is close to that of Lemma 1. For more detail see the above cited papers by Gärtner.

EXAMPLE 5. Let us consider equation (7) for $f(x, u) = u(1 - u)(u - \mu(x))$. For $\mu(x)$ with values from (0, 1), such a function $f(x, u)$ belongs to class $\mathscr{F}_2$. It is easy to see that

$$\int_0^1 f(x, u)\, du = \frac{1}{12} - \frac{1}{6}\mu(x).$$

Therefore $\mu = ½$ is a boundary value at which the velocity sign changes. For the sake of brevity, we will assume that $0 < \mu(x) < ½$ for $x \in R^r$. To compute $\alpha^*(x) = \alpha^*[f(x, \cdot)]$, let us consider the equation for the wave profile:

$$(25) \quad ½q''(\xi) + \alpha^* q'(\xi) + q(1 - q)(q - \mu(x)) = 0, \quad q(-\infty) = 1, \quad q(\infty) = 0.$$

It turns out that, for the above indicated function $f(x, u)$, problem (25) may be solved explicitly. By direct substitution in the equation, one can verify that problem (25) is solvable for

$$\alpha^* = \alpha^*(x) = ½ - \mu(x).$$

For such an α^* the function

$$q(\xi) = [1 + e^{\xi}]^{-1}$$

is a solution of problem (25). For the velocity field $v(x, e)$, we obtain the expression

$$v(x, e) = ½(1 - 2\mu(x))(\textstyle\sum_{i,j=1}^r a_{ij}(x)e^i e^j)^{-1/2}.$$

In particular, in the case of the homogeneous and isotropic diffusion ($a^{ij}(x) = D\delta_{ij}$), we have $v(x, e) = \sqrt{D}(½ - \mu(x))$. When $\mu(x)$ converges to ½, the velocity tends to zero. It is not difficult to give a number of results concerning the case where $\mu(x) > ½$ for some $x \in R^r$ and $\mu(x) < ½$ for others. It is an interesting question to study the effect of the asymptotic behavior of the front on the surface $\mu(x) = ½$ where $\alpha^*(x) = 0$. This effect is connected with the formation of the so-called dissipative structures in R-D equations.

Denote by $\mathscr{F}_3$ the class of continuously differentiable functions $f(u)$, $u \in R^1$, such that $f(0) = f(1) = 0$, $f(u) > 0$ for $u \in (0, 1)$ and $f(u) < 0$ for $u \notin [0, 1]$, for which $f'(0) \neq \max_{0 \le u \le 1} f'(u)$.

Analysis of equation (7) as $\varepsilon \downarrow 0$ for nonlinearities of this class seems to be the most complicated, because in this case one has to overcome difficulties characteristic of both classes $\mathscr{F}_1$ and $\mathscr{F}_2$. Just as in class $\mathscr{F}_1$, the appearance of new sources is possible in this case; that is, the problem generally cannot be localized. On the other hand, for $f \in \mathscr{F}_3$ the main contribution in the right-hand side of equation (9) is given by the trajectories passing near the wave front. However, if one supposes that $f'_u(x, 0) = c$ does not depend on x, then the propagation of the disturbance will be of local nature. In this case ($f(x, \cdot) \in \mathscr{F}_3$, $f'_u(x, 0) = \text{const}$), the local behavior of $u^\varepsilon(t, x)$ may be described by the phenomenological model with the velocity field defined by (24), where

$$\alpha^*[f(x, \cdot)] = \inf_{\rho \in Y_{01}} \sup_{0<s<1} \{\rho'(s) + f(x, s)/\rho(s)\}.$$

Here Y_{01} is the totality of continuously differentiable functions $\rho(s)$, $s \in [0, 1]$, such that $\rho(1) = 0$, $\rho'(1) < 0$. (Compare with Rothe (1981), where the space-homogeneous case is considered.)

4. Remarks and generalizations. 1. First of all, we emphasize the changes which occur if in equation (7) the drift is not zero. Suppose for the present that the drift does not depend on x: $b(x) = b = (b^1, \cdots, b^r)$. If $u^\epsilon(t, x)$ is the solution of problem (7) without drift, then $v^\epsilon(t, x) = u^\epsilon(t, x + bt)$ is the solution of the problem with the drift b. This can be seen by straightforward substitution in the equation. Suppose say, that $f(x, \cdot)$ belongs to $\mathscr{F}_2^+$ for every $x \in R^r$. Then by Theorem 3, for each point $x \in R^r$ one can associate the ellipsoid Γ_x which is traced out by the end of the vector $ev(x, e)$ while the vector e runs over the unit sphere. Translate the ellipsoid by the vector $-b$. If after this translation the origin remains inside of the ellipsoid, then propagation of the disturbance at the point x occurs in all directions. The velocity of the propagation of the disturbance in the direction of the unit vector e is $\tilde{v}(x, e) = |ev(x, e) - b|$. If after the translation of the ellipsoid Γ_x by the vector $-b$ the origin 0 lies outside the ellipsoid or on its boundary, then the wave front will propagate from the point x in the directions lying inside the solid angle bounded by the cone K_x with vertex at the point 0 and tangent to the translated ellipsoid. If $b \in \partial\Gamma_x$, then the cone degenerates into a half-space. For a vector e lying inside K_x, a straight line starting from the origin and collinear to e intersects the ellipsoid at two points. Let $\underline{v}(x, e)$, $\bar{v}(x, e)$ denote the distances from the nearer and farther of these points to 0. If at a point x there is a wave front and the vector e is orthogonal to this front and points in the direction where $u^\epsilon(t, x) \to 0$, then the front will propagate with the velocity $\bar{v}(x, e)$. If on the other hand, the vector e points in the direction where $u^\epsilon(t, x) \to 1$, then the front will propagate in the direction of the vector e with velocity $\underline{v}(x, e)$.

If the vector field $b(x)$ depends on $x \in R^r$ in a sufficiently smooth way, then to every point $x \in R^r$ one can associate the ellipsoid which is obtained from Γ_x by translation by the vector $-b(x)$ and the cone K_x, provided the vector $b(x)$ lies outside of Γ_x. In accordance with the locality principle, the propagation of the wave front is described by the velocity field which at each point $x \in R^r$ is given by the coefficients at this point. Just as before let $T_g(x)$ denote the time necessary for the disturbance to reach the point x starting from the region G_+. If $b(y) \notin \Gamma_y$ for some $y \in R^r$ then, when defining $T_g(x)$, one should take the infimum over the functions φ such that $\dot{\varphi}_s$ belongs to K_{φ_s}.

Note that for $f(x, \cdot) \in \mathscr{F}_2^+$ the drift may cause the wave front to stop. For example, in the one-dimensional case with $g(x) = \chi^-(x)$ and $f(x, u) = f(u) \in \mathscr{F}_2$ the front propagates to the right with velocity $\alpha^*[f] - b(x)$ as long as this difference is positive. If at a point $a > 0$ this difference first changes sign from + to −, then as a is approached, the velocity of the front tends to zero and the point a will never be reached by the front. If in the same example $f(x, u) = f(u) \in \mathscr{F}_1$ and $\sqrt{2f'(0)} - b(0) > 0$, then the velocity of the front propagation will also be positive up to a point a^* where the difference $\sqrt{2f'(0)} - b(x)$ vanishes for the first time. If $\sqrt{2f'(0)} < b(x)$ everywhere in the region $x > a^*$, then the wave front will never extend past the point a^*. However, if somewhere to the right of the point a^* the inequality $\sqrt{2f'(0)} > b(x)$ is valid, then unlike the case $f \in \mathscr{F}_2^+$, a "new source" will appear to the right of the point a^* after some time (just as in Example 3). The wave front will propagate in both directions from the source and will in particular reach the point a^* from the right. An exact description of the behavior of the wave front for $f \in \mathscr{F}_1$ can be obtained with the help of a function $V(t, x)$ which is analogous to that introduced in Theorem 1, but there the action functional needs to be written in its general form with drift. In particular, one can in such a way calculate when and where the "new source is born" in the above example.

2. Now we dwell on the boundary problems for equation (7). Let D be a region in R^r which, for definiteness, will be assumed bounded and to have sufficiently smooth boundary. In the cylinder $[0, \infty) \times D$, we consider the equation

$$\frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^r \frac{\partial}{\partial x^i}\left(a^{ij}(x) \frac{\partial u^\varepsilon}{\partial x^j}\right) + \frac{1}{\varepsilon} f(x, u^\varepsilon), \tag{26}$$

$$t > 0, \quad x \in D, \quad u^\varepsilon(0, x) = g(x).$$

To the equation and initial condition (26), one needs to specify boundary conditions; we will assume here that

$$u^\varepsilon(t, x)\,|_{t>0, x\in\partial D} = 0 \tag{27}$$

or

$$\left.\frac{\partial u^\varepsilon(t, x)}{\partial n}\right|_{t>0, x\in\partial D} = 0, \tag{28}$$

$n = n(x)$ is the field of conormals.

Under minor regularity conditions, problems (26), (27) and (26), (28) can be solved in a unique way. If one sets $\tau^\varepsilon = \inf\{t: X_t^\varepsilon \notin D\}$, then the solution of problem (26), (27) obeys the equation

$$u^\varepsilon(t, x) = E_x g(X_t^\varepsilon)\chi_{\tau^\varepsilon > t} \exp\left\{\frac{1}{\varepsilon} \int_0^t c(X_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon))\, ds\right\}, \tag{29}$$

where $\chi_{\tau^\varepsilon > t}$ is the indicator of the set $\{\omega: \tau^\varepsilon > t\}$. The solution of problem (26), (28) satisfies equation (9), but in this case by the process (X_t^ε, P_x) one means the process in $D \cup \partial D$ which is governed by the operator L^ε inside the domain and on the boundary is subject to reflection along the conormal.

It follows from (29) that in the case of problem (26), (27), if $f(x), \cdot) \in \mathscr{F}_1$ for $x \in D$, then the limiting behavior of $u^\varepsilon(t, x)$ as $\varepsilon \downarrow 0$ is defined by the same functional $\int_0^t c(\varphi_s)\, ds - S_{0t}(\varphi)$. However the upper bound in the definition of $V(t, x)$ should be taken over the set of the functions $\varphi \in C_{0t}$, $\varphi_0 = x$, $\varphi_t \in G_0 = \operatorname{supp} g$, which do not leave the region D during the time interval $(0, t)$. Allowing for these modifications, Theorem 1 remains valid for the solution of problem

(26), (27). The proof follows the same scheme. If $f(x, \cdot) \in \mathscr{F}_1$, and $c(x) = c =$ const, then the limiting behavior of the solution of problem (26), (27) admits a phenomenological description with the same velocity field as in Example 1, provided the lower bound is taken along trajectories not leaving the region D.

The behavior for problem (26), (27) when $f(x, \cdot) \in \mathscr{F}_2$ for $x \in D$ is analogous. Here the limit function is also defined by the phenomenological model with the same velocity field as in the case of Cauchy's problem, but the lower bound in the definition of $T_g(x)$ should be taken over the functions not leaving D.

In the case of problem (26), (28) and $f(x, \cdot) \in \mathscr{F}_1$ for $x \in D$, all the statements on the limiting behavior remain valid if in place of $S_{0t}(\varphi)$ one takes the action functional $\tilde{S}_{0t}(\varphi)$ for the family of processes (X_t^ε, P_x) with reflection of ∂D. This functional is calculated in Anderson and Orey (1976), and Freidlin and Wentzell, (1984).

If instead of the boundary condition (27), the condition $u^\varepsilon(t, x)\,|_{x \in \partial D} = \psi(x)$ with a nonnegative function ψ is considered, then in the definition of the corresponding function $V(t, x)$ one should take the supremum over the functions φ_s, $0 \le s \le t$, such that $\varphi_0 = x$, $\varphi_t \in G_0 = \operatorname{supp} q \cup \operatorname{supp} \psi$. The functions φ are not allowed to leave the region D before time t.

3. Consider the Cauchy problem for the R-D equation without small parameter:

$$(30) \qquad \partial u(t, x)/\partial t = \tfrac{1}{2}(\partial/\partial x)(a(x)(\partial u/\partial x)) + f(x, u), \quad u(0, x) = \chi^-(x).$$

Suppose that $f(x, \cdot) \in \mathscr{F}_1$, for every $x \in R^1$. When t grows, the region occupied by the disturbance will propagate from left to right. Certainly, if the diffusion coefficients and nonlinear term depend on x in an arbitrary way, then one cannot expect that the velocity of the boundary of the region where $u(t, x)$ is close to 1 will stabilize. For the velocity to stabilize for large t, one must require homogeneity in some sense. The simplest way is to suppose that equation (30) is translation invariant. This case has been studied in detail. Weaker assumptions under which the velocity of the wave front stabilizes consist of considering functions $a(x)$ and $f(x, u)$ which are periodic in x or in supposing that these functions are random fields which are homogeneous in space. These problems for $f(x, \cdot) \in \mathscr{F}_1$ are considered in Gärtner and Freidlin (1979), Gärtner (1982), and Freidlin (1983). The results of these papers are also based on examining probabilities of large deviations. In the case $f \in \mathscr{F}_2$, the question of propagation of disturbances for equation (30) remains open for both periodic and random coefficients.

There are other variants of the problem on the propagation of disturbances in periodic or random medium. For example, one can consider problem (7) with the nonlinear term $f(x, u) = \tilde{f}(\xi(x), u)$, where $\xi(x)$ is a real-valued random field. Let $\tilde{f}(a, \cdot) \in \mathscr{F}_2$ for $a \in R^1$ and, for some $A \in R^1$, $\int_0^1 \tilde{f}(a, u)\, du > 0$ for $-\infty < a < A$, $\int_0^1 \tilde{f}(a, u)\, du < 0$ for $A < a < \infty$. It follows from the results of Section 3 that for small ε the disturbance will propagate in the region $\mathfrak{A}^+ = \{x \in R^r: \xi(x) > A\}$ and die out in the region $\mathfrak{A}^- = \{x \in R^r: \xi(x) < A\}$. Let $G_0 = \{x \in R^r: g(x) > 0\}$ be a bounded region. Will the disturbance expand to infinity as $t \to \infty$ or will only a bounded part of the space be disturbed? This question can be answered via the results of percolation theory.

4. It follows from Example 2 that for $f \in \mathscr{F}_1$, the wave front propagation may in a sense be of a non-Markov nature. As usual, in order to turn a process into Markov process it is helpful to extend the state space. The position of the wave front shows where the solution $u^\varepsilon(t, x)$ of problem (7) tends to 0 as $\varepsilon \downarrow 0$, and where it tends to 1. We shall describe $u^\varepsilon(t, x)$ as $\varepsilon \downarrow 0$ via its logarithmic asymptotics. Let $v(t, x) = \lim_{\varepsilon\downarrow 0}\varepsilon \ln u^\varepsilon(t, x)$. (Of course, the existence of this limit needs proof.) It is possible to prove that if $v(t, x) = 0$, then $u^\varepsilon(t, x) \to 1$ as $\varepsilon \downarrow 0$ and therefore the description of $u^\varepsilon(t, x)$ with the help of the function $v(t, x)$ is in fact more precise than the indication of the position of the wave front at time t. It is also convenient to assume that the initial function depends on ε. Let $g(x) = g^\varepsilon(x)$ be a nonnegative function which is continuous for each $\varepsilon > 0$ and $g^\varepsilon(x) \asymp \exp\{-(\alpha(x)/\varepsilon)\}$. The function $\alpha(x)$ is assumed to be nonnegative and continuous everywhere except maybe on a set G_- for which $\alpha(x) = +\infty$.

We put $\varkappa(a) = 1$ for $a < 0$ and $\varkappa(a) = 0$ for $a \geq 0$. Consider the following equation with respect to the unknown function $v(t, x)$:

$$v(t, x) = \sup\left\{-\alpha(\varphi_t) - S_{0t}(\varphi) + \int_0^t \varkappa(v(t-s, \varphi_s))c(\varphi_s)\, ds \colon \varphi \in C_{0t},\ \varphi_0 = x\right\}, \tag{31}$$

$$t \geq 0, \quad x \in R^r, \quad c(x) = \left.\frac{\partial f(x, u)}{\partial u}\right|_{u=0}.$$

Let $v(t, x)$ be a solution of this equation which is continuous for $t > 0$, $x \in R^r$. Then $v(0, x) = -\alpha(x)$, $v(t, x) \leq 0$ for all $t \geq 0$, $x \in R^r$. For a fixed x the function $v(t, x)$ does not decrease in t.

THEOREM 4. *Suppose that* $f(x, \cdot) \in \mathscr{F}_1$ *for* $x \in R^r$, *and let* $g(x) = g^\varepsilon(x)$ *satisfy the above conditions. Assume that the function*

$$v(t, x) = 0 \wedge \sup\left\{-\alpha(\varphi_t) - S_{0t}(\varphi) + \int_0^t c(\varphi_s)\, ds \colon \varphi \in C_{0t},\ \varphi_0 = x\right\} \tag{32}$$

is a continuous solution of equation (31). *Then for the solution* $u^\varepsilon(t, x)$ *of problem* (7):

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t, x) = v(t, x).$$

If a point (t, x) *is such that* $v(s, y)$ *vanishes in a neighborhood of the point* (t, x), *then* $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$.

PROOF. The proof of the theorem is analogous to that of Theorem 1. From (9) and (11) it follows that for small ε one can bound $\ln u^\varepsilon(t, x)$ from above by a quantity which is equivalent to $(1/\varepsilon)v(t, x)$ as $\varepsilon \downarrow 0$. This implies that

$\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t, x) = v(t, x)$ in the region $Q_- = \{(t, x): t > 0, x \in R^r, v(t, x) < 0\}$. From (31) and (32) one can conclude that for $(t, x) \in [Q_-]$ the following analogue of condition (N) holds: the upper bounds in (32) coincides with

$$\sup\left\{-\alpha(\varphi_t) - S_{0t}(\varphi) + \int_0^t c(\varphi_s)\, ds: \right.$$
$$\left. \varphi \in C_{0t}, \varphi_0 = x, v(t-s, \varphi_s) < 0 \text{ for } 0 < s < t\right\}.$$

This implies an analogue of the bound (14): if $(t, x) \in \partial Q_-$ and $t > 0$, then for any $\delta > 0$ one can find $\varepsilon_0 > 0$ such that

$$u^\varepsilon(t, x) > \exp\{-\delta/\varepsilon\},$$

provided $\varepsilon < \varepsilon_0$. With the help of a construction similar to that in the final part of Theorem 1 and relying on the above bound one can establish the last claim of Theorem 4.

Note that the assumption that the solution of equation (31) can be represented in the form (32) is in fact similar to condition (N). One can formulate conditions similar to those of Theorem 2, under which the solution of equation (31) defines the logarithmic asymptotics of $u^\varepsilon(t, x)$ as $\varepsilon \downarrow 0$. The solution of equation (31) seems to also describe the behavior of $\ln u^\varepsilon(t, x)$ in more general situations.

EXAMPLE 6. Suppose that $r = 1$, $a^{11}(x) \equiv 1$, $f \in \mathscr{F}_1$, $c(x) = c = \text{const}$ and let $g^\varepsilon(x) = 1$ for $x \le 0$ and $g^\varepsilon(x) = e^{-\alpha x}$, $\alpha > 0$, for $x > 0$. Then $v(t, x) = 0 \wedge \bar{v}(t, x)$, where

$$\bar{v}(t, x) = ct - \inf\left\{\alpha\varphi_t + \frac{1}{2}\int_0^t \dot{\varphi}_s^2\, ds: \varphi \in C_{0t}, \varphi_0 = x\right\}.$$

Simple calculations show that

$$\bar{v}(t, x) = \begin{cases} ct - \alpha x + \alpha^2 t/2 & \text{for } x/t > \alpha \\ ct - x^2/2t & \text{for } x/t \le \alpha. \end{cases}$$

In particular, at time t the wave front (the boundary of the region where $u^\varepsilon(t, x) \to 1$) is located at

$$X_t^* = \begin{cases} t(c/\alpha + \alpha/2) & \text{for } \alpha < \sqrt{2c} \\ t\sqrt{2c} & \text{for } \alpha \ge \sqrt{2c}. \end{cases}$$

From this it follows that the front velocity may be arbitrarily large, provided α is small enough. The velocity decreases as α grows and for $\alpha > \sqrt{2c}$ takes the constant value $\sqrt{2c}$, the same as for the initial function $g(x) = \chi^-(x)$.

5. Consider the R-D equation with the nonlinear term depending on t:

$$\frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^r \frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial u^\varepsilon}{\partial x^j}\right) + \frac{1}{\varepsilon} f(t, u^\varepsilon),\quad u^\varepsilon(0, x) = g(x).$$

The same assumptions as in Theorem 1 are made concerning the coefficients

$a^{ij}(x)$ and the initial function. Denote $(\partial f(t, u)/\partial u)|_{u=0} = c(t)$ and suppose that $f(t, \cdot) \in \mathscr{F}_1$ for $t \geq 0$.

THEOREM 5. *Suppose that for all $t \geq 0$ the inequality*

$$c(t) \geq \frac{1}{t} \int_0^t c(s)\, ds$$

holds. Then

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x) = \begin{cases} 1, & \text{if } d(x, G_0) < t\sqrt{(2/t) \int_0^t c(s)\, ds}, \\ 0, & \text{if } d(x, G_0) > t\sqrt{(2/t) \int_0^t c(s)\, ds}. \end{cases}$$

where $G_0 = \operatorname{supp} g$ and $d(x, G_0)$ is the distance from x to G_0 in the Riemannian metric

$$ds^2 = \textstyle\sum_{i,j=1}^r a_{ij}(x)\, dx^i\, dx^j, \quad (a_{ij}(x)) = (a^{ij}(x))^{-1}.$$

PROOF. Noting that

$$\inf\left\{\int_0^t \textstyle\sum_{i,j=1}^r a_{ij}(\varphi_s)\dot\varphi_s^i\dot\varphi_s^j\, ds \colon \varphi \in C_{0t},\ \varphi_0 = x,\ \varphi_t = y\right\} = \frac{1}{t}\, d^2(x, y),$$

the proof of this theorem can be developed just as that of Theorem 1.

Note that the conditions of Theorem 5 are fulfilled for increasing functions $c(t)$. If $c(t)$ decreases, then the front propagates accoding to the Huygens principle along the velocity field

$$v(t, x, e) = \sqrt{2c(t)}\left(\textstyle\sum_{i,j=1}^r a_{ij}(x)e^ie^j\right)^{-1/2}.$$

6. Equation (7) describes the process which is obtained as the result of the interaction of two factors—particle transport which in this case is described by the diffusion process, and multiplication (killing) of particles which is governed by the nonlinear term.

It is not necessary to choose a diffusion as the transport process. For example, one can choose any Markov process in R^r. Under minor assumptions on this process, one can write equation (9) and derive results similar to those for a diffusion process. It should be borne in mind that to a different transport process corresponds a different action functional. The particle transport can also be described by non-Markov processes. In this case it is in general impossible to write down differential equations, but equation (9) remains valid. If a transport process is a component of a multidimensional Markov process, then the corresponding differential equations may be degenerate.

Let X_t be the Wiener process in the interval $[-1, 1]$ with reflection at the endpoints. As the transport process, we will take the random process Y_t^ε defined by the differential equation

$$\dot Y_t^\varepsilon = b(X_{t/\varepsilon}, Y_t^\varepsilon), \quad Y_0^\varepsilon = y.$$

Then for the concentration $u^\varepsilon(t, x, y)$, we have the problem

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2\varepsilon}\frac{\partial^2 u^\varepsilon}{\partial x^2} + b(x, y)\frac{\partial u^\varepsilon}{\partial y} + \frac{1}{\varepsilon} f(x, y, u^\varepsilon),$$

(33)

$$x \in (-1, 1), \quad y \in R^1, \quad u^\varepsilon(0, x, y) = \chi^-(y), \quad \left.\frac{\partial u^\varepsilon(t, x, y)}{\partial x}\right|_{x=\pm 1} = 0.$$

For the sake of brevity, we are assuming in (33) that at the initial time the concentration does not depend on x and is equal to 1 for $y < 0$. The averaging principle implies (see, e.g., Freidlin and Wentzell, 1984) that for any $T > 0$ and $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} P_y\{\sup_{0 \le t \le T} | Y_t^\varepsilon - \bar{Y}_t | > \delta\} = 0,$$

where $\bar{Y}_t$ is the trajectory of the averaged equation

$$\dot{\bar{Y}}_t = \bar{b}(\bar{Y}_t), \quad \bar{Y}_0 = y, \quad \bar{b}(y) = \frac{1}{2}\int_{-1}^{1} b(x, y)\, dx.$$

Deviations of Y_t^ε of order 1 from $\bar{Y}_t$ have probabilities of order $\exp\{-\text{const}/\varepsilon\}$; they are therefore the main component of the right-hand side of equation (9), which in our case has the form:

$$u^\varepsilon(t, x) = E_{x,y} g(Y_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_{s/\varepsilon}, Y_s^\varepsilon, u^\varepsilon(t - s, X_{s/\varepsilon}, Y_s^\varepsilon))\, ds\right\},$$

(34)

$$c(x, y, u) = u^{-1} f(x, y, u).$$

Let

$$c(x, y) = c(x, y, 0), \quad Z_t^\varepsilon = \int_0^t c(X_{s/\varepsilon}, Y_s^\varepsilon)\, ds.$$

The action functional for the family of processes $(Y_s^\varepsilon, Z_s^\varepsilon)$ is as follows (Freidlin and Wentzell, 1984):

$$\tilde{S}_{0t}(\varphi^1, \varphi^2) = \int_0^t L(\varphi_s^1, \dot{\varphi}_s^1, \dot{\varphi}_s^2)\, ds,$$

where $L(y, \alpha^1, \alpha^2)$ is the Legendre transform of $\lambda(y, \beta_1, \beta_2)$ in the variables (β_1, β_2) and $\lambda = \lambda(y, \beta_1, \beta_2)$ is the eigenvalue of the problem

$$\frac{1}{2}\frac{d^2 v(x)}{dx^2} + (\beta_1 b(x, y) + \beta_2 c(x, y)) v(x) = \lambda v(x),$$

$$x \in (-1, 1), \quad (dv/dx)|_{x=\pm 1} = 0, \quad y \in R^1; \quad \beta_1, \beta_2 \in R^1,$$

which corresponds to the positive eigenfunction. In the case of $f(x, y, \cdot) \in \mathscr{F}_1$, $\bar{b}(y) \equiv 0$, if a number of supplementary conditions are fulfilled, then the location of the wave front is defined by the equation $V(t, y) = 0$, where

$$V(t, y) = \sup\left\{\varphi_t^2 - \int_0^t L(\varphi_s^1, \dot{\varphi}_s^1, \dot{\varphi}_s^2)\, ds : \varphi_0^1 = y, \varphi_t^1 \le 0, \varphi_0^2 = 0\right\}.$$

This assertion is similar to that of Theorem 1. The condition $\bar{b}(y)$ is analogous to assuming that there is no drift.

For $f(x, y, \cdot) \in \mathscr{F}_1$, problem (33) is studied in Sarafian and Safarian (1980). For $f(x, y, \cdot) \in \mathscr{F}_2$, the question of the behavior of the solution of problem (33) as $\varepsilon \downarrow 0$ is open.

7. Consider the mixed problem with the nonlinear boundary condition in the region $t > 0$, $x \in R^r_+ = \{x \in R^r: x^1 > 0\}$:

$$(35)\qquad \begin{gathered} \frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2}\Delta u^\varepsilon, \quad u^\varepsilon(0, x) = g(x) \geq 0, \\ \left.\frac{\partial u^\varepsilon(t, x)}{\partial x^1}\right|_{x^1=0} = \frac{1}{\varepsilon} f(x, u^\varepsilon). \end{gathered}$$

Such problems arise, for example, in simulation of some biological processes (Freidlin and Sivak, 1979).

We will suppose that $f(x, \cdot) \in \mathscr{F}_1$ and $f(x, u) = uc(x, u)$. Let (X^ε_t, P_x) be the Markov process in R^r_+ which, inside of this half-space, is governed by the operator $(\varepsilon/2)\Delta$, and on the boundary is subject to the reflection along the normal. Denote by ξ^ε_t the local time on the boundary of R^r_+ associated with the process (X^ε_t, P_x) (Watanabe, 1971).

It is possible to prove that the solution of problem (35) obeys the following relation

$$(36)\qquad u^\varepsilon(t, x) = E_x g(X^\varepsilon_t)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X^\varepsilon_s, u^\varepsilon(t-s, X^\varepsilon_s))\, d\xi^\varepsilon_s\right\}.$$

To examine (36), one must know the action functional for the pair $(X^\varepsilon_t, \xi^\varepsilon_t)$. From the results of Anderson and Orey (1976) and Freidlin and Wentzell (1984), one can deduce that this functional has the form

$$(36a)\qquad S^{X,\xi}_{0t}(\varphi, \mu) = \frac{1}{2}\int_0^t [\textstyle\sum_{i=2}^r (\dot\varphi^i_s)^2 + (\dot\varphi^1_s - \chi_0(\varphi^1_s)\dot\mu_s)^2]\, ds.$$

Here $\varphi_s = (\varphi^1_s, \cdots, \varphi^r_s)$ is an absolutely continuous function, $\chi_0(0) = 1$, $\chi_0(x) = 0$ for $x \neq 0$, μ_s being a nondecreasing function.

We will confine ourselves to the case $c(x, 0) = c = \text{const}$ and set $R^+_{0t} = c\mu_t - S^{X,\xi}_{0,t}(\varphi, \mu)$. From (36), noting that $\dot\xi^\varepsilon_t \geq 0$, we conclude that

$$0 \leq u^\varepsilon(t, x) \leq E_x g(X^\varepsilon_t)\exp\{c\xi^\varepsilon_t/\varepsilon\}.$$

The action functional properties imply that right-hand side of the last equality is logarithmically equivalent to

$$\exp\{(1/\varepsilon)\sup\{R^+_{0t}(\varphi, \mu): \varphi_0 = x, \varphi_t \in G_0, \dot\mu_s \geq 0 \text{ for } s \leq t\}\},$$

where G_0 is the support of the initial function. It is easily checked that this supremum is equal to

$$V(t, x) = \sup_{\varphi_0=x, \varphi_t\in G_0}\left\{\frac{1}{2}\int_0^t \left(\chi_0(\varphi^1_s)[(c + \dot\varphi^1_s) \vee 0]^2 - \textstyle\sum_{i=2}^r (\dot\varphi^i_s)^2\right) ds\right\}.$$

In a manner analogous to Theorem 1, one can obtain the following result from the above bounds.

THEOREM 6. *Suppose that* $f(x, \cdot) \in \mathscr{F}_1$, $c(x, 0) = c = \text{const}$, *and the function* $g(x)$ *is continuous. Then for the solution* $u^\varepsilon(t, x)$ *of problem* (35) *the following relations hold*:

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = g(x) \quad \text{if} \quad x \in R^r_+, \quad t > 0,$$

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 0 \quad \text{if} \quad x \in \partial R^r_+, \quad t > 0 \quad \text{and} \quad V(t, x) < 0,$$

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1 \quad \text{if} \quad x \in \partial R^r_+, \quad t > 0 \quad \text{and} \quad V(t, x) > 0.$$

A number of results and examples concerning the behavior as $\varepsilon \downarrow 0$ of the solution of problem (35) are available in Korostelev and Freidlin (1980).

Similar results seem to also be valid in the case when the particle "multiplication" takes place on some manifold Γ of co-dimension 1 lying inside the region rather than on the boundary.

5. Systems of reaction-diffusion equations. Reasoning as in Section 1, it comes as no surprise that small parameters must be involved in a system of equations in the following fashion:

$$\frac{\partial u_k^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^r \frac{\partial}{\partial x^i}\left(a_k^{ij}(x)\,\frac{\partial u_k^\varepsilon}{\partial x^j}\right) + \frac{1}{\varepsilon} f_k(x, u_1^\varepsilon, \cdots, u_n^\varepsilon)$$

$$(37) \qquad = \varepsilon L_k u_k^\varepsilon + \frac{1}{\varepsilon} f(x, u_1^\varepsilon, \cdots, u_n^\varepsilon),$$

$$t > 0, \quad x \in R^r, \quad u_k^\varepsilon(0, x) = g_k(x); \quad k = 1, 2, \cdots, n.$$

As we have seen in Section 3 in the case of a single equation, for sufficiently small t and $\varepsilon \downarrow 0$, the solution $u^\varepsilon(t, x)$ first of all approaches the equilibrium point of the local system (in the case of $n = 1$ this is the equation $\dot{u} = f(x, u)$) whose domain of attraction the point $u^\varepsilon(0, x) = g(x)$ belongs to. For $f \in \mathscr{F}_2$, there are two such attracting points: 0 and 1. These points are separated by the unstable equilibrium point $\mu(x) \in (0, 1)$. According to Lemma 2, for small $t > 0$, $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x)$ equals 0 or 1 depending on whether $u^\varepsilon(0, x) = g(x) < \mu(x)$ or $> \mu(x)$. If the initial function takes the value $\mu(x)$ on an open set $\mathscr{E} \subset R^r$, then $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = \mu(x)$ for each fixed $x \in \mathscr{E}$ and t small enough. As a matter of fact, we faced this case for $f \in \mathscr{F}_1$, with $\mu(x) \equiv 0$. If the entire set $\{u\colon u = g(x), x \in R^r\}$ belongs to the domain of attraction of some equilibrium point, say of the point $u = 1$, then for a fixed $t > 0$ we have $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = 1$.

For system (37), the situation is far more complicated. Certainly, this is in part caused by the fact that for $n > 1$ the corresponding local system may have more complicated ω-limit sets. But even if these ω-limit sets have a simple structure, a number of new effects appear for $n > 1$.

For example, for $n > 1$ the so-called "diffusion instability" arises (Turing, 1952). Suppose that $f(x, u) = f(u) = (f_1(u), \cdots, f_n(u))$ and $u^0 = (u_1^0, \cdots, u_n^0)$ is an asympotically stable equilibrium point for the vector field $f(u)$. Then the functions

$u_k^\varepsilon(t, x) \equiv u_k^0$ are the solution of the Cauchy problem (37) with initial conditions $u_k^\varepsilon(0, x) = u_k^0$. It turns out that in spite of the fact that u^0 is an asymptotically stable equilibrium point of the corresponding local system, this solution may be unstable for the Cauchy problem (37). Under appropriate conditions, one can find functions $\delta_1(x), \cdots, \delta_n(x)$ which are arbitrarily small in modulus such that the solution of problem (37) with the initial conditions $g_k^\delta(x) = u_k^0 + \delta_k(x)$ will move away from u^0 as t increases (ε is assumed to be fixed).

Diffusion instability arises only if the operators L_k controlling the transport of the particles of kth type are different for different k. For identical L_k, the asymptotically stable equilibrium points of the field $f(u)$ will be stable equilibrium points of the Cauchy problem (37). However, in this case too, one needs to impose some convexity conditions so that u will be attracted to the constant function $u(t, x) \equiv u^0$ as $t \to \infty$ under initial conditions in the domain of attraction of u^0 (Freidlin, 1968).

For the present, assume that $L_1 = L_2 = \cdots = L_n$. Consider the system of the following form

$$\partial u_k^\varepsilon/\partial t = \varepsilon L u_k^\varepsilon + (1/\varepsilon)[c_{kk}(u^\varepsilon)u_k^\varepsilon + \textstyle\sum_{j:j\neq k} c_{kj}u_j^\varepsilon]; \quad k = 1, 2, \cdots, n. \tag{38}$$

We assume that $c_{kj} = \text{const} > 0$ for $k \neq j$ and set

$$f_k(u) = c_{kk}(u)u_k + \textstyle\sum_{j:j\neq k} c_{kj}u_j;$$

$$c_k(u) = c_{kk}(u) + \textstyle\sum_{j:j\neq k} c_{kj};$$

$$c_{kk} = c_{kk}(0), \quad c_k = c_k(0).$$

Let ν_t, $t \geq 0$, be the Markov process with the finite number of states $1, 2, \cdots, n$ for which

$$P\{\nu_{t+\Delta} = j \mid \nu_t = i\} = c_{ij}\Delta + 0(\Delta), \quad \Delta \downarrow 0, \quad i \neq j.$$

It is not difficult to prove that the solution of problem (38) with the initial conditions $u_k^\varepsilon(0, x) = g_k(x)$, $k = 1, \cdots, n$, can be represented as follows (Freidlin, 1983b):

$$u_k^\varepsilon(t, x) = E_{x,k} g_{\nu(t/\varepsilon)}(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c_{\nu(s/\varepsilon)}(u^\varepsilon(t-s, X_s^\varepsilon))\, ds\right\}, \tag{39}$$

where X_t^ε is a diffusion process in R^r governed by the operator εL and the indices x and k in the symbol of the expectation indicate that $X_0^\varepsilon = x$, $\nu_0 = k$.

Formula (39) is convenient to use for examining wave fronts in the system (38). The front velocity is then defined by the large deviations of both the diffusion process X_t^ε and the process $\nu(t/\varepsilon)$.

Suppose that in the region $R_+^n = \{u = (u_1, \cdots, u_n): u_1 \geq 0, \cdots, u_n \geq 0\}$ the field $f(u) = (f_1(u), \cdots, f_n(u))$ has two equilibrium points: an unstable one at the point $0 = (0, \cdots, 0)$ and an asymptotically stable one at a point $a = (a_1, \cdots, a_n)$. Assume that all the integral curves in the region $R_+^n\setminus\{0\}$ do not leave R_+^n and are attracted to the point a. Moreover, we will assume that for some $\alpha_1, \alpha_2 > 0$

$$f_1(u) > \alpha_1, \cdots, f_n(u) > \alpha_1 \quad \text{for} \quad u \in \{u \in R_+^n: u \neq 0, \textstyle\sum_1^n u_i < \alpha_2\},$$

and in the domain $B_{\alpha_2/2} = \{u \in R_+^n: \sum u_i > \alpha_2/2\}$ the convex function $V(u)$ (the

Liapounov function) is defined such that $V(u) > 0$ for $u \in B_{\alpha_2/2}\backslash\{a\}$, $V(a) = 0$, $(\nabla V(u), f(u)) < 0$ for $u \in B_{\alpha_2/2}\backslash\{a\}$. We also suppose that for $k = 1, 2, \cdots, n$

(40) $$c_{kk}(0) = c_{kk} = \max\{c_{kk}(u) : u = (u_1, \cdots, u_n), 0 \le u_i \le a_i; i = 1, \cdots, n\}.$$

Consider problem (38) with the initial conditions

(41) $$u_k(0) = g_k(x) \ge 0, \quad k = 1, \cdots, n.$$

For brevity, we will assume that the functions $g_k(x)$ are bounded and have common support G_0. As usual, we suppose that the closure of G_0 coincides with the closure of the interior of G_0.

Denote by λ the eigenvalue of the matrix (c_{ij}) possessing the largest real part. By the Frobenius theorem, such a λ is real (and, therefore, unique).

Let $d(\cdot, \cdot)$ be the metric in R^r corresponding to the metric form $ds^2 = \sum_{i,j=1}^r a_{ij}(x)\, dx^i\, dx^j$, $(a_{ij}) = (a^{ij})^{-1}$, where $a^{ij}(x)$ are the coefficients of the operator L.

THEOREM 7. *Suppose that the above conditions are fulfilled. Then for the solution* $(u_1^\varepsilon(t, x), \cdots, u_n^\varepsilon(t, x))$ *of problem* (38) *and* (41),

$$\lim_{\varepsilon\downarrow 0} u_k^\varepsilon(t, x) = \begin{cases} a_k, & \text{if } d(x, G_0) < t\sqrt{2\lambda} \\ 0, & \text{if } d(x, G_0) > t\sqrt{2\lambda}. \end{cases}$$

We will outline the proof of this theorem. From (39), relying on (40) and (41), we obtain

(42) $$\begin{aligned} 0 \le u_k^\varepsilon(t, x) &\le E_{x,k} g_{\nu(t/\varepsilon)}(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^t c_{\nu(s/\varepsilon)}\, ds\right\} \\ &\le \sup_{k=1,\cdots,n; x\in R^r} |g_k(x)| \cdot P_x\{X_t^\varepsilon \in G_0\} E_k \exp\left\{\frac{1}{\varepsilon}\int_0^t c_{\nu(s/\varepsilon)}\, ds\right\}. \end{aligned}$$

It follows from the previously cited properties of the action functional for the family of the processes X_t^ε that the probability on the right-hand side of inequality (42) is as $\varepsilon \downarrow 0$ logarithmically equivalent to

$$\exp\left\{-\frac{1}{2\varepsilon}\inf\left\{\int_0^t \sum_{i,j=1}^r a_{ij}(\varphi_s)\dot\varphi_s^i\dot\varphi_s^j\, ds : \varphi_0 = x, \varphi_t \in G_0\right\}\right\} = \exp\left\{-\frac{d^2(x, G_0)}{2\varepsilon t}\right\}.$$

Using the fact that the family of operators

$$T_t f(k) = E_k f(\nu(t)) \exp\left\{\int_0^t c_{\nu(s)}\, ds\right\}$$

is a semigroup, one can deduce that the expectation on the right-hand side of (42) is as $\varepsilon \downarrow 0$ logarithmically equivalent to $\exp\{\lambda t\varepsilon^{-1}\}$ (a version of the Frobenius theorem; see, e.g., Freidlin and Wentzell, 1984, Chapter 7). Gathering these bounds together, we get that the right-hand side in (42) is logarithmically equivalent to

$$\exp\{-(1/\varepsilon)(\lambda t - d^2(x, G_0)/2t)\},$$

which implies that $\lim_{\varepsilon\downarrow 0} u_k^\varepsilon(t, x) = 0$ for $d(x, G_0) > t\sqrt{2\lambda}$.

The proof that $\lim_{\varepsilon\downarrow 0} u_k^\varepsilon(t, x) = a_k$ for $d(x, G_0) < t\sqrt{2\lambda}$ can be divided into two parts. First, one proves that $u^\varepsilon(t, x) = (u_1^\varepsilon(t, x), \cdots, u_n^\varepsilon(t, x)) \in B_{\alpha_2/2}$ for sufficiently small ε and $d(x, G_0) < t\sqrt{2\lambda}$. Then one checks that the solution of the boundary value problem (38) in the region $\mathscr{E} = \{(s, x): s > 0, x \in R^r, d(x, G_0) < s\sqrt{2\lambda}\}$ with boundary values lying in $B_{\alpha_2/2}$, tends to the equilibrium point $a = (a_1, \cdots, a_n)$ as $\varepsilon \downarrow 0$. The proof of the first of these statements is analogous to that of the final part of Theorem 1.

To prove the second statement, it is sufficient to verify that $V(u^\varepsilon(t, x)) \to 0$ as $\varepsilon \downarrow 0$ if $(t, x) \in \mathscr{E}$. Denote by $\tau = \tau^\varepsilon$ the first exit time of the "heat" process $(t - s, X_s^\varepsilon)$ from the region $\mathscr{E}$: $\tau^\varepsilon = \inf\{s: (t - s, X_s^\varepsilon) \notin \mathscr{E}\}$. One can deduce from (39) and the strong Markov property that

$$u_k^\varepsilon(t, x) = E_{x,k} u_{\nu(\tau/\varepsilon)}^\varepsilon(t - \tau, X_\tau^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^\tau c_{\nu(s/\varepsilon)}(u^\varepsilon(t - s, X_s^\varepsilon))\, ds\right\}$$

$$\text{for} \quad (t, x) \in \mathscr{E}; \quad k = 1, \cdots, n.$$

Since the Liapounov function is convex, we have

$$\begin{aligned} V(u^\varepsilon(t, x)) &= V(E_{x,1}\zeta_0^\tau, \cdots, E_{x,n}\zeta_0^\tau) \\ &\le E_x V(E_1\zeta_0^\tau, \cdots, E_n\zeta_0^\tau), \end{aligned} \tag{43}$$

where

$$\zeta_0^\tau = u_{\nu(t/\varepsilon)}(t - \tau, X_\tau^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^\tau c_{\nu(s/\varepsilon)}(u^\varepsilon(t - s, X_s^\varepsilon))\, ds\right\}.$$

We denote by $E_k\zeta_0^\tau$ the expectation of ζ_0^τ for a fixed trajectory X^ε under the assumption that $\nu_0 = k$, E_x being the expectation with respect to the measure which corresponds to the process X_t^ε, $X_0^\varepsilon = x$. It is clear that $E_{x,k}\zeta_0^\tau = E_x(E_k\zeta_0^\tau)$.

Next note that the transformation

$$M_t: z = (z_1, \cdots, z_n) \to M_t(z) = (z_1(t), \cdots, z_n(t)),$$

$$z_k(t) = E_k z_{\nu(t)} \exp\left\{\int_0^t c_{\nu(s)}(z(t - s))\, ds\right\},$$

is a shift along the trajectories of the dynamical system $\dot{z}_t = f(z_t)$, $f(z) = (f_1(z), \cdots, f_n(z))$, $f_k(z) = \sum_{i=1}^n c_{ki}(z) z_i$. Noting that $V(z)$ is the Liapounov function for this system, one can deduce from inequality (43) that $V(u^\varepsilon(t, x)) \to 0$ as $\varepsilon \downarrow 0$.

This implies that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x) = a$ provided $d(x, G_0) < t\sqrt{2\lambda}$.

If the operators L_k responsible for the particle transport are different for different k, then it is necessary to impose more stringent conditions on the local system in order to avoid effects caused by the diffusion instability. Some results in the case of different L_k are available in Freidlin (1983b). There is a defect in the proof of Theorem 2 of this paper, but the formula for the wave front velocity is correct. The correction is published in a later issue of *TAMS*.

One can also consider the case when the nonlinear terms depend on x. Just as in Example 3, effects due to the appearance of new sources are possible in this case.

The generalization considered in Theorem 7 is a natural generalization of the nonlinearities of class $\mathscr{F}_1$. On the other hand, if the local system has two stable equilibrium points separated by a saddle point (an analogy with $\mathscr{F}_2$), then there are no general results concerning the velocity of the wave front.

In the conclusion of this section, we will clarify how one can write down the integral equation for the solution of problem (37). For the sake of brevity, we will assume $n = 2$, $\varepsilon = 1$ and drop the superscript. Moreover, we make the following assumption on homogeneity: $f_1(x, 0) = f_2(x, 0) = 0$. In this case the functions $f_k(x, u)$ can be written as follows:

$$f_k(x, u_1, u_2) = c_{k1}(x, u)u_1 + c_{k2}(x, u)u_2.$$

Suppose for the present that $c_{12}(x, u)$ and $c_{21}(x, u)$ are nonnegative. Then just as in the case of the system (38), one can associate with problem (37) a Markov process (X_t, ν_t) which is defined by the relations:

$$X_t = x + \int_0^t \sigma_{\nu(s)}(X_s)\, dW_s,$$

where the matrix $\sigma_k(x)$ is such that $\sigma_k(x)\sigma_k^*(x) = (a_k^{ij}(x))$; ν_t is a right continuous process with two states 1 and 2 for which

$$P\{\nu_{t+\Delta} = j \mid \nu_t = i,\ X_t = x\} = c_{ij}(x, u(t, x))\Delta + 0(\Delta), \quad \Delta \downarrow 0.$$

For the solution $u(t, x) = (u_1(t, x), u_2(t, x))$ of problem (37) one can write down a relation which is a generalization of equation (39):

$$u_k(t, x) = E_{x,k} g_{\nu(t)}(X_t) \exp\left\{ \int_0^t c_{\nu(s)}(X_s, u(t - s, X_s))\, ds \right\},$$

where $c_i(x, u) = c_{i1}(x, u) + c_{i2}(x, u)$. It is possible to write the equation for the process ν_t, but for the sake of brevity, we will not.

Therefore, for the variables $(X_t, \nu_t, u(t, x))$ one can obtain a system of three equations. This system is not very convenient to examine since each of these equations has two or more variables. In place of the variables $(X_t, \nu_t, u(t, x))$ we introduce new variables $(Z_t, \lambda_t, u(t, x))$ for which the system is reduced to triangular form.

Let λ_t be a right continuous Markov process with two states 1 and 2 for which

$$P\{\lambda_{t+\Delta} = j \mid \lambda_t = i\} = \Delta + 0(\Delta), \quad \Delta \downarrow 0, \quad \text{if} \quad i \neq j.$$

The process Z_t is defined by the equation

$$Z_t = x + \int_0^t \sigma_{\lambda_s}(Z_s)\, dW_s.$$

It is readily checked that the measure μ_1 which is induced in the space of trajectories of the process (X_t, ν_t) is absolutely continuous with respect to the measure μ_2 corresponding to the process (Z_t, λ_t). The density function $d\mu_1/d\mu_2$

has the form

$$\frac{d\mu_1}{d\mu_2}(Z_., \lambda_.) = \prod_{i=0}^{\eta_0^t} [c_{\lambda_{t_i}\bar\lambda_{t_i}}(Z_{t_i}, u(t - t_i, Z_{t_i}))]$$

$$\cdot \exp\left\{- \int_{t_i}^{t_{i+1}} [c_{\lambda_{t_i}\bar\lambda_{t_i}}(Z_s, u(t - s, Z_s)) - 1]\, ds\right\},$$

where $t_i = t_i[\lambda_.]$ are times at which the trajectory λ_s has jumps, η_0^t is the number of such jumps before the time t, $t_0 = 0$, $t_{\eta_0^t+1} = t$; $\bar\lambda = 1$ if $\lambda = 2$ and $\bar\lambda = 2$ if $\lambda = 1$.

Using this expression for $d\mu_1/d\mu_2$, we obtain the following equations for $u_k(t, x)$:

$$u_k(t, x) = E_{x,k} g_{\lambda_t}(Z_t) \exp\left\{\int_0^t [c_{\lambda_s\lambda_s}(Z_s, u(t - s, Z_s)) + 1]\, ds\right\}$$

$$\times \prod_{i=0}^{\eta_0^t} c_{\lambda_{t_i}\bar\lambda_{t_i}}(Z_{t_i}, u(t - t_i, Z_{t_i})), \quad k = 1, 2.$$

Here the process (Z_t, λ_t) is defined independently of $u(t, x)$.

These equations hold no matter what the signs of the functions $c_{ij}(x, u)$ are. If $c_{12}(x, u)$ or $c_{21}(x, u)$ is negative, then the proof of these equations is also not difficult.

6. Some other ways of introducing a small parameter. This section is concerned with a number of asymptotic problems for the propagation of wave fronts of R-D equations. We shall not strive here for the utmost generality, but merely consider the simplest representatives of the corresponding classes of problems.

1. We have been considering R-D equations in the case when the diffusion coefficients are small. Then after going over to the new time scale we obtained equation (7). Now we suppose that only some of the diffusion coefficients are small. More precisely let $D = \{(x, y) \in R^2: -\infty < x < \infty,\ |y| < a\}$. We will consider the following problem (after going over to the new time scale):

$$\frac{\partial u^\varepsilon(t, x, y)}{\partial t} = \frac{1}{2\varepsilon}\frac{\partial^2 u^\varepsilon}{\partial y^2} + \frac{\varepsilon}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon} f(u^\varepsilon), \tag{44}$$

$$t > 0, \quad (x, y) \in D, \quad u^\varepsilon(0, x, y) = g(x, y), \quad u^\varepsilon(t, x, \pm a) = 0.$$

With regard to $f(u)$, we shall assume that $f(0) = 0$, $c(u) = u^{-1}f(u)$ is strictly decreasing for $u \ge 0$ from $c = c(0) = \lim_{u\downarrow 0} c(u)$, $0 < c < \infty$, to $c(\infty) = \lim_{u\to\infty} c(u) \ge -\infty$. Let (X_t^ε, P_x) be a process governed by the operator $(\varepsilon/2)(d^2/dx^2)$, and $(Y_t^\varepsilon, \tilde P_Y)$ be a process governed by the operator $(1/2\varepsilon)(d^2/dy^2)$, which is independent of (X_t^ε, P_x). We put $\tau^\varepsilon = \inf\{t: |Y_t^\varepsilon| = a\}$. One can write down the following equation for the solution $u^\varepsilon(t, x, y)$ of problem (44)

$$u^\varepsilon(t, x, y) = E_{x,y} g(X_t^\varepsilon)\chi_{\tau^\varepsilon > t} \exp\left\{\frac{1}{\varepsilon}\int_0^t c(u^\varepsilon(t - s, X_s^\varepsilon, Y_s^\varepsilon))\, ds\right\}, \tag{45}$$

where $E_{x,y}$ denotes the integration with respect to the measure $P_x \times P_y$. It follows from (45) since $c(u) \le c(0) = c$, that

$$u^\varepsilon(t, x, y) \le e^{ct} P_x\{X_t^\varepsilon \le 0\} \tilde{P}_Y\{\tau^\varepsilon > t\}. \tag{46}$$

We shall denote by λ^ε the largest eigenvalue of the problem

$$(1/2\varepsilon)\varphi''_{yy}(y) = \lambda^\varepsilon \varphi(y), \quad -a < y < a, \quad \varphi(\pm a) = 0.$$

A simple calculation shows that $\lambda^\varepsilon = (-\pi^2)/8\varepsilon a^2$, the corresponding eigenfunction $\varphi(y)$ being equal to $\cos(\pi y/2a)$. It is readily checked that $\tilde{P}_Y\{\tau^\varepsilon > t\} \sim \varphi(y)\exp\{-\lambda^\varepsilon t\}$ as $\varepsilon \downarrow 0$. Therefore remembering (46) we conclude

$$u^\varepsilon(t, x, y) \le \exp\left\{\frac{1}{\varepsilon}\left(ct - \frac{x^2}{2t} - \frac{\pi^2 t}{8a^2}\right)\right\}.$$

From this it results that if $x > t\sqrt{2c - \pi^2/4a^2}$ and $c > \pi^2/8a^2$, then $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x, y) = 0$.

We shall denote by u^* a positive solution of the equation $c(u^*) = \pi^2/8a^2$. Such a u^* exists, provided $c(\infty) < \pi^2/8a^2 < c$, and it is always unique. We note that $c(u) < c$ for $u > u^*$. It is therefore possible to verify that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x, y) = u^*$ for $x < t\sqrt{2c - \pi^2/4a^2}$ by reasoning as in Theorem 1.

Consequently for $c < \pi^2/8a^2$, the solution $u^\varepsilon(t, x, y)$ of problem (44) for small ε has the shape of a step with the height u^* which travels along the x-axis from left to right with velocity $\alpha^* = \sqrt{2c - \pi^2/4a^2}$. If $c < \pi^2/8a^2$, then $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t, x, y) = 0$ for all x, y and $t > 0$. In particular, if the band is narrow enough ($a < \pi/2\sqrt{2c}$), then the wave does not propagate.

In the same band D we consider the problem

$$\text{(47)}\quad \begin{aligned} \frac{\partial v^\varepsilon(t, x, y)}{\partial t} &= \frac{1}{2\varepsilon}\frac{\partial^2 v^\varepsilon}{\partial y^2} + \frac{\varepsilon a(y)}{2}\frac{\partial 2 v^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon} f(y, v^\varepsilon), \\ v^\varepsilon(0, x, y) &= g(x, y), \quad \left.\frac{\partial v^\varepsilon(t, x, y)}{\partial y}\right|_{y=\pm a} = 0. \end{aligned}$$

Let $f(y, \cdot) \in \mathscr{F}_1$ for every $y \in R_1$, $c(y, u) = u^{-1} f(y, u)$, $c(y) = c(y, 0)$. Denote by $(\xi_s, \tilde{P}_y)$ the Wiener process with reflection at the endpoints of the interval $[-a, a]$; as before, (X_t^ε, P_x) is the process in R^1 corresponding to the operator $(\varepsilon/2)(d^2/dx^2)$. Probabilistic representation of the solution of problem (47) shows that

$$\text{(48)}\quad \begin{aligned} v^\varepsilon(t, x, y) &\le \tilde{E}_Y P_x\{X^\varepsilon_{\int_0^t a(\xi_{s/\varepsilon})ds} \le 0\} \exp\left\{\frac{1}{\varepsilon}\int_0^t c(\xi_{s/\varepsilon})\, ds\right\} \\ &\asymp \tilde{E}_y \exp\left\{\frac{1}{\varepsilon}\left[\int_0^t c(\xi_{s/\varepsilon})\, ds - \frac{x^2}{2\int_0^t a(\xi_{s/\varepsilon})\, ds}\right]\right\}. \end{aligned}$$

The logarithmic asymptotics of the right-hand side of (48) are defined by the action functional for the pair of processes $(\int_0^t c(\xi_{s/\varepsilon})\, ds, \int_0^t a(\xi_{s/\varepsilon})\, ds)$. It can be expressed in terms of the first eigenvalue of the problem (compare with

Section 4):

$$\tfrac{1}{2}\varphi''_{yy}(y) + (\beta_1 c(y) + \beta_2 a(y))\varphi(y) = \lambda(\beta_1, \beta_2)\varphi(y),$$
$$|y| < a, \quad \varphi'(\pm a) = 0; \quad \beta_1, \beta_2 \in R^1.$$

Using this action functional, one can write down the expression for the velocity of the wave front along the x-axis.

2. Consider the Cauchy problem

$$(49) \qquad \frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^r a^{ij}(x) \frac{\partial^2 u^\varepsilon}{\partial x^i \partial x^j} + \sum_{i=1}^r b^i(x) \frac{\partial u^\varepsilon}{\partial x^i} + f(u^\varepsilon)$$
$$= L^\varepsilon u^\varepsilon + f(u^\varepsilon), \quad u^\varepsilon(0, x) = g(x), \quad x \in R^r.$$

Let $f(u) = cu(u - 1)$, $c > 0$; $g(x) = 1$ in a bounded neighborhood G_0 of the origin and $g(x) = 0$ for $x \notin G_0$. Clearly $-f(u) \in \mathscr{F}_1$; therefore, for $b^i(x) \equiv 0$, $i = 1, \cdots, n$, the question of the behavior of the solution of problem (49) as $\varepsilon \downarrow 0$ reduces to the problem dealt with in Section 2. After dividing by $\sqrt{\varepsilon}$, we obtain a problem of the same type as in (7). By Theorem 1, for every $t > 0$ the function $u^\varepsilon(t, x)$ tends to the step-function as $\varepsilon \downarrow 0$. The wave front propagates with the velocity of order of $\sqrt{\varepsilon}$.

The inclusion of the nonzero drift in the equation leads to a change in the order of the front velocity as $\varepsilon \downarrow 0$. Assume the field $b(x) = (b^1(x), \cdots, b^r(x))$ possesses an asymptotically stable equilibrium point only at the origin 0 and suppose that all the integral curves of the field $b(x)$ are attracted to 0. Denote by $T(x)$ the time which is necessary for the integral curve starting from a point $x \in R^r$ to reach the set G_0. We put

$$W = W(G_0) = \inf\left\{\frac{1}{2}\int_0^t \sum_{i,j=1}^r a_{ij}(\varphi_s)(\dot\varphi_s^i - b^i(\varphi_s))(\dot\varphi_s^j - b^j(\varphi_s))\,ds: \right.$$
$$\left. \varphi \in C_{0t}, \varphi_0 = 0, \varphi_t \notin G_0, t > 0\right\}.$$

THEOREM 8. *Under the above assumptions for the solution $u^\varepsilon(t, x)$ of problem* (49), *the following relations hold*:

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t(\varepsilon), x) = \begin{cases} 0, & \text{if } t(\varepsilon) < T(x), \\ 1, & \text{if } T(x) < t(\varepsilon) < W/c\varepsilon \quad \text{beginning from some } \varepsilon > 0, \\ 0, & \text{if } W/c\varepsilon < t(\varepsilon) \quad \text{beginning from some } \varepsilon > 0. \end{cases}$$

PROOF. The proof of this theorem uses the representation of the solution of problem (49) in the form of the mean value of an appropriate functional of the corresponding branching process with diffusion. One can construct a branching diffusion process whose particles move along the trajectories of the diffusion process X_t^ε governed by the operator L^ε. Every particle independently splits into two particles by time t with probability $1 - e^{-ct}$ counting from the time of birth. These particles move and multiply according to the same rules as the original particle. Let $\nu(t)$ denote the number of these particles by time t. Then the solution of problem (49) can be written in the form

$$u^\varepsilon(t, x) = E_x g(X_t^{\varepsilon,1}) \cdots g(X_t^{\varepsilon,\nu(t)}), \tag{50}$$

where $X_t^{\varepsilon,k}$ is the position of kth particle at the time t, $k \le \nu(t)$. Such a representation has been used (Skorohod, 1964; McKean, 1975; Bramson, 1978) for examining R-D equations.

It follows from (50) that $u^\varepsilon(t(\varepsilon), x) \to 0$ as $\varepsilon \downarrow 0$, provided the probability that at time $t(\varepsilon)$ at least one of the particles is outside of the domain G_0 tends to 1. Such is the case, first of all, if $t(\varepsilon) < T(x)$, i.e. if the particles have not had time to reach G_0. This is also the case if at least one of the particles which has reached G_0 has had time to leave G_0. Since our branching process cannot become extinct, it is not difficult to prove that

$$P\{\lim_{t\to\infty}(1/t)\ln \nu(t) = c\} = 1. \tag{51}$$

We will choose a small $\delta > 0$ and consider the spheres γ and Γ of radii $\delta/2$ and δ, respectively, about the origin. For any $h > 0$, one can choose $\delta > 0$ small enough so that the probability of the event: "starting from $x \in \gamma$, the trajectory X_t^ε reaches ∂G_0 without returning to γ after hitting Γ," lies between $\exp\{-(1/\varepsilon)(W + h)\}$ and $\exp-(1/\varepsilon)(W - h)\}$ (Freidlin and Wentzell, 1984, Chapter 4).

By the time $t(\varepsilon)$, there are $\nu(t(\varepsilon)) \asymp \exp\{ct(\varepsilon)\}$ particles. In this time, each of them will make one or more (but not more than const $\times$ $[t(\varepsilon)]$) attempts to hit ∂G_0 without returning to γ after reaching Γ. These attempts occur "almost independently." (Here we have reasoning analogous to that in Freidlin and Wentzell, 1984, Section 6.5.) Simple estimates show that if $\lim\sup_{\varepsilon\downarrow 0}\varepsilon t(\varepsilon) < W/c$, then the probability that at least one particle is not in G_0 at time $t(\varepsilon)$ tends to zero. If $\lim\inf_{\varepsilon\downarrow 0}\varepsilon t(\varepsilon) > W/c$, then this probability tends to 1, which implies the claim of Theorem 8.

3. One can also study R-D systems by introducing a small parameter in only some of the equations forming the system. Consider the example

$$\frac{\partial u^\varepsilon(t, x)}{\partial t} = \frac{\varepsilon}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon} f_1(u^\varepsilon, v) \quad \frac{\partial v(t, x)}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + f_2(v). \tag{52}$$

Suppose that $f_1(\cdot, v), f_2 \in \mathscr{F}_2$ for every $v > 0$. Denote by $q_2(\xi) = q_2^{f_2}(\xi)$ the profile of the wave with the conditions $q_2(-\infty) = 1$, $q_2(\infty) = 0$ for the second equation, $\alpha_2^* = \alpha^*[f_2]$ being the corresponding propagation velocity. For the sake of definiteness, let $\alpha_2^* > 0$, $q_2(0) = a$. In the first equation, for every $v \in [0, 1]$ the wave velocity $\alpha_1^* = \alpha_1^*(v)$ stabilizes, which will also be assumed to be positive. We shall consider the solution of (52) with initial conditions

$$u^\varepsilon(0, x) = \chi^-(x), \quad v(0, x) = q_2(x). \tag{53}$$

Consider the equation

$$\alpha_1^*(v) = \alpha_2^*.$$

Suppose that v^* is a root of this equation in which the sign of the function

$\alpha_2^* - \alpha_1^*(v)$ changes from plus to minus. Further suppose that $\alpha_2^* - \alpha_1^*(v) > 0$ for $v \in [a, v^*)$ and that $(d\alpha_1^*/dv)(v^*) < \infty$. Then from the results of Section 3 one can conclude that for the solution of problem (52), (53) the following relation is valid:

$$(u^\varepsilon(t, x), v(t, x)) \to (\chi^-(x - s(t)), q_2(x - \alpha_2^* t)), \quad \varepsilon \downarrow 0.$$

Here $s(t)$ together with some function $b(t)$ is the solution of the system of ordinary differential equations

$$db/dt = q_2'(b)(\alpha_2^* - \alpha_1^*(b), \quad ds/dt = \alpha_1^*(b),$$
$$s(0) = 0, \quad b(0) = a.$$

Note that $\lim_{t\to\infty} t^{-1}s(t) = \alpha_2^*$. Therefore, for large t and small ε, the wave velocity tends to α_2^*.

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