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ON LIMITING DISTRIBUTIONS OF ORDER STATISTICS WITH VARIABLE RANKS FROM STATIONARY SEQUENCES

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Let $\{\xi_n\}$ be a stationary sequence and $\xi_1^{(n)} \le \cdots \le \xi_n^{(n)}$ be the order statistics of $\xi_1, \dots, \xi_n$. In this paper the limiting distribution of $\{\xi_{k_n}^{(n)}\}$, where $\{k_n\}$ satisfies $\min(k_n, n - k_n) \to \infty$, is determined under appropriate conditions. Further results for some special $\{k_n\}$ that satisfy $k_n/n \to \lambda \in [0,1]$ are also obtained. These results are applied to discussing the limiting distributions of corresponding order statistics from m-dependent stationary sequences and stationary normal sequences.

1. Introduction. Let $\{\xi_n\}$ be a sequence of random variables and $\xi_1^{(n)} \le \cdots \le \xi_n^{(n)}$ be the order statistics of $\xi_1, \dots, \xi_n$. Then $\{\xi_{k_n}^{(n)}\}$ is called the sequence of order statistics with rank sequence $\{k_n\}$ from the $\{\xi_n\}$. Write $\lambda_n = k_n/n$ and $\Lambda_n = [n\lambda_n(1-\lambda_n)]^{1/2}$. A rank sequence $\{k_n\}$ is said to be variable if $\min(k_n, n - k_n) \to \infty$, or equivalently $\Lambda_n \to \infty$. Two kinds of variable ranks are of special interest: *the intermediate rank sequences* that satisfy $\lambda_n \to 0$ or 1 and *the central rank sequences* that satisfy $\lambda_n \to \lambda, \lambda \in (0,1)$. In the case when $\{\xi_n\}$ is i.i.d. and $\{k_n\}$ is variable, Smirnov (1952) gave a necessary and sufficient condition for the normalized sequence $\{(\xi_{k_n}^{(n)} - b_n)/a_n\}$ to converge weakly to a certain d.f. Based on this condition, the limiting distributions of the sequences of order statistics with the intermediate and central rank sequences from i.i.d. sequences were extensively studied by Smirnov (1952), Cheng (1965), Wu (1966) and others [cf., Leadbetter, Lindgren, and Rootzén (1983)]. Recently, the above works have been extended to stationary sequences. For the intermediate rank sequences, Watts (1977) and Watts, Rootzén, and Leadbetter (1982) have shown that the results of Wu (1966) are still true for the stationary sequences under some condition $A(u_n)$ and especially for the stationary normal sequences under some condition on their correlation sequences. The central rank sequences have been discussed by Cheng (1980), but the mixing condition used there is hard to check even for the stationary normal sequences.

The present paper is devoted to the same objective. We will be concerned with the general variable rank sequences, in order to deal with the intermediate and central rank sequences simultaneously. In Section 2, a central limit theorem is given for stationary indicator arrays. This theorem is applied to discussing the limiting distributions of the sequences of order statistics with variable rank sequences from stationary sequences in Section 3. Further results for some special intermediate and central rank sequences are also given in Section 3. In Section 4, we discuss two special kinds of stationary sequences, the m-dependent stationary

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sequences and the stationary normal sequences. It will be seen that our conditions are easily checked for these special sequences.

2. A central limit theorem for stationary indicator arrays. Let $\{I_{n,i}, i = 1,\dots, m_n, n = 1,2,\dots\}$ be an array of random variables on some probability space $(\Omega, \mathscr{F}, P)$. We call $\{I_{n,i}\}$ a *stationary indicator array* if the following conditions are satisfied: (1) for each n, $\{I_{n,1},\dots, I_{n,m_n}\}$ is stationary; (2) for each n, $P(I_{n,i} = 1) + P(I_{n,i} = 0) = 1$, $i = 1,\dots, m_n$; (3) $m_n \to \infty$ as $n \to \infty$. Denote $p_n = P(I_{n,i} = 1)$, $q_n = 1 - p_n$, $d_n = (m_n p_n q_n)^{1/2}$, and $S_n = \sum_{i=1}^{m_n} I_{n,i}$. In the special case when $I_{n,1},\dots, I_{n,m_n}$ are independent for each n, we have such a central limit theorem: If

$$d_n \to \infty \quad \text{as } n \to \infty, \tag{2.1}$$

then

$$\lim_{n\to\infty} P((S_n - m_n p_n)/d_n \le x) = \Phi(x),$$

where $\Phi(x)$ is the standard normal d.f. In the following, we will extend this theorem to the general stationary indicator arrays.

Denote $p_n^{(i)} = P(I_{n,j} = I_{n,i+j} = 1)$, $1 \le j \le i+j \le m_n$. It is easily seen that if $i = 0$, then $\{I_{n,j} = I_{n,i+j} = 1\} = \{I_{n,j} = 1\}$, and therefore $p_n^{(0)} = p_n$. Let $\mathscr{F}_{n,k}$ be the σ field generated by $I_{n,1},\dots, I_{n,k}$, i.e., $\mathscr{F}_{n,k} = \sigma\{I_{n,1},\dots, I_{n,k}\}$, $k = 1,\dots, m_n$. For all positive integers $s \le m_n$, the *mixing coefficients* are defined as

$$\alpha_n(s) = \sup\Big\{|P\big(\{I_{n,i} = I_{n,j} = 1\} \cap B\big) - p_n^{(j-i)}P(B)|\colon B \in \mathscr{F}_{n,k},$$
$$1 \le k < k+s \le i \le j \le m_n\Big\}.$$

Then we prove the following lemma.

LEMMA 2.1. *Suppose that* (2.1) *hold. If there exist positive integers* $s_n = o(d_n)$ *such that*

$$m_n \alpha_n(s_n)/d_n \to 0, \tag{2.2}$$

$$\delta_n = 1 + 2m_n \sum_{i=1}^{s_n} \big(p_n^{(i)} - p_n^2\big)/d_n^2 \to \sigma^2 < \infty, \tag{2.3}$$

and for some positive integers $\{t_n\}$ *satisfying* $s_n = o(t_n)$ *and* $t_n = o(d_n)$,

$$\gamma_n = m_n \sum_{i=1}^{s_n} i\big(p_n^{(i)} - p_n^2\big)/\big(d_n^2 t_n\big) \to 0, \tag{2.4}$$

then we have

$$P\big((S_n - m_n p_n)/d_n \le x\big) \to_w \Phi(x/\sigma), \tag{2.5}$$

where $\Phi(x/0)$ *is defined to be* 1 *for* $x \ge 0$ *and* 0 *for* $x < 0$.

PROOF. Denote $\hat{t}_n = t_n + s_n$ and $N_n = [m_n/\hat{t}_n]$ where [ ] is the integer part of a real number. Split set $\{1,\dots, m_n\}$ into $2N_n + 1$ parts: $J_{n,k} = \{(k-1)\hat{t}_n + 1,\dots,(k-1)\hat{t}_n + t_n\}$, $\tilde{J}_{n,k} = \{(k-1)\hat{t}_n + t_n + 1,\dots, k\hat{t}_n\}$, $k = 1,\dots, N_n$ and $\bar{J}_n = \{N_n\hat{t}_n + 1,\dots, m_n\}$. Let $y_{n,k} = \sum_{i\in J_{n,k}}(I_{n,i} - p_n)/d_n$, $\tilde{y}_{n,k} = \sum_{i\in \tilde{J}_{n,k}}(I_{n,i} -$

$p_n)/d_n$, $k = 1, \ldots, N_n$ and $\bar{y}_n = \Sigma_{i \in \bar{J}_n}(I_{n,i} - p_n)/d_n$. Then we have

$$(S_n - m_n p_n)/d_n = \sum_{k=1}^{N_n} y_{n,k} + \sum_{k=1}^{N_n} \tilde{y}_{n,k} + \bar{y}_n.$$

Since $|\bar{y}_n| \le 2\hat{t}_n/d_n \to 0$, (2.5) will hold if we show both

$$P\left(\sum_{k=1}^{N_n} y_{n,k} \le x\right) \to {}_w\Phi(x/\sigma) \tag{2.6}$$

and $P(\Sigma_{k=1}^{N_n} \tilde{y}_{n,k} \le x) \to {}_w\Phi(x/0)$. The latter can be shown in the same way as (2.6). Hence we need only to show (2.6).

Let $\mathscr{B}_{n,k} = \mathscr{F}_{n,(k-1)\hat{t}_n + t_n}$ for $1 \le k \le N_n$ and $\mathscr{B}_{n,0} = \{\phi, \Omega\}$. We will consider a martingale difference array $\{X_{n,k}, \mathscr{B}_{n,k},\ k = 1, \ldots, N_n,\ n = 1, 2, \ldots\}$ instead of $\{y_{nk}\}$, where $X_{n,k} = y_{n,k} - E(y_{n,k}|\mathscr{B}_{n,k-1})$. By Lemma 5.2 of Dvoretzky (1972), it follows that

$$\begin{aligned} E\left|\sum_{k=1}^{N_n} y_{n,k} - \sum_{k=1}^{N_n} X_{n,k}\right| &\le \sum_{k=1}^{N_n} E|E(y_{n,k}|\mathscr{B}_{n,k-1})| \\ &\le \sum_{k=1}^{N_n} \sum_{i \in J_{n,k}} E|E(I_{n,i}|\mathscr{B}_{n,k-1}) - EI_{n,i}|/d_n \\ &\le 4m_n\alpha_n(s_n)/d_n \to 0. \end{aligned}$$

This shows that (2.6) is equivalent to

$$P\left(\sum_{k=1}^{N_n} X_{n,k} \le x\right) \to {}_w\Phi(x/\sigma). \tag{2.7}$$

Using Lemma 5.2 of Dvoretzky (1972) again, we have

$$\begin{aligned} &E\left|\sum_{i=1}^{N_n} E(X_{n,k}^2|\mathscr{B}_{n,k-1}) - N_n E y_{n,1}^2\right| \\ &\quad \le \sum_{k=1}^{N_n} \sum_{i,j \in J_{n,k}} \Big[E\big|E(I_{n,i}I_{n,j}|\mathscr{B}_{n,k-1}) - EI_{n,i}I_{n,j}\big| \\ &\qquad\qquad + E\big|E(I_{n,i}|\mathscr{B}_{n,k-1}) - EI_{n,i}\big| \\ &\qquad\qquad + E\big|E(I_{n,j}|\mathscr{B}_{n,k-1}) - EI_{n,j}\big|\Big]/d_n^2 \\ &\quad \le 12 m_n t_n \alpha_n(s_n)/d_n^2 \to 0. \end{aligned}$$

Furthermore, noticing that $|N_n \Sigma_{i=s_n+1}^{t_n}(t_n - i)(p_n^{(i)} - p_n^2)/d_n^2| \le N_n t_n^2 \alpha_n(s_n)/d_n^2 \to 0$, and using conditions (2.4) and (2.5), we obtain

$$\begin{aligned} N_n E y_{n,1}^2 &= N_n t_n \delta_n/m_n - 2N_n t_n \gamma_n/m_n + 2N_n \sum_{i=s_n+1}^{t_n} (t_n - i)(p_n^{(i)} - p_n^2)/d_n^2 \\ &\to \sigma^2. \end{aligned}$$

Hence it has been shown that

$$\sum_{k=1}^{N_n} E\left(X_{n,k}^2 | \mathscr{B}_{n,k-1}\right) \to_p \sigma^2.$$

Since $|X_{n,k}| \le 2t_n/d_n \to 0$, it is also seen that

$$\sum_{k=1}^{N_n} E\left[X_{n,k}^2 I(|X_{n,k}| > \varepsilon) | \mathscr{B}_{n,k-1}\right] \to_p 0,$$

where $I(\cdot)$ denotes the indicator of a set. According to Hall and Heyde [(1980), Corollary 3.1 and its remark], (2.8) follows, so that the lemma is proved. □

Now the central limit theorem we need follows:

THEOREM 2.1. *Suppose that a stationary indicator array satisfies* (2.1), *and*

$$\sup_{n\ge 1} m_n \sum_{i=1}^{[d_n]} |p_n^{(i)} - p_n^2|/d_n^2 < \infty \tag{2.8}$$

and

$$1 + 2m_n \sum_{i=1}^{[d_n]} \left(p_n^{(i)} - p_n^2\right)/d_n^2 \to \sigma^2. \tag{2.9}$$

If there exist positive integers $s_n = o(d_n)$ *such that* (2.2) *holds, then* (2.5) *holds.*

PROOF. If (2.2) holds, then (2.9) is equivalent to (2.3) since

$$\left| m_n \sum_{i=s_n+1}^{[d_n]} \left(p_n^{(i)} - p_n^2\right)/d_n^2 \right| \le m_n \alpha_n(s_n)/d_n \to 0.$$

Moreover, by choosing $t_n = [(s_n d_n)^{1/2}]$, (2.4) follows from the fact that

$$\left| m_n \sum_{i=1}^{s_n} i\left(p_n^{(i)} - p_n^2\right)/\left(d_n^2 t_n\right) \right|$$

$$\le (s_n/t_n) \sup_{n\ge 1} m_n \sum_{i=1}^{[d_n]} |p_n^{(i)} - p_n^2|/d_n^2 \to 0.$$

Hence (2.5) is obtained by using Lemma 2.1. □

At the end of this section, we prove a law of large numbers for stationary indicator arrays, which will also be used in the next section.

THEOREM 2.2. *If there exist positive integers* $s_n = o(m_n)$ *such that* $\alpha_n(s_n) \to 0$, *then*

$$(S_n - m_n p_n)/m_n \to_p 0. \tag{2.10}$$

PROOF. Let $t_n = [(s_n m_n)^{1/2}]$ and denote $\hat{t}_n$, N_n, $J_{n,k}$, $\tilde{J}_{n,k}$, $\bar{J}_n$, and $\mathscr{B}_{n,k}$ as in the proof of Lemma 2.1. Let $Z_{n,k} = \sum_{i \in J_{n,k}} (I_{n,i} - p_n)/m_n$, $\tilde{Z}_{n,k} = \sum_{i \in \tilde{J}_{n,k}} (I_{n,i} - p_n)/m_n$, $k = 1, \dots, N_n$, and $\bar{Z}_n = \sum_{i \in \bar{J}_n} (I_{n,i} - p_n)/m_n$. Then it follows that $|\bar{Z}_n| \le 2\hat{t}_n/m_n \to 0$ and $|\sum_{k=1}^{N_n} \tilde{Z}_{n,k}| \le 2N_n s_n/m_n \to 0$. Noticing that $N_n E Z_{n,1}^2 \le (1 + 4t_n)/m_n \to 0$, by the same way as proved in Lemma 2.1, we can also show that

$$P\left(\sum_{k=1}^{N_n} Z_{n,k} \le x\right) \to_w \Phi(x/0), \quad \text{i.e.,} \quad \sum_{k=1}^{N_n} Z_{n,k} \to_p 0.$$

Hence (2.10) holds. □

3. The results for general stationary sequences. Let $\{\xi_n, n = 1, 2, \dots\}$ be a stationary sequence with one dimensional d.f. $F(x)$ and two dimensional d.f. $F^{(i)}(x, y) = p(\xi_n \le x, \xi_{n+i} \le y)$, $i = 1, 2, \dots$. For convenience, denote $F^{(0)}(x) = F(x)$ and $F^{(i)}(x) = F^{(i)}(x, x)$. Let $\{k_n\}$ be a variable rank sequence and denote $\lambda_n = k_n/n$, $\Lambda_n = [n\lambda_n(1 - \lambda_n)]^{1/2}$ as in Section 1. In the special case when $\{\xi_n\}$ is an i.i.d. sequence, Smirnov (1952) has shown that there exist constants $a_n > 0$, b_n such that

$$p\left(\xi_{k_n}^{(n)} \le a_n x + b_n\right) \to_w \Psi(x), \qquad \text{for some d.f. } \Psi(x), \tag{3.1}$$

if and only if

$$v_n(x) = n\left[F(a_n x + b_n) - \lambda_n\right]/\Lambda_n \to_w v(x) \tag{3.2}$$

for some nondecreasing, right continuous, and extended real function $v(\cdot)$ satisfying $\lim_{x \to -\infty} v(x) = -\infty$, $\lim_{x \to \infty} v(x) = \infty$. The relation between $\Psi(x)$ and $v(x)$ is

$$\Psi(x) = \Phi(v(x)). \tag{3.3}$$

We will extend the above result for the general stationary sequences. Unless otherwise stated, we assume that the limiting d.f. $\Psi(x)$ is not degenerate, i.e., for the function $v(\cdot)$ appearing on the right hand of (3.2), there exists a real number x, such that $-\infty < v(x) < \infty$.

Let $\{u_n\}$ be a real sequence satisfying

$$v_n = n\left[F(u_n) - \lambda_n\right]/\Lambda_n \to v, \tag{3.4}$$

where v may be finite or infinite. Then it is obvious that $\{I_{n,i} = I(\xi_i \le u_n),$ $i = 1, \dots, n,\ n = 1, 2, \dots\}$ is a stationary indicator array with $m_n = n$, $p_n = F(u_n)$, and $p_n^{(i)} = F^{(i)}(u_n)$. We will use all of the notation in Section 2 for such a stationary indicator array. For example, the mixing coefficients are now

$$\alpha_n(s) = \sup\Big\{\left|p\left(\{\xi_i \le u_n, \xi_j \le u_n\} \cap B\right) - F^{(j-i)}(u_n)P(B)\right| : B \in \mathscr{F}_{n,k},$$
$$1 \le k < k + s \le i \le j \le n\Big\}.$$

We need a new notation $A(\lambda)$ defined as follows. It is easily seen that for each $\lambda \in (0, 1)$, there exists a real number $a(\lambda)$ such that $F(a(\lambda) - 0) \le \lambda \le F(a(\lambda))$.

We define $A(\lambda)$ to be the set $\{x: x < a(\lambda)\}$ if $F(a(\lambda)) - \lambda \geq \lambda - F(a(\lambda) - 0)$ and the set $\{x: x \leq a(\lambda)\}$ if $F(a(\lambda)) - \lambda < \lambda - F(a(\lambda) - 0)$.

LEMMA 3.1. *Let $\{u_n\}$ be a real sequence. Suppose that there exist positive integers $s_n = o(\Lambda_n)$ such that*

$$n\alpha_n(s_n)/\Lambda_n \to 0. \tag{3.5}$$

If $d_n/\Lambda_n \to 1$, then (2.8) *and* (2.9) *are, respectively, equivalent to*

$$\sup_{n\geq 1} n \sum_{i=1}^{[\Lambda_n]} \left|F^{(i)}(u_n) - F^2(u_n)\right|/\Lambda_n^2 < \infty \tag{3.6}$$

and

$$1 + 2n \sum_{i=1}^{[\Lambda_n]} \left[F^{(i)}(u_n) - F^2(u_n)\right]/\Lambda_n^2 \to \sigma^2. \tag{3.7}$$

Furthermore, if (3.4) *holds with a finite v, then $p_n/\lambda_n \to 1$, $q_n/(1-\lambda_n) \to 1$, and* (3.6) *and* (3.7) *are, respectively, equivalent to*

$$\sup_{n\geq 1} n \sum_{i=1}^{s_n} \left|p(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) - \lambda_n^2\right|/\Lambda_n^2 < \infty \tag{3.8}$$

and

$$1 + 2n \sum_{i=1}^{s_n} \left[p(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) - \lambda_n^2\right]/\Lambda_n^2 \to \sigma^2. \tag{3.9}$$

PROOF. If $d_n/\Lambda_n \to 1$, then (3.5) and (2.2) are equivalent. Moreover, (2.8) is equivalent to $\sup_{n\geq 1} n\sum_{i=1}^{[d_n]}|p_n^{(i)} - p_n^2|/\Lambda_n^2 < \infty$. The latter is equivalent to $\sup_{n\geq 1} n\sum_{i=1}^{s_n}|p_n^{(i)} - p_n^2|/\Lambda_n^2 < \infty$ under the mixing condition (3.5) or (2.2), and so is (3.6). Hence (2.8) and (3.6) are equivalent. Similarly, (2.9) and (3.7) are equivalent. Noticing that $\Lambda_n \to \infty$ iff $\min(k_n, n-k_n) \to \infty$, we have $k_n/\Lambda_n \geq k_n^{1/2} \to \infty$. Hence if (3.4) holds with a finite v, i.e., $k_n(p_n/\lambda_n - 1)/\Lambda_n = n(p_n - \lambda_n)/\Lambda_n \to v$, then $p_n/\lambda_n \to 1$. Similarly we have $q_n/(1-\lambda_n) \to 1$. From the definition of $A(\lambda)$, it follows that

$$\begin{aligned}
&\big|\,|P(\xi_1 \leq u_n, \xi_{i+1} \leq u_n) - F^2(u_n)| - |P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) - \lambda_n^2|\,\big| \\
&\quad \leq |P(\xi_1 \leq u_n, \xi_{i+1} \leq u_n) - P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n))| + |F^2(u_n) - \lambda_n^2| \\
&\quad \leq 2\left[|F(u_n) - P(\xi_1 \in A(\lambda_n))| + |F(u_n) - \lambda_n|\right] \\
&\quad \leq 4|F(u_n) - \lambda_n| + 2|P(\xi_1 \in A(\lambda_n)) - \lambda_n| \leq 6|F(u_n) - \lambda_n|.
\end{aligned}$$

If (3.4) holds with a finite v, then it follows that

$$\begin{aligned}
&\left| n \sum_{i=1}^{[\Lambda_n]} |F^{(i)}(u_n) - F^2(u_n)|/\Lambda_n^2 - n \sum_{i=1}^{s_n} |P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) - \lambda_n^2|/\Lambda_n^2 \right| \\
&\quad \leq 6ns_n|F(u_n) - \lambda_n|/\Lambda_n^2 + n\alpha_n(s_n)/\Lambda_n \to 0,
\end{aligned}$$

so that (3.6) and (3.8) are equivalent. Similarly, (3.7) and (3.9) are equivalent. The lemma is now proved. □

THEOREM 3.1. *Let* $\{u_n\}$ *be a real sequence. Suppose that* $d_n/\Lambda_n \to 1$ *and* (3.4) *holds with finite or infinite* v. *If there exist positive integers* $s_n = o(\Lambda_n)$ *such that* (3.5), (3.6), *and* (3.7) *hold with* $\sigma^2 > 0$, *then as* $n \to \infty$,

$$P\left(\xi_{k_n}^{(n)} \le u_n\right) \to \Phi(v/\sigma).$$

Furthermore, if v *in* (3.4) *is finite, the above conclusion still holds by replacing* (3.6) *and* (3.7) *by* (3.8) *and* (3.9).

PROOF. By Theorem 2.1 and Lemma 3.1, the conclusion of this theorem follows from the continuity of $\Phi(x/\sigma)$ when $\sigma^2 > 0$ and the fact that

$$\begin{aligned} P\left(\xi_{k_n}^{(n)} \le u_n\right) &= P(S_n \ge k_n) \\ &= 1 - P((S_n - np_n)/\Lambda_n < -v_n) \to 1 - \Phi(-v/\sigma) = \Phi(v/\sigma). \end{aligned}$$

□

Denote $B(v(\cdot)) = \{x\colon\ v(\cdot) \text{ is finite and continuous at } x\}$. We state three corollaries of Theorem 3.2 as follows:

COROLLARY 3.1. *Suppose that there exist constants* $a_n > 0$ *and* b_n *such that* (3.2) *holds for a finite valued function* $v(\cdot)$. *If there exist positive integers* $s_n = o(\Lambda_n)$ *such that* (3.8) *and* (3.9) *hold with* $\sigma^2 > 0$ *and such that* (3.5) *holds for all* $u_n = a_n x + b_n, x \in B(v(\cdot))$, *then* (3.1) *holds with*

$$\Psi(x) = \Phi(v(x)/\sigma). \tag{3.10}$$

PROOF. Use Theorem 3.1. □

COROLLARY 3.2. *Suppose that there exist constants* $a_n > 0$ *and* b_n *such that* (3.2) *holds for a continuous* $v(\cdot)$. *If there exist positive integers* $s_n = o(\Lambda_n)$ *such that* (3.8) *and* (3.9) *hold with* $\sigma^2 > 0$ *and such that* (3.5) *holds for all* $u_n = a_n x + b_n, x \in B(v(\cdot))$, *then* (3.1) *holds with* (3.10).

PROOF. By Corollary 3.1, we need only to show that (3.1) holds if $v(\cdot)$ is continuous and with infinite values at some points. Let $x_0 = \sup\{x\colon\ v(x) < \infty\}$. If $v(x) = \infty$, then $x \ge x_0$. By taking $x_n \in B(v(\cdot)), x_n \uparrow x_0$ and using the continuity of $\Phi(\cdot/\sigma)$ and $v(\cdot)$, it follows that

$$\begin{aligned} \liminf_{n\to\infty} P\left(\xi_{k_n}^{(n)} \le a_n x + b_n\right) &\ge \liminf_{n\to\infty} P\left(\xi_{k_n}^{(n)} \le a_n x_0 + b_n\right) \\ &\ge \lim_{n\to\infty} P\left(\xi_{k_n}^{(n)} \le a_n x_n + b_n\right) = \lim_{n\to\infty} \Phi(v(x_n)/\sigma) = 1. \end{aligned}$$

This shows that $\lim_{n\to\infty} P(\xi_{k_n}^{(n)} \le a_n x + b_n) = \Phi(v(x)/\sigma)$ still holds for x satisfying $v(x) = \infty$. Similarly, we can show the above equation for x satisfying $v(x) = -\infty$, to complete the proof of Corollary 3.2. □

COROLLARY 3.3. *Let* $\{k_n\}$ *be a rank sequence satisfying* $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1$. *Suppose that there exist constants* $a_n > 0$ *and* b_n *such that* (3.2) *holds. If there exist positive integers* $s_n = o(n^{1/2})$ *such that* (3.8) *and* (3.9) *hold with* $\sigma^2 > 0$, *and such that* $n^{1/2}\alpha_n(s_n) \to 0$ *for all* $u_n = a_n x + b_n$, *then* (3.1) *holds with* (3.10).

PROOF. It is easily seen that for a rank sequence satisfying $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1$, $\Lambda_n = O(n^{1/2})$ and therefore that (3.5) is equivalent to $n^{1/2}\alpha_n(s_n) \to 0$. By Theorem 3.1, it is sufficient to show that $\lim_{n\to\infty}P(\xi_{k_n}^{(n)} \le a_n x + b_n) = 1$ for x satisfying $v(x) = \infty$ and $\lim_{n\to\infty}P(\xi_{k_n}^{(n)} \le a_n x + b_n) = 0$ for x satisfying $v(x) = -\infty$. If there exists a x_0 satisfying $v(x_0) = \infty$ such that $\lim_{n\to\infty}P(\xi_{k_n}^{(n)} \le a_n x_0 + b_n) \ne 1$, then we could choose a subsequence such that $\lim_{n'\to\infty}P(\xi_{k_{n'}}^{(n')} \le a_{n'}x_0 + b_{n'}) < 1$. Taking $x_1 \in B(v(\cdot))$, we have

$$0 < \lim_{n'\to\infty} P\left(\xi_{k_{n'}}^{(n')} \le a_{n'}x_1 + b_{n'}\right) \le \lim_{n'\to\infty} P\left(\xi_{k_{n'}}^{(n')} \le a_{n'}x_0 + b_{n'}\right) < 1,$$

and therefore for $u_{n'} = a_{n'}x_0 + b_{n'}$,

$$0 < \lim_{n'\to\infty} P\left((S_{n'} - n'F(u_{n'}))/n' \ge \lambda_{n'} - F(u_{n'})\right) < 1.$$

Hence from Theorem 2.2, it follows that $\lambda_{n'} - F(u_{n'}) \to 0$. This implies $F(u_{n'})/\lambda_{n'} \to 1$ and $[1 - F(u_{n'})]/(1 - \lambda_{n'}) \to 1$ since $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 1$. By Theorem 2.1, we would obtain

$$1 = \Phi(v(x_0)/\sigma) = \lim_{n'\to\infty} \Phi(v_{n'}(x_0)/\sigma) = \lim_{n'\to\infty} P\left(\xi_{k_{n'}}^{(n')} \le a_{n'}x_0 + b_{n'}\right) < 1.$$

Thus it is impossible that $\lim_{n\to\infty}P(\xi_{k_n}^{(n)} \le a_n x_0 + b_n) \ne 1$ for an x_0 satisfying $v(x_0) = \infty$. A similar discussion can be done for the case when $v(x) = -\infty$, to complete the proof of this corollary. □

Now let us discuss the cases with the intermediate rank sequences. In these cases, $\lambda_n \to 0$ or 1. We will assume that $\lambda_n \to 0$ since the case $\lambda_n \to 1$ can be easily transformed to the former. It is easy to see that in all of the equations (3.2), (3.5)–(3.9), Λ_n may be replaced by $k_n^{1/2}$ under the assumption $\lambda_n \to 0$. The following theorem is the extension of Wu's result.

THEOREM 3.2. *Let* $\{k_n\}$ *be an intermediate rank sequence such that* $k_n \uparrow \infty$ *and* $\lambda_n \to 0$. *Suppose that there exist constants* $a_n > 0$, b_n *such that* (3.2) *holds. If there exist positive integers* $s_n = o(k_n^{1/2})$ *such that*

$$\sup_{n\ge 1} n \sum_{i=1}^{s_n} |P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) - \lambda_n^2|/k_n < \infty, \tag{3.8'}$$

$$1 + 2n \sum_{i=1}^{s_n} \left[P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) - \lambda_n^2\right]/k_n \to \sigma^2 > 0, \tag{3.9'}$$

and such that

$$n\alpha_n(s_n)/k_n^{1/2} \to 0 \tag{3.5'}$$

for all $u_n = a_n x + b_n$, $x \in B(v(\cdot))$, *then* (3.1) *holds with* (3.10).

PROOF. It has been proved by Wu (1966) that the only possibilities for $v(\cdot)$ that is defined by (3.2) are

$$v_1(x) = \begin{cases} -\alpha \log|x|, & x < 0, \\ \infty, & x \geq 0, \alpha > 0, \end{cases}$$

$$v_2(x) = \begin{cases} -\infty, & x \leq 0 \\ \alpha \log x, & x > 0, \alpha > 0, \end{cases}$$

$$v_3(x) = x,$$

and other functions which can be obtained by replacing x by $ax + b$ ($a > 0$ and b are constants) in the above functions $v_i(x)$, $i = 1, 2, 3$. All of these functions are continuous. Hence this theorem follows from Corollary 3.2. □

The above theorem shows that, under the conditions of the theorem, the only possible types of nondegenerate limiting laws of $\{\xi_{k_n}^{(n)}\}$ are the same as in the i.i.d. case, i.e., $\Phi(v_i(x))$, $i = 1, 2, 3$, where $v_i(\cdot)$ are indicated in the proof of theorem 3.2. It is also shown by the theorem that the limiting distributions of normalized sequences $\{(\xi_{k_n}^{(n)} - b_n)/a_n\}$ and $\{(\hat{\xi}_{k_n}^{(n)} - b_n)/a_n\}$ may be different, where $\{\hat{\xi}_{k_n}^{(n)}\}$ is the corresponding order statistics from the associated independent sequence $\{\hat{\xi}_n\}$. Anyway, the two limiting distributions will be the same if $\sigma^2 = 1$, especially if (3.8) and (3.9) are replaced by one equation:

$$\lim_{n \to \infty} n \sum_{i=1}^{[k_n^{1/2}]} |F^{(i)}(u_n) - F^2(u_n)|/k_n = 0.$$

[See Watts, Rootzén and Leadbetter (1982).]

Now we discuss the cases with the central rank sequences. The general result follows from Corollary 3.3. Here we discuss only the case when the ranks $\{k_n\}$ satisfy

(3.11) $$n^{1/2}(\lambda_n - \lambda) \to t \quad \text{for some} \quad \lambda \in (0,1) \quad \text{and} \quad t \in R.$$

If $\{\xi_n\}$ is i.i.d., Smirnov (1952) has shown that the only possible nondegenerate types of limiting laws of $\{\xi_{k_n}^{(n)}\}$ are $\Phi(w_i(x) - t/[\lambda(1-\lambda)]^{1/2})$, $i = 1, 2, 3, 4$, where

$$w_1(x) = \begin{cases} -\infty, & x < 0, \\ cx^\alpha, & x \geq 0, c > 0, \alpha > 0, \end{cases}$$

$$w_2(x) = \begin{cases} -c|x|^\alpha, & x < 0, \\ \infty, & x \geq 0, c > 0, \alpha > 0, \end{cases}$$

$$w_3(x) = \begin{cases} -c_1|x|^\alpha, & x < 0, \\ c_2 x^\alpha, & x \geq 0, c_1, c_2 > 0, \alpha > 0, \end{cases}$$

$$w_4(x) = \begin{cases} -\infty, & x < -1, \\ 0, & -1 \leq x < 1, \\ \infty, & x \geq 1. \end{cases}$$

We will extend this result for stationary sequences.

It is easily seen that if the ranks $\{k_n\}$ satisfy (3.11), then (3.2) and (3.4) are, respectively, equivalent to

$$w_n(x) = n^{1/2}[F(a_n x + b_n) - \lambda]/[\lambda(1-\lambda)]^{1/2} \to w(x) \tag{3.2''}$$

and

$$w_n = n^{1/2}[F(u_n) - \lambda]/[\lambda(1-\lambda)]^{1/2} \to w, \tag{3.4''}$$

and our mixing condition may be restated as follows: There exist positive integers $s_n = o(n^{1/2})$ such that

$$n^{1/2}\alpha_n(s_n) \to 0. \tag{3.5''}$$

LEMMA 3.2. *Let $\{k_n\}$ be a rank sequence satisfying* (3.11). *Suppose that* (3.4″) *hold with a finite w. If there exist positive integers $s_n = o(n^{1/2})$ such that* (3.5″) *holds, then* (3.8) *and* (3.9) *hold iff*

$$\sum_{n=1}^{\infty} |P(\xi_1 \in A(\lambda), \xi_{n+1} \in A(\lambda)) - \lambda^2| < \infty, \tag{3.12}$$

and it can be found that

$$\sigma^2 = 1 + 2\sum_{n=1}^{\infty} [P(\xi_1 \in A(\lambda), \xi_{n+1} \in A(\lambda)) - \lambda^2]/[\lambda(1-\lambda)]. \tag{3.13}$$

PROOF. It has been seen, by Lemma 3.1, that (3.8) is equivalent to

$$\sup_{n\ge 1} n \sum_{i=1}^{t_n} |F^{(i)}(u_n) - F^2(u_n)|/k_n < \infty,$$

where $s_n \le t_n \le n^{1/2}$. Let $t_n = [s_n^{1/2} n^{1/4}]$. Noting that $k_n/n \to \lambda \in (0,1)$ and $t_n \to \infty$, to show this lemma we need only to prove

$$\lim_{n\to\infty} \sum_{i=1}^{t_n} |F^{(i)}(u_n) - F^2(u_n)| = \lim_{n\to\infty} \sum_{i=1}^{t_n} |P(\xi_1 \in A(\lambda), \xi_{i+1} \in A(\lambda)) - \lambda^2|.$$

This follows from the fact that

$$\begin{aligned} &\left| \sum_{i=1}^{t_n} |F^{(i)}(u_n) - F^2(u_n)| - \sum_{i=1}^{t_n} |P(\xi_1 \in A(\lambda), \xi_{i+1} \in A(\lambda)) - \lambda^2| \right| \\ &\quad \le \sum_{i=1}^{t_n} \left[|F^{(i)}(u_n) - P(\xi_1 \in A(\lambda), \xi_{i+1} \in A(\lambda))| + |F^2(u_n) - \lambda^2| \right] \\ &\quad \le 6t_n |F(u_n) - \lambda| \to 0, \end{aligned}$$

so that (3.8) and (3.12) are equivalent. If (3.12) holds, by the same way as above, we can show (3.9) with (3.13). □

THEOREM 3.3. *Let $\{\xi_n\}$ be a stationary sequence and $\{k_n\}$ satisfy* (3.11). *Suppose that* (3.12) *holds and $\sigma^2 > 0$ in* (3.13). *If there exist constants $a_n > 0$*

and b_n *such that* (3.2″) *holds and there exist positive integers* $s_n = o(n^{1/2})$ *such that* (3.5″) *holds for all* $u_n = a_n x + b_n$, *then we have* (3.1) *with* (3.10).

PROOF. Use Corollary 3.3 and Lemma 3.2. □

4. Two examples: The *m*-dependent case and the normal case.. In this section, we apply our general results to two special cases: the m-dependent stationary sequences and the stationary normal sequences. For these special cases, one could obtain simpler conditions under which (3.1) holds. And sometimes the variance σ^2 appearing in (3.10) can easily be determined.

It is easily seen that if $\{\xi_n\}$ is a m-dependent stationary sequence, (3.5′) and (3.8′) hold by choosing $s_n = m$. Furthermore, for a intermediate rank sequence $\{k_n\}$, (3.9′) will hold if

$$(4.1)\qquad P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n))/\lambda_n \to \sigma_i, \qquad i = 1, \dots, m.$$

At the same time, the variance σ^2 is determined by

$$(4.2)\qquad \sigma^2 = 1 + 2\sum_{i=1}^{m} \sigma_i,$$

which is always positive.

THEOREM 4.1. *Let* $\{\xi_n\}$ *be a m-dependent stationary sequence and* $\{k_n\}$ *be an intermediate rank sequence satisfying* $k_n \uparrow \infty$. *Suppose that there exist constants* $a_n > 0$ *and* b_n *such that* (3.2) *holds. If*

$$(4.3)\qquad \lim_{x \to \omega(F)} F^{(i)}(x)/F(x) = \sigma_i, \qquad i = 1, \dots, m,$$

then (3.1) *holds with* (3.10) *and* (4.2), *where* $\omega(F) = \inf\{x\colon F(x) > 0\}$ *and other notation is the same as in Section* 3.

PROOF. Noticing the remarks before this theorem and using Theorem 3.2, we need only to show that (4.3) implies (4.1). Choose $x \in B(v(\cdot))$ [such a number x must exist since $v(\cdot)$ is nondegenerate as stated in Section 3] and let $u_n = a_n x + b_n$. Then, from (3.2), it follows that $F(u_n)/\lambda_n \to 1$ and that $F^{(i)}(u_n)/P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n)) \to 1$. Note that both $\lambda_n \to 0$ and $F(u_n)/\lambda_n \to 1$ imply $u_n \to \omega(F)$ as $n \to \infty$. Hence we obtain that

$$\lim_{n\to\infty} P(\xi_1 \in A(\lambda_n), \xi_{i+1} \in A(\lambda_n))/\lambda_n = \lim_{n\to\infty} F^{(i)}(u_n)/F(u_n) = \lim_{x\to\omega(F)} F^{(i)}(x)/F(x) = \sigma_i,$$

as desired. □

The above theorem shows us that if the limiting distribution of $\{(\hat{\xi}_{k_n}^{(n)} - b_n)/a_n\}$ exists, then the limiting distribution of $\{(\xi_{k_n}^{(n)} - b_n)/a_n\}$ will exist under only one condition (4.3), where, as in Section 3, $\{\hat{\xi}_{k_n}^{(n)}\}$ is the corresponding order statistics from the associated i.i.d. sequence $\{\hat{\xi}_n\}$. The similar conclusion holds for central rank sequence satisfying (3.11). We state it as follows:

THEOREM 4.2. *Let* $\{\xi_n\}$ *be a m-dependent stationary sequence and* $\{k_n\}$ *satisfy* (3.11). *Suppose that there exist constants* $a_n > 0$ *and* b_n *such that* (3.2″) *holds. Then* (3.1) *holds with*

$$\sigma^2 = 1 + 2\sum_{i=1}^{m} \left[P(\xi_1 \in A(\lambda), \xi_{i+1} \in A(\lambda)) - \lambda^2\right]/[\lambda(1-\lambda)] \tag{4.4}$$

if σ^2 *in* (4.4) *is positive. Furthermore, if* $F(x)$ *is continuous,* (4.4) *can be replaced by*

$$\sigma^2 = 1 + 2\sum_{i=1}^{m} \left[F^{(i)}(a(\lambda)) - \lambda^2\right]/[\lambda(1-\lambda)],$$

where $a(\lambda)$ *is a solution of the equation* $F(x) = \lambda$.

PROOF. This follows from Theorem 3.3 and the definition of set $A(\lambda)$. □

Now we turn our attention to discussing the stationary normal sequence. The case with intermediate rank sequence has been discussed by Watts, Rootzén, and Leadbetter (1982). Here we consider only the case with central rank sequence.

Let $\{\xi_n\}$ be a stationary normal sequence with $E\xi_n = 0$, $E\xi_n^2 = 1$, and $E\xi_1\xi_{n+1} = r_n$, $n = 1, 2, \dots$. The one dimensional d.f. of $\{\xi_n\}$ is now the standard normal d.f. $\Phi(x)$. For any $\lambda \in (0,1)$, we will denote the solution of the equation $\Phi(x) = \lambda$ by $a(\lambda)$. Let $\{k_n\}$ be a rank sequence satisfying $\lambda_n \to \lambda \in (0,1)$ and denote $b_n = a(\lambda_n)$ and $a_n = k_n^{1/2}(1-\lambda)^{1/2}/[n\phi(b_n)]$ where $\phi(x)$ is the density function of $\Phi(x)$. According to Theorem 3.5 of Cheng (1965), we know (3.2) holds with the following form:

$$n^{1/2}[\Phi(a_n x + b_n) - \lambda_n]/[\lambda(1-\lambda)]^{1/2} \to x, \qquad x \in R. \tag{4.5}$$

LEMMA 4.1. *If for some positive integers* $\{s_n\}$,

$$n^{1/2}\sum_{i=s_n}^{n} |r_i| \to 0 \quad \text{as } n \to \infty, \tag{4.6}$$

then (3.5″) *holds for any* $u_n = a_n x + b_n$, $x \in R$.

PROOF. By a fact quoted from Watts, Rootzén, and Leadbetter (1982), we have

$$\left|P\left(\{\xi_i \le u, \xi_j \le u\} \cap B\right) - \Phi^{(j-i)}(u)P(B)\right| \le K\sum_{t=s}^{n} |r_t| \exp\left[-u^2/(1+|r_t|)\right]$$

for any real number u, set $B \in \sigma(\{\xi_i \le u\}, i = 1, \dots, k)$, integers $1 \le k < k + s \le i \le j \le n$, and some constant K. Then it follows that

$$\begin{aligned} n^{1/2}\alpha_n(s_n) &\le Kn^{1/2}\sum_{i=s_n}^{n} |r_i| \exp\left[-u_n^2/(1+|r_i|)\right] \\ &\le K\exp\left(-u_n^2/2\right) \cdot n^{1/2}\sum_{i=s_n}^{n} |r_i|. \end{aligned}$$

For any $u_n = a_n x + b_n$, $-\infty < x < \infty$, we have $\Phi(u_n) \to \lambda$ and therefore $u_n \to a(\lambda)$ which implies $\exp(-u_n^2/2) \to \exp[-a^2(\lambda)/2]$. Hence (4.6) follows, to complete the proof of this lemma. □

LEMMA 4.2. *Suppose that* $\sum_{n=1}^{\infty}|r_n| < \infty$. *Then*

$$\sup_{n\geq 1} \sum_{i=1}^{s_n} \left|\Phi^{(i)}(b_n) - \lambda_n^2\right| < \infty \tag{4.7}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{s_n} \left[\Phi^{(i)}(b_n) - \lambda_n^2\right] \\ &\quad = (2\pi)^{-1} \sum_{n=0}^{\infty} \int_0^{r_n} \exp[-a^2(\lambda)/(1+r)]/(1-r^2)^{1/2}\, dr \end{aligned} \tag{4.8}$$

for any positive integers s_n *satisfying* $s_n \to \infty$.

PROOF. Let $\phi_r(x_1, x_2)$ be the joint density function of the normal random vector (η_1, η_2) with $E\eta_1 = E\eta_2 = 0$, $E\eta_1^2 = E\eta_2^2 = 1$, and $E\eta_1\eta_2 = r$. Then it is well known that $\partial/(\partial r)\phi_r(x_1, x_2) = \partial^2/(\partial x_1\, \partial x_2)\phi_r(x_1, x_2)$ [see Cramér and Leadbetter (1967), p. 26]. Hence it is easily seen that

$$\begin{aligned} \Phi^{(i)}(b) - \Phi^2(b) &= \int_0^{r_i} \phi_r(b, b)\, dr \\ &= (2\pi)^{-1} \int_0^{r_i} \exp[-b^2/(1+r)]/(1-r^2)^{1/2}\, dr \end{aligned} \tag{4.9}$$

for any real number b. Using (4.9) and noticing that $\Phi(b_n) = \lambda_n$, we have, for any $n \geq 1$,

$$\begin{aligned} &\sum_{i=1}^{s_n} \left|\Phi^{(i)}(b_n) - \lambda_n^2\right| \\ &\quad \leq (2\pi)^{-1} \sum_{i=1}^{s_n} \int_0^{|r_i|} (1-r^2)^{-1/2}\, dr \\ &\quad \leq (2\pi)^{-1} \sum_{i=1}^{\infty} \arcsin |r_i|. \end{aligned}$$

This shows (4.7) since $\sum_{i=1}^{\infty} \arcsin|r_i| < \infty$ iff $\sum_{n=1}^{\infty}|r_n| < \infty$. Using (4.9) again, we have

$$\begin{aligned} &\left| \sum_{i=1}^{s_n} \left[\Phi^{(i)}(b_n) - \lambda_n^2\right] - (2\pi)^{-1} \sum_{i=1}^{s_n} \int_0^{r_i} \exp[-a^2(\lambda)/(1+r)]/(1-r^2)^{1/2}\, dr \right| \\ &\quad \leq K\left|b_n^2 - a^2(\lambda)\right| \sum_{i=1}^{\infty} \arcsin|r_i| \to 0, \end{aligned}$$

where K is a constant. This proves (4.8). □

THEOREM 4.3. *Let* $\{\xi_n\}$ *be a stationary normal sequence with* $E\xi_n = 0$, $E\xi_n^2 = 1$, *and* $E\xi_1\xi_{n+1} = r_n$, $n = 1, 2, \ldots,$ *and* $\{k_n\}$ *be a rank sequence satisfying* $\lambda_n \to \lambda$, $0 < \lambda < 1$. *If*

$$\sum_{n=1}^{\infty} n^p |r_n| < \infty \tag{4.10}$$

for some $p > 1$ *and*

$$\sum_{n=1}^{\infty} \arcsin r_n > -\pi\lambda(1-\lambda)\exp a^2(\lambda), \tag{4.11}$$

then there exist constants $a_n > 0$ *and* b_n *such that*

$$\lim_{n\to\infty} P\left(\xi_{k_n}^{(n)} \le a_n x + b_n\right) = \Phi(x/\sigma), \qquad x \in R,$$

where

$$\sigma^2 = 1 + [\pi\lambda(1-\lambda)]^{-1} \sum_{n=1}^{\infty} \int_0^{r_n} \exp[-a^2(\lambda)/(1+r)]/(1-r^2)^{1/2}\,dr \tag{4.12}$$

is positive.

PROOF. It is easily seen that (4.10) implies $\sum_{n=1}^{\infty}|r_n| < \infty$. Hence the conclusions of Lemma 4.2 hold. Furthermore, letting $s_n = [n^{1/(2p)}]$, we have $s_n = o(n^{1/2})$ and

$$n^{1/2} \sum_{i=s_n}^{n} |r_i| \le 2 s_n^p \sum_{i=s_n}^{n} |r_i| \le 2 \sum_{i=s_n}^{n} i|r_i|.$$

This shows that (4.10) implies (4.6). Hence the conclusion of Lemma 4.1 holds. By Corollary 3.3, to prove this theorem, it is sufficient to show that (4.11) implies $\sigma^2 > 0$. Notice that for any $\rho \in (-1, 1)$,

$$\int_0^{\rho} \exp[-a^2(\lambda)/(1+r)]/(1-r^2)^{1/2}\,dr \ge \exp[-a^2(\lambda)]\arcsin\rho.$$

Then we have

$$\sigma^2 \ge 1 + \left[\pi\lambda(1-\lambda)\exp a^2(\lambda)\right]^{-1} \sum_{n=1}^{\infty} \arcsin r_n > 0$$

if (4.11) holds. □

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