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# ON THE CENTRAL LIMIT QUESTION UNDER ABSOLUTE REGULARITY[1]

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Davydov showed that in a classic central limit theorem of Ibragimov under the strong mixing condition, the combinations of mixing rate and moment assumption were practically as weak as permissible. Here some constructions will be given to show that Davydov's observation still holds under the additional assumption that the growth of $\operatorname{var}(X_1 + \cdots + X_n)$ be asymptotically linear. Our constructions will be similar to a recent one of Herrndorf.

**1. Introduction.** Suppose $X := (X_k,\ k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $-\infty \le J \le L \le \infty$ let $\mathscr{F}_J^L$ denote the $\sigma$-field of events generated by $(X_k,\ J \le k \le L)$. For any two $\sigma$-fields $\mathscr{A}$ and $\mathscr{B}$, define

$$\alpha(\mathscr{A}, \mathscr{B}) := \sup_{A \in \mathscr{A},\, B \in \mathscr{B}} |P(A \cap B) - P(A)P(B)|.$$

For each $n = 1, 2, \ldots$ define $\alpha(n) := \alpha(\mathscr{F}_{-\infty}^{0}, \mathscr{F}_n^{\infty})$. The sequence $X$ is said to be "strongly mixing" [Rosenblatt (1956)] if $\alpha(n) \to 0$ as $n \to \infty$. For each $n = 1, 2, \ldots$ define the partial sum $S_n := X_1 + \cdots + X_n$. The following statement combines two classic theorems of Ibragimov (1962) [see Theorems 18.5.3 and 18.5.4 of Ibragimov and Linnik (1971)]:

THEOREM 0 (IBRAGIMOV). *Suppose $X := (X_k)$ is strictly stationary, $EX_0 = 0$, and at least one of the following two conditions holds*:

(i) *for some $\delta > 0$, $E|X_0|^{2+\delta} < \infty$ and $\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty$; or*

(ii) *for some $C < \infty$, $|X_0| < C$ a.s. and $\sum_{n=1}^{\infty} \alpha(n) < \infty$.*

*Then $\sigma^2 = EX_0^2 + 2\sum_{k=1}^{\infty} E(X_0 X_k)$ exists, the sum being absolutely convergent. If in addition $\sigma^2 > 0$, then $S_n/(n^{1/2}\sigma) \to N(0,1)$ in distribution as $n \to \infty$.*

Davydov (1969, 1973) constructed some (nontrivial) counterexamples to the central limit theorem under strong mixing. He made the observation that in Theorem 0 the assumptions are very nearly as weak as they can be permitted to be. [See Examples 1 and 2 of Davydov (1973).]

Davydov (1969, 1973) also showed that even under conditions not much weaker than in Theorem 0, $n^{-1}\operatorname{var} S_n$ does not need to approach a finite limit or even be "slowly varying." In his examples the rate of growth of $\operatorname{var} S_n$ was at least $n^d$ for some $d$, $1 < d < 2$. In a counterexample to the clt constructed in

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Bradley (1983) which had no (finite) moments of higher than second order, the mixing rate was arbitrarily fast and the growth of $\operatorname{var} S_n$ almost quadratic, in the sense that $n^{-2}\operatorname{var} S_n \to 0$ arbitrarily slowly.

Herrndorf (1983) also constructed a counterexample to the clt which has an arbitrarily fast mixing rate; his example satisfies the condition

$$(1.1) \qquad \lim_{n\to\infty} n^{-1}\operatorname{var} S_n = \sigma^2 \quad \text{for some } \sigma^2, 0 < \sigma^2 < \infty.$$

The main purpose of the present note is to use a modified version of Herrndorf's construction in order to show, for $0 < \delta \le \infty$, that even under the additional assumption of (1.1) Davydov's observation still holds: The assumptions in Theorem 0 are very nearly as weak as permissible. This will be done in Theorems 1 and 2.

The following proposition was alluded to by Gordin (1969), stated in Ibragimov and Linnik [(1971), p. 420], and given again in Hall and Heyde [(1980), p. 139, Corollary 5.3(ii)]: Suppose $0 \le \delta < \infty$, $X := (X_k)$ is strictly stationary, $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, $\sum_{n=1}^{\infty}\alpha(n)^{(1+\delta)/(2+\delta)} < \infty$, and (1.1) holds; then $S_n/(n^{1/2}\sigma) \to N(0,1)$ in distribution as $n \to \infty$. This proposition is false, for each $\delta$, $0 \le \delta < \infty$; Herrndorf's (1983) construction is a counterexample for the case $\delta = 0$, and Theorem 1 below describes a similar counterexample for the case $0 < \delta < \infty$. This proposition was derived in Hall and Heyde (1980) as a corollary of Theorem 5.4 (on p. 136 there); but Herrndorf (1983) pointed out a fatal error in that derivation. Corollary 5.3(i) of Hall and Heyde (1980) is correct and is an easy consequence of Ibragimov [(1975), Theorem 2.2] and Ibragimov and Linnik [(1971), Theorem 17.2.3]. Also Theorem 5.4 of Hall and Heyde (1980), which was announced by Gordin (1973) and is a clt for some stationary ergodic sequences without the assumption of finite second moments, is correct. An error in its proof in Hall and Heyde (1980) was pointed out by Herrndorf (1983); however, Esseen and Janson [(1985), Theorem 1], corrected that error.

As mentioned, our constructions will be modified versions of Herrndorf's (1983). A key feature of Herrndorf's construction is the use of moving averages of i.i.d. r.v.s in which the coefficients add up to zero. We shall replace the i.i.d. r.v.s there by dependent ones, the dependence being such that in any of those moving averages there can be at most one nonzero term. This change (which is somewhat hidden in our constructions but will become clear in the proof of Lemma 2.4(v) below) will make it easy to get adequate bounds on the $\mathscr{L}_p$ norms, and especially the $\mathscr{L}_\infty$ norms, of r.v.s when necessary. The only other major change will be the obvious one of choosing different parameters to achieve different moment properties and mixing rates.

Here are some properties of Herrndorf's (1983) construction:

$$(1.2) \qquad 0 < \operatorname{var} X_0 < \infty \quad \text{and} \quad \operatorname{corr}(X_0, X_n) = 0, \qquad \forall\, n \ge 1,$$

$$(1.3) \qquad \inf_{n\ge 1} P(S_n = 0) > 0,$$

$$(1.4) \qquad \lim_{C\to\infty}\left[\sup_{n\ge 1} P(|S_n| > C)\right] = 0.$$

Equation (1.4) simply says that the family of distributions of $S_n$, $n = 1, 2, \ldots$ is tight. Equation (1.3) shows that $S_n$ cannot be asymptotically normal under any kind of normalization. Equation (1.2) implies (1.1). In our examples, properties (1.3) and (1.4) [and sometimes (1.2)] will occur in the same way as in Herrndorf's example.

A stronger mixing condition known as "absolute regularity" [Volkonskii and Rozanov (1959)] has recently been used frequently in limit theory; see, e.g., Yoshihara (1978), Berbee (1979), or Dehling and Philipp (1982). Thus in our examples it seems worthwhile to discuss absolute regularity, for as it turns out that will be just as easy as to discuss strong mixing. Herrndorf's (1983) construction also satisfies absolute regularity, with essentially the same (arbitrarily fast) mixing rate as for strong mixing. In Examples 1 and 2 of Davydov (1973) (his very sharp counterexamples) the mixing rate for absolute regularity is essentially the same as for strong mixing (see p. 328 there).

For any two σ-fields $\mathscr{A}$ and $\mathscr{B}$ define the measure of dependence

$$\beta(\mathscr{A}, \mathscr{B}) := \sup \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} \left| P(A_i \cap B_j) - P(A_i)P(B_j) \right|,$$

where this sup is taken over all pairs of partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of $\Omega$ such that $A_i \in \mathscr{A}$ $\forall$ $i$ and $B_j \in \mathscr{B}$ $\forall$ $j$. For a given strictly stationary sequence $X$ define for each $n = 1, 2, \ldots$ $\beta(n) := \beta(\mathscr{F}_{-\infty}^{0}, \mathscr{F}_{n}^{\infty})$. $X$ is said to satisfy "absolute regularity" if $\beta(n) \to 0$ as $n \to \infty$. Clearly $\alpha(m) \le \beta(m)$ $\forall$ $m$.

In what follows, $\ll$ means $O(\cdot)$.

THEOREM 1. *Suppose $\delta > 0$. Then there exists a strictly stationary sequence $X := (X_k)$ such that $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, $\beta(m) \ll ((\log m)^3/m)^{(2+\delta)/\delta}$ as $m \to \infty$,* (1.2) *holds, and* (1.3) *and* (1.4) *also hold.*

THEOREM 2. *There exists a strictly stationary sequence $X := (X_k)$ such that $EX_0 = 0$, $|X_0| < C$ a.s. for some $C < \infty$, $\beta(m) \ll (\log m)^3/m$ as $m \to \infty$,* (1.2) *holds, and* (1.3) *and* (1.4) *also hold.*

It will become clear from the proofs of these two theorems that in the mixing rate for $\beta(m)$ in both theorems the factor $(\log m)^3$ can be replaced by $(\log m)^{2+\varepsilon}$ where $\varepsilon > 0$ is fixed arbitrarily small. But this still leaves a slight gap, essentially a certain power of $(\log m)$, between the mixing rates in Theorems 1 and 2 and those in Theorem 0. There is no obvious way of narrowing this gap with just the strong mixing coefficients $\alpha(m)$. Similarly there remains a tiny gap between Davydov's (1973) constructions and Theorem 0.

Theorems 1 and 2 will be proved from a construction given in Section 3, after some preliminary work in Section 2. Other information about the central limit question, under different combinations of mixing rates and moment assumptions, can be gained without much extra work from the same construction by varying certain parameters. Here we shall give five additional results of this nature; their proofs will also be given in Section 3.

In some limit theorems under mixing conditions, it is assumed that $m^{-1}\operatorname{var} S_m$ is slowly varying as $m \to \infty$; see, e.g., Theorem 18.4.1 in Ibragimov and Linnik (1971), the main result of Herrndorf (1984), or Theorem 2.3 of Peligrad (1984). In the context of Theorem 0, one might hope that in the case where $\sigma^2 = 0$, if $m^{-1}\operatorname{var} S_m$ is slowly varying (while approaching 0) as $m \to \infty$, the clt might hold. But Theorems 3 and 4 show that this is not always the case. Here we shall use the notation $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$.

THEOREM 3. *Suppose $\delta > 0$. Then there exists a strictly stationary sequence $X := (X_k)$ such that $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, $\sum_{m=1}^{\infty}\beta(m)^{\delta/(2+\delta)} < \infty$, $\operatorname{var} S_m \sim (\log m)^{-4} m$ as $m \to \infty$, and* (1.3) *and* (1.4) *both hold.*

THEOREM 4. *There exists a strictly stationary sequence $X := (X_k)$ such that $EX_0 = 0$, $|X_0| < C$ a.s. for some $C < \infty$, $\sum_{m=1}^{\infty}\beta(m) < \infty$, $\operatorname{var} S_m \sim (\log m)^{-4} m$ as $m \to \infty$, and* (1.3) *and* (1.4) *both hold.*

The next theorem was motivated by some questions raised by M. Peligrad concerning possible very slow rates of growth of $\operatorname{var} S_n$ under strong mixing.

THEOREM 5. *Suppose $L_1, L_2, L_3, \dots$ is a sequence of positive integers such that for each $n \geq 2$, $L_n \geq nL_{n-1}$. Then there exists a strictly stationary sequence $X := (X_k)$ such that $EX_0 = 0$, $|X_0| < C$ a.s. for some $C < \infty$, $\beta(m) \ll (\log m)^3 \cdot m^{-2}$ as $m \to \infty$, $\operatorname{var} S_m$ is nondecreasing as $m$ increases, $\operatorname{var} S_{L(n)} \sim n$ as $n \to \infty$, and also* (1.3) *and* (1.4) *hold.*

Here $L(n)$ means $L_n$, for typographical convenience.

REMARK 1. Suppose $X := (X_k)$ is strictly stationary, $EX_0 = 0$, $|X_0| < C$ a.s. for some $C < \infty$, and $\sum_{m=1}^{\infty} m \cdot \alpha(m) < \infty$. Then $\sum_{m=1}^{\infty} m|E(X_0 X_m)| < \infty$ by Theorem 17.2.1 of Ibragimov and Linnik (1971). It follows from simple calculations that either $\sup_n \operatorname{var} S_n < \infty$ or (1.1) holds, and in the latter case $S_n/(n^{1/2}\sigma) \to N(0,1)$ in distribution as $n \to \infty$ by Theorem 0.

Problem 4 of Ibragimov and Linnik [(1971), p. 393] in essence reads as follows: Suppose $\delta > 0$, $X := (X_k)$ is strictly stationary, $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, and $\operatorname{var} S_n \to \infty$ as $n \to \infty$; what is the "slowest" mixing rate for $\alpha(n)$ that will insure that $S_n$ is asymptotically normally distributed as $n \to \infty$? (Compare this question with Theorem 0 and Theorem 3.) In the case where $X_0$ is bounded, Theorem 5 and Remark 1 together show that the rate $\sum_{m=1}^{\infty} m \cdot \alpha(m) < \infty$ is essentially just fast enough to imply asymptotic normality. In the case $E|X_0|^{2+\delta} < \infty$ where $0 < \delta < \infty$, Theorem 6 below and an exact analog of Remark 1 [using Theorem 17.2.2 of Ibragimov and Linnik (1971)] show that the rate $\sum_{m=1}^{\infty} m \cdot \alpha(m)^{\delta/(2+\delta)} < \infty$ is essentially just fast enough to imply asymptotic normality. The answer to the above question seems to be pinpointed to within certain powers of $\log m$; a small gap remains.

THEOREM 6. *Suppose* $0 < \delta < \infty$. *Then there exists a strictly stationary sequence* $X := (X_k)$ *such that* $EX_0 = 0$, $E|X_0|^{2+\delta} < \infty$, $\operatorname{var} S_n \to \infty$ *as* $n \to \infty$, $\beta(m) \ll ((\log m)^3 \cdot m^{-2})^{(2+\delta)/\delta}$ *as* $m \to \infty$, *and* (1.3) *and* (1.4) *both hold.*

Now let us return to the context of Theorems 1 and 2. Herrndorf [(1985), Theorem 1] gave a broad generalization of Theorem 0 under the additional assumption of (1.1). A natural question is whether the assumptions there (i.e., "moment" conditions and mixing rates) are essentially as weak as permissible. Herrndorf (private communication) suggested that if this were so, then one might be able to verify this by using the constructions in the present paper with careful choices of parameters. To avoid extra complications, we shall pursue this suggestion in only the special case treated in the corollary to Theorem 2 in Herrndorf (1985). There the clt (and weak invariance principle) were given under the "moment" condition $EX_0^2(\log^+|X_0|)^a < \infty$ for some $a > 1$ and the mixing rate $\alpha(m) \ll r^{-m}$ for some $r > 1$. Here $\log^+ x := \max\{0, \log x\}$. The theorems in that paper did not assume stationarity, and in a nonstationary construction in Example 1 of that paper Herrndorf showed that that particular clt (without the assumption of stationarity) does not extend to the case $a = 1$. The following stationary example is almost as sharp:

THEOREM 7. *There exists a strictly stationary sequence* $X := (X_k)$ *such that* $EX_0 = 0$, $EX_0^2(\log^+|X_0|)^a < \infty \ \forall\, a \in (0,1)$, $\beta(m) \ll e^{-m}$ *as* $m \to \infty$, (1.2) *holds, and* (1.3) *and* (1.4) *also hold.*

REMARK 2. For the $\rho$-mixing (maximal correlation) condition (whose formal definition need not be mentioned here), the basic clts are due to Ibragimov [(1975), Theorems 2.1 and 2.2]. In those results the assumptions ("moment" condition, mixing rate) are essentially as weak as permissible for the clt, as is shown by stationary $\rho$-mixing counterexamples (satisfying barely weaker conditions) in Bradley (1984). It is an open question whether stationary $\rho$-mixing counterexamples to the clt exist that also satisfy (1.1), and, if so, whether any such examples are sharp.

**2. Preliminaries.** We shall first mention two elementary lemmas that will be needed later on.

LEMMA 2.1. *Suppose* $\mathscr{A}_n$, $n = 1, 2, \ldots$ *and* $\mathscr{B}_n$, $n = 1, 2, \ldots$ *are* $\sigma$*-fields, and the* $\sigma$*-fields* $\mathscr{A}_n \vee \mathscr{B}_n$, $n = 1, 2, \ldots$ *are independent. Then* $\beta(\bigvee_{n=1}^{\infty} \mathscr{A}_n, \bigvee_{n=1}^{\infty} \mathscr{B}_n) \le \sum_{n=1}^{\infty} \beta(\mathscr{A}_n, \mathscr{B}_n)$.

To prove Lemma 2.1, first show that $\beta(\mathscr{A}_1 \vee \mathscr{A}_2, \mathscr{B}_1 \vee \mathscr{B}_2) \le \beta(\mathscr{A}_1, \mathscr{B}_1) + \beta(\mathscr{A}_2, \mathscr{B}_2)$, which is easy; then use induction and an approximation argument.

REMARK. Under the same conditions one can obtain $\beta(\mathscr{A}_1 \vee \mathscr{A}_2, \mathscr{B}_1 \vee \mathscr{B}_2) \le \beta(\mathscr{A}_1, \mathscr{B}_1) + \beta(\mathscr{A}_2, \mathscr{B}_2) - \beta(\mathscr{A}_1, \mathscr{B}_1) \cdot \beta(\mathscr{A}_2, \mathscr{B}_2)$ by an elementary but slightly longer calculation analogous, e.g., to the proof of Lemma 2.2 of Bradley (1980);

and this leads to the inequality $\beta(\bigvee_{n=1}^{\infty}\mathscr{A}_n, \bigvee_{n=1}^{\infty}\mathscr{B}_n) \le 1 - \prod_{n=1}^{\infty}[1 - \beta(\mathscr{A}_n, \mathscr{B}_n)]$. But Lemma 2.1 is sufficient for our purposes.

LEMMA 2.2. *Suppose $\mathscr{A}$ is a $\sigma$-field and $F$ is an atom of $\mathscr{A}$. Then* $\beta(\mathscr{A}, \mathscr{A}) \le 2 \cdot [1 - P(F)]$.

Herrndorf (1983) used an analogous lemma involving $\alpha(\mathscr{A}, \mathscr{A})$. A related property of entropy (for finite $\sigma$-fields) played a similar role in Theorem 2 of Bradley (1983).

PROOF OF LEMMA 2.2. Suppose $\{A_1, A_2, \dots, A_I\}$ is a partition of $\Omega$ such that each $A_i \in \mathscr{A}$ and $A_I = F$. It suffices to show that

$$\sum_{i=1}^{I} \sum_{j=1}^{I} |P(A_i \cap A_j) - P(A_i)P(A_j)| \le 4[1 - P(F)]. \tag{2.1}$$

By a simple argument,

$$\begin{aligned}(\text{l.h.s. of (2.1)}) &= 2 \sum_{i=1}^{I} P(A_i)[1 - P(A_i)] \\ &\le 2[P(A_I)(1 - P(A_I))] + 2 \sum_{i=1}^{I-1} P(A_i) \le 4P(F^c).\end{aligned}$$

This completes the proof. □

Now let us turn our attention to some random sequences that will be used as "building blocks" for the random sequences $X$ to be constructed later on for Theorems 1–7.

DEFINITION 2.3. Suppose $L$ is a positive integer and $0 < p < 1$. A random sequence is said to have the $\mathscr{S}(L, p)$ distribution if it has the same distribution as the random sequence $W$ defined as follows:

Let $U := (U_k,\ k \in \mathbb{Z})$ be i.i.d. with $P(U_0 = 1) = 1 - P(U_0 = 0) = p$. Let $V := (V_k,\ k \in \mathbb{Z})$ be i.i.d. and independent of the sequence $U$, such that $P(V_k = 1) = P(V_k = -1) = \frac{1}{2}$. Define the random sequence $W := (W_k,\ k \in \mathbb{Z})$ as follows: For each $k$,

$$\begin{aligned} W_k := &\sum_{j=k-L+1}^{k} V_j \cdot I(U_j = 1 \text{ and } U_{j+1} = U_{j+2} = \cdots = U_{j+2L-1} = 0) \\ &- \sum_{j=k-2L+1}^{k-L} V_j \cdot I(U_j = 1 \text{ and } U_{j+1} = U_{j+2} = \cdots = U_{j+2L-1} = 0). \end{aligned} \tag{2.2}$$

Here $I(\cdot)$ denotes the indicator function. This definition is more complicated than it needs to be, but only in order to make it easier to verify some tedious, elementary technical properties. Of the $2L$ terms on the r.h.s. of (2.2), at most one can be nonzero for any given sample point $\omega \in \Omega$.

In what follows, the absolute regularity coefficients $\beta(m)$ for a given stationary sequence $X$ will henceforth be denoted $\beta_m(X)$, in order to avoid confusion when other stationary sequences are present.

Also, the $\sigma$-field of events generated by a family $(Y_s,\ s \in S)$ of r.v.s will henceforth be denoted by $\mathscr{B}(Y_s,\ s \in S)$.

LEMMA 2.4. *Suppose $L$ is a positive integer and $0 < p < 1$. If $W := (W_k,\ k \in \mathbb{Z})$ is a random sequence with the $\mathscr{S}(L, p)$ distribution, then the following statements hold*:

(i) *$W$ is strictly stationary*;

(ii) $P(W_0 = 1) = P(W_0 = -1) = Lp(1-p)^{2L-1}$,

$P(W_0 = 0) = 1 - 2Lp(1-p)^{2L-1}$,

$EW_0 = 0$, *and* $EW_0^2 = 2Lp(1-p)^{2L-1}$;

(iii) $\forall\ m \geq 1$, $P(W_1 + \cdots + W_m \neq 0) \leq 4Lp(1-p)^{2L-1}$;

(iv) $\beta_1(W) \leq 8Lp$ *and* $\beta_{4L-1}(W) = 0$;

(v) $$EW_0W_m = \begin{cases} (2L - 3m)p(1-p)^{2L-1}, & \text{if } 1 \leq m \leq L, \\ (m - 2L)p(1-p)^{2L-1}, & \text{if } L \leq m \leq 2L, \\ 0, & \text{if } m \geq 2L; \end{cases}$$

(vi) $\forall\ m = 1, 2, \ldots, L$,

$$E(W_1 + \cdots + W_m)^2 = (-m^3 + 2Lm^2 + m)p(1-p)^{2L-1};$$

(vii) $\forall\ m = L, L+1, \ldots, 2L$

$$E(W_1 + \cdots + W_m)^2 = \left(4L^2m - \frac{4L^3}{3} + \frac{4L}{3} - 2Lm^2 + \frac{m^3}{3} - \frac{m}{3}\right) \cdot p(1-p)^{2L-1};$$

(viii) $\forall\ m \geq 2L$,

$$E(W_1 + \cdots + W_m)^2 = \left(\frac{4L^3}{3} + \frac{2L}{3}\right)p(1-p)^{2L-1};$$

(ix) $E(W_1 + \cdots + W_m)^2$ *is nondecreasing as $m$ increases.*

PROOF. Without loss of generality, we shall assume that $W$ is accompanied by random sequences $U$ and $V$ as in Definition 2.3, and that $\forall\ \omega \in \Omega$, $U_k(\omega) \in \{0, 1\}\ \forall\ k$ and $V_k(\omega) \in \{-1, 1\}\ \forall\ k$.

Property (i) is obvious, and (ii) is a simple consequence of Definition 2.3.

To verify (iii), first note that for each $\omega \in \Omega$, the sequence $(W_k(\omega),\ k \in \mathbb{Z})$ consists of 0s interrupted occasionally by strings of $+1, \ldots, +1, -1, \ldots, -1$ (with $L$ $+1$s and $L$ $-1$s) or $-1, \ldots, -1, +1, \ldots, +1$. (Two such strings can be contiguous, without zeros between them.) Hence $\forall\ m \geq 1$, $\{W_1 + \cdots + W_m \neq 0\} \subset \{W_1 \neq 0\} \cup \{W_m \neq 0\}$, and (iii) now follows from (ii).

To prove (iv), first note that (2.2) remains unchanged if each $U_i$ is replaced by $|U_iV_i|$ and each $V_i$ by $U_iV_i$. Hence for each fixed $k$, $W_k$ is really a function of only the r.v.s $(U_jV_j,\ k - 2L + 1 \le j \le k + 2L - 1)$. Now $\mathscr{B}(W_k,\ k \le 0) \subset \mathscr{A}_1 \vee \mathscr{G}$ and $\mathscr{B}(W_k,\ k \ge 1) \subset \mathscr{B}_1 \vee \mathscr{G}$, where $\mathscr{A}_1 = \mathscr{B}(U_kV_k,\ k \le -2L + 1)$, $\mathscr{B}_1 = \mathscr{B}(U_kV_k,\ k \ge 2L)$, and $\mathscr{G} = \mathscr{B}(U_kV_k,\ -2L + 2 \le k \le 2L - 1)$. Hence by Lemma 2.1, $\beta_1(W) \le \beta(\mathscr{G}, \mathscr{G})$. Now $\mathscr{G}$ has an atom $\{U_kV_k = 0\ \forall\ k,\ -2L + 2 \le k \le 2L - 1\}$ which has probability $(1 - p)^{4L-2} \ge 1 - 4Lp$. Hence by Lemma 2.2, $\beta_1(W) \le 8Lp$. The equation $\beta_{4L-1}(W) = 0$ follows from the fact that $\mathscr{B}(W_k,\ k \le 0)$ and $\mathscr{B}(W_k,\ k \ge 4L - 1)$ are sub-$\sigma$-fields of the independent $\sigma$-fields $\mathscr{B}(U_kV_k,\ k \le 2L - 1)$ and $\mathscr{B}(U_kV_k,\ k \ge 2L)$, respectively.

Next, to prove (v), first define the random sequence $(T_k,\ k \in \mathbb{Z})$ by $T_k := V_k \cdot I(\{U_k = 1\} \cap \{U_{k+1} = U_{k+2} = \cdots = U_{k+2L-1} = 0\})$. Then $ET_k = 0$, $ET_k^2 = p(1 - p)^{2L-1}$, and $ET_kT_l = 0$ for $k \ne l$. (If $k \ne l$ and $|k - l| \le 2L - 1$ then $T_kT_l = 0$.)

If $1 \le m \le L - 1$, then $W_0W_m = \sum_{k=m-L+1}^{0} T_k^2 - \sum_{k=-L+1}^{m-L} T_k^2 + \sum_{k=m-2L+1}^{-L} T_k^2 + Y$ where $Y$ is a linear combination of finitely many r.v.s of the form $T_kT_l$, $k \ne l$, and hence $EW_0W_m = (2L - 3m)p(1 - p)^{2L-1}$.

If $L \le m \le 2L - 1$, then $W_0W_m = -\sum_{k=m-2L+1}^{0} T_k^2 + Y$ where $Y$ has the same form as above, and $EW_0W_m = (m - 2L)p(1 - p)^{2L-1}$.

If $m \ge 2L$, then obviously $EW_0W_m = 0$. Thus (v) holds.

Substituting the equations in (v) into the formula $E(W_1 + \cdots + W_m)^2 = mEW_0^2 + 2\sum_{k=1}^{m-1}(m - k)EW_0W_k$ and carrying out some simple, tedious arithmetic, we obtain (vi), (vii), and (viii).

One can prove (ix) by direct calculations, but there is an easier way. For each $m \ge 2$, $E(W_1 + \cdots + W_m)^2 = q_m + E(W_1 + \cdots + W_{m-1})^2$ where $q_m = EW_0^2 + 2\sum_{k=1}^{m-1} EW_0W_k$. By (viii), $q_m = 0\ \forall\ m \ge 2L + 1$. By (v), $EW_0W_m > 0$ if $1 \le m < 2L/3$, and $EW_0W_m \le 0$ if $m \ge 2L/3$. Hence, if $2 \le m < 2L/3$ then $q_m > 0$, and if instead $m \ge 2$ and $2L/3 \le m \le 2L$ then $q_m = q_m - q_{2L+1} = -2\sum_{k=m}^{2L} EW_0W_k \ge 0$. Thus $q_m \ge 0\ \forall\ m \ge 2$, and (ix) follows. This completes the proof of Lemma 2.4. □

**3. Proofs of Theorems 1–7.** The random sequences $X$ for Theorems 1–7 will be constructed as follows:

PARAGRAPH A. (Construction of X): Let $L_1, L_2, L_3, \ldots$ be a nondecreasing sequence of positive integers such that $\lim_{n\to\infty} L_n = \infty$. Let $p_1, p_2, p_3, \ldots$ be positive numbers satisfying

$$\sum_{n=1}^{\infty} L_n p_n \le \frac{1}{8}. \tag{3.1}$$

Let $C_1, C_2, C_3, \ldots$ be positive numbers such that

$$\sum_{n=1}^{\infty} L_n C_n^2 p_n < \infty. \tag{3.2}$$

For each $n = 1, 2, \ldots$ let $X^{(n)} := (X_k^{(n)},\ k \in \mathbb{Z})$ be a (strictly stationary) random sequence with the $\mathscr{S}(L_n, p_n)$ distribution. Let these sequences $X^{(1)}, X^{(2)}, \dots$ be independent of each other. Define the sequence $X := (X_k,\ k \in \mathbb{Z})$ as follows: for each $k$,

$$X_k := \sum_{n=1}^{\infty} C_n X_k^{(n)}. \tag{3.3}$$

For each $k$ this sum converges a.s. and in $\mathscr{L}_2$ by Lemma 2.4(ii) and (3.2).

LEMMA 3.1. *Suppose $X := (X_k,\ k \in \mathbb{Z})$ is defined as in* (3.3), *with all assumptions in Paragraph A satisfied. For each $m = 1, 2, \dots$ define the partial sum $S_m = X_1 + \cdots + X_m$. Then the following statements hold*:

(i) *$X$ is strictly stationary*;

(ii) *$EX_0 = 0$ and $0 < EX_0^2 < \infty$*;

(iii) *$\|X_0\|_\infty \le \sum_{n=1}^{\infty} C_n$ and for each $\nu \ge 1$,*

$$\|X_0\|_\nu \le \sum_{n=1}^{\infty} C_n (2L_n p_n)^{1/\nu};$$

(iv) *equations* (1.3) *and* (1.4) *both hold*;

(v) *$\forall\, m \ge 1$, $\beta_m(X) \le \sum_{\{n: 4L(n) \ge m\}} 8 L_n p_n$*;

(vi) *$\forall\, m \ge 1$,*

$$\begin{aligned}
\operatorname{var} S_m = {} & \sum_{\{n:\, m \le L(n)\}} (-m^3 + 2L_n m^2 + m) C_n^2 p_n (1-p_n)^{2L(n)-1} \\
& + \sum_{\{n:\, L(n) < m \le 2L(n)\}} \left( 4L_n^2 m - \frac{4L_n^3}{3} + \frac{4L_n}{3} \right. \\
& \qquad \left. - 2L_n m^2 + \frac{m^3}{3} - \frac{m}{3} \right) C_n^2 p_n (1-p_n)^{2L(n)-1} \\
& + \sum_{\{n:\, 2L(n) < m\}} \left( \frac{4L_n^3}{3} + \frac{2L_n}{3} \right) C_n^2 p_n (1-p_n)^{2L(n)-1}.
\end{aligned}$$

*Here $L(n)$ means $L_n$, for typographical convenience. In both parts of* (*iii*) *of course, the r.h.s.* (*and even the l.h.s.*) *can perhaps be $\infty$.*

PROOF OF LEMMA 3.1. The proofs of (i), (ii), and (iii) are elementary consequences of the assumptions in Paragraph A [using Minkowski's inequality for (iii)].

Proof of (iv): For each $n \ge 1$ and $m \ge 1$, $P(X_1^{(n)} + \cdots + X_m^{(n)} \ne 0) \le 4L_n p_n$ by Lemma 2.4(iii). Hence $\forall m \ge 1$, $P(S_m \ne 0) \le \sum_{n=1}^{\infty} 4L_n p_n \le \frac{1}{2}$ by (3.1). This proves (1.3). This argument was like that in Herrndorf (1983), and the argument for (1.4) is also like that in Herrndorf [(1983), starting with line $-7$ of p. 812 there].

Part (v) is an easy consequence of Lemma 2.4(iv) and Lemma 2.1.

Part (vi) is an easy consequence of Lemma 2.4(vi), (vii), (viii), and the equality $\operatorname{var} S_m = \sum_{n=1}^{\infty} C_n^2 \operatorname{var}(X_1^{(n)} + \cdots + X_m^{(n)})$. This completes the proof.

PROOF OF THEOREM 1. For each $n = 1, 2, 3, \ldots$ define $L_n := 2^{n-1}$. Define the sequence of numbers $p_1, p_2, \ldots$ by $p_n := q \cdot 2^{-n} \cdot (2^{-n} \cdot n^3)^{(2+\delta)/\delta}$ where the constant $q > 0$ is fixed sufficiently small that (3.1) holds. For each $n = 1, 2, \ldots$ define $C_n := [(1 - p_n)^{1-2L(n)}]^{1/2} \cdot 2^{n/\delta} \cdot n^{-(3/2)(2+\delta)/\delta}$. Then (3.2) holds. Define the random sequence $X := (X_k)$ as in (3.3).

First note that $C_n(L_n p_n)^{1/(2+\delta)} \ll n^{-3/2}$ as $n \to \infty$, and hence $E|X_0|^{2+\delta} < \infty$ by Lemma 3.1(iii).

Next, by Lemma 3.1(v) and elementary calculations,

$$\begin{aligned}\beta_m(X) &\le \sum_{n=N(m)}^{\infty} 4q \cdot (2^{-n} \cdot n^3)^{(2+\delta)/\delta} \\ &\ll \left[2^{-N(m)} \cdot N(m)^3\right]^{(2+\delta)/\delta} \\ &\ll \left[(\log m)^3/m\right]^{(2+\delta)/\delta},\end{aligned}$$

where for each $m = 1, 2, \ldots,$ $N(m) := \min\{n: 2^{n+1} \ge m\}$.

Except for (1.2), all properties in Theorem 1 either have been verified or follow immediately from Lemma 3.1. To verify (1.2), let $m \ge 1$ be arbitrary but fixed. Let $N$ be the positive integer such that $2^{N-1} \le m < 2^N$. Then by Lemma 2.4(v),

$$\begin{aligned}EX_0X_m &= \sum_{n=1}^{\infty} C_n^2 EX_0^{(n)}X_m^{(n)} \\ &= C_N^2 EX_0^{(N)}X_m^{(N)} + \sum_{n=N+1}^{\infty} C_n^2 EX_0^{(n)}X_m^{(n)} \\ &= (m - 2L_N)q \cdot 4^{-N} + \sum_{n=N+1}^{\infty} (2L_n - 3m)q \cdot 4^{-n} \\ &= 0.\end{aligned}$$

This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. In essence, carry out the proof of Theorem 1 with $\delta = \infty$; i.e., with $L_n := 2^{n-1}$, $p_n := q \cdot 4^{-n} \cdot n^3$, and $C_n := [(1 - p_n)^{1-2L(n)}]^{1/2} \cdot n^{-3/2}$. □

PROOF OF THEOREM 3. For each $n \ge 1$ define $L_n := [\exp(n^{5/6})]$ where $[x]$ denotes the greatest integer $\le x$. Define the numbers $p_1, p_2, \ldots$ by $p_n := qn^{-(7/6)(2+\delta)/\delta}\exp(-((2 + 2\delta)/\delta)n^{5/6})$ where $q > 0$ is fixed sufficiently small that (3.1) holds. For each $n \ge 1$ define $C_n := n^{(7/6)(1-\delta)/\delta}\exp((1/\delta)n^{5/6})$. Then (3.2) holds. Define the sequence $X := (X_k)$ as in (3.3).

In verifying the mixing rate on $\beta(m)$ and the rate of growth of $\operatorname{var} S_m$, one can use Lemma 3.1(v), (vi) and elementary calculus, including (i) $\lim_{n\to\infty}\exp(s(n +$

$1)^\gamma)/\exp(sn^\gamma) = 1$ when $s$ is real and $0 < \gamma < 1$, and also (ii) such formulas as $\sum_{n=M+1}^{N} n^r \exp(sn^\gamma) \sim (s\gamma)^{-1}[N^{r+1-\gamma}\exp(sN^\gamma) - M^{r+1-\gamma}\exp(sM^\gamma)]$ as $N > M \to \infty$, whenever the constants $r$, $s$, and $\gamma$ satisfy $0 < \gamma < 1$ and $s \neq 0$. From Lemma 3.1(v) one obtains $\beta_m(X) \ll (\log m)^{(1/5)(1-7(2+\delta)/\delta)} m^{-(2+\delta)/\delta}$ and hence $\sum_{m=1}^{\infty}(\beta_m(X))^{\delta/(2+\delta)} < \infty$. From Lemma 3.1(vi) and much arithmetic one obtains $\operatorname{var} S_m \sim C \cdot (\log m)^{-4} m$ as $m \to \infty$ for some constant $C > 0$. Simply by rescaling the process $X$ one can make $C = 1$ without affecting the other properties of $X$ stated in Theorem 3. The remaining properties are easy to verify from Lemma 3.1. □

Proof of Theorem 4. In essence, carry out the proof of Theorem 3 with $\delta = \infty$; i.e., with $L_n := [\exp(n^{5/6})]$, $p_n := qn^{-7/6}\exp(-2n^{5/6})$, and $C_n := n^{-7/6}$. □

Proof of Theorem 5. Let $L_1, L_2, \ldots$ be as in the statement of Theorem 5. Define the numbers $p_1, p_2, \ldots$ by $p_n := (3/4)qn^3L_n^{-3}$ with the constant $q > 0$ fixed sufficiently small that (3.1) holds. For each $n = 1, 2, \ldots$ define $C_n := [(1 - p_n)^{1-2L(n)} \cdot n^{-3}q^{-1}]^{1/2}$. Then (3.2) holds. Define the sequence $X := (X_k)$ as in (3.3).

The properties in Theorem 5 can be verified from Lemma 3.1 pretty easily; we shall just discuss the rate of growth of $\operatorname{var} S_n$. By Lemma 2.4(vi), (viii) and the hypothesis of Theorem 1, for each $n \geq 2$ we have $\operatorname{var}(\sum_{k=1}^{2L(n-1)} C_n X_k^{(n)}) \leq 6/n^2$, and for each $n \geq 1$ we have $\operatorname{var}(\sum_{k=1}^{m} C_n X_k^{(n)}) = 1 + L_n^{-2}/2\ \forall\ m \geq 2L_n$. Using the formula $\operatorname{var} S_m = \sum_{n=1}^{\infty}\operatorname{var}(\sum_{k=1}^{m} C_n X_k^{(n)})\ \forall\ m \geq 1$, and using Lemma 2.4(ix), we obtain that $\operatorname{var} S_m$ is nondecreasing and that $\operatorname{var} S_{2L(N)} \sim N$ as $N \to \infty$. Now $\operatorname{var} S_{L(N)} \sim N$ follows. This completes our argument. □

Proof of Theorem 6. Choose $L_n := 2^{n-1}$, $p_n := q \cdot 2^{-n}(2^{-2n}n^3)^{(2+\delta)/\delta}$, and $C_n := 2^{2n/\delta} n^{-(3/2)(2+\delta)/\delta}$, and simply apply Lemma 3.1 with elementary calculations. □

Proof of Theorem 7. Choose $L_n := 2^{n-1}$, $p_n := q \cdot 4^{-n}\exp(-2^{n+1})$, and $C_n := [(1 - p_n)^{1-2L(n)}]^{1/2}\exp(2^n)$. Now one can imitate the proof of Theorem 1. For each $a$, $0 < a < 1$, one has $EX_0^2(\log^+|X_0|)^a < \infty$ by a simple calculation after one shows that (to put it loosely) for large $N$, for $\omega \in \{X_0^{(N)} \neq 0,\ X_0^{(n)} = 0\ \forall\ n > N\}$, one has $|X_0(\omega)| \approx C_N$. □

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