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ON THE DISTRIBUTION OF FIRST PASSAGE AND RETURN TIMES FOR SMALL SETS

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For a Harris recurrent Markov chain with invariant initial distribution π, we consider the return times τ_ε to state sets A_ε with $0 < \pi(A_\varepsilon) \to 0$ as $\varepsilon \to 0$ and show that, provided the probability of early returns to A_ε approaches 0, the τ_ε, multiplied by suitable scaling factors, are asymptotically exponentially distributed.

1. Introduction. Mark Kac in a lecture at the University of New Mexico in 1983 noted that, for many stationary ergodic processes, the return times to sets of small probability are asymptotically exponentially distributed. His remark inspired this small study. News of his death came during the process of revision and extension of these results, and this paper is respectfully dedicated to his memory.

Of course an exponential distribution for the return time would provide information going beyond Kac's famous formula: $\int_A \tau_A \, dP = 1$ for any stationary ergodic probability P and event A with $P(A) > 0$, where τ_A is the return time to A [Kac (1947)].

Results of this type were first established by T. E. Harris (1952) for a positive recurrent Markov chain on a denumerable state space. Harris considered first passage times for a sequence of states, x_n, converging to infinity. The first passage times to these states, starting from a fixed state, are asymptotically exponentially distributed. The return times to x_n (for the process started at x_n) will have this property only if the probability of an early return to x_n converges to 0. Otherwise visits to x_n tend to occur in clusters and the first return may have any sort of distribution.

In this study we make no attempt to tackle the general stationary ergodic sequence, but do consider a Harris recurrent Markov chain on a general state space $(\mathbb{X}, \mathbb{A})$ with finite invariant measure. In effect, we generalize the return time result of Harris (1952) to general state spaces, with a sequence of small sets replacing the states x_n.

Let π denote the invariant initial distribution on $(\mathbb{X}, \mathbb{A})$ and P^n denote the n-step transition probability. For any initial distribution ϕ, P_ϕ denotes the resulting distribution on the chain $X_0, X_1, X_2, \ldots$, and E_ϕ denotes the corresponding expectation. When $\phi = \delta_x$, we write P_x, E_x for P_ϕ, E_ϕ, respectively. Also, for any event A with $\pi(A) > 0$, π_A is defined by $\pi_A(B) = \pi(AB)/\pi(A)$.

Our approach depends on the recurrence times defined by Athreya and Ney (1978). They assume a Harris recurrent Markov chain possessing a C set:

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$C \in \mathbb{A}$ *is a C set if* $\pi(C) > 0$ *and there exists an* n *and* $\varepsilon > 0$ *such that* $P^n(x, A) \geq \varepsilon\pi(A)$ *for all* $x \in C$ *and* $A \in \mathbb{A}$ *with* $A \subset C$. Provided $\mathbb{A}$ is countably generated, a Harris recurrent Markov chain always possesses a C set. We do not need to assume $\mathbb{A}$ countably generated for our result since, if the asserted convergences failed as $\varepsilon \to 0$, then they would fail for some sequence $\varepsilon_n \to 0$. But if we consider only the sequence $\{A_{\varepsilon_n}\}$, then it is possible to form an admissible, countably generated sub-σ field of $\mathbb{A}$ containing this sequence [see Orey (1971)], and the theorem would then hold for the A_{ε_n}s. Thus, without loss of generality, we assume existence of a C set.

We will let C denote a fixed C set in what follows. Then Athreya and Ney (1978) show there exist times $0 = \rho_0 < \rho_1 < \rho_2 < \cdots$ such that, starting the process with distribution π_C, the random variables $X_{\rho_0}, X_{\rho_1}, X_{\rho_2}, \ldots$ are independent and identically distributed. Combining this with the strong Markov property for discrete parameter Markov chains it is also true that the random vectors $Y_n = (X_{\rho_{n-1}+1}, \ldots, X_{\rho_n})$, $n = 1, 2, \ldots$ are independent and identically distributed. Moreover, in our case where π is finite, it is true that $E_{\pi_C}(\rho_1) < \infty$. It may help to explain that the ρ_ks are chosen from the successive passages of C by a supplementary randomizing distribution. As Nummelin (1978) shows, we can in effect treat the ρ_ks as passage times of a positive atom. Starting with any distribution ϕ, it will be true that $P_\phi[X_{\rho_k} \in A] = \pi_C(A)$, $A \in \mathbb{A}$, $k \geq 1$.

Now consider a family of sets $\{A_\varepsilon \in \mathbb{A}, \varepsilon > 0\}$ with $0 < \pi(A_\varepsilon) \to 0$ as $\varepsilon \to 0$. This family remains fixed, and we let τ_ε be the first $k \geq 1$ such that $x_k \in A_\varepsilon$, i.e., the first passage times of A_ε (or return times if $X_0 \in A_\varepsilon$). Set

$$p_\varepsilon = P_{\pi_C}[\tau_\varepsilon \leq \rho_1] \quad \text{and} \quad \mu = E_{\pi_C}\rho_1.$$

The following two theorems describe the basic results:

THEOREM 1. *For any initial distribution* $\phi \ll \pi$ *and* $t > 0$,

$$P_\phi\left[p_\varepsilon\tau_{A_\varepsilon}/\mu > t\right] \to e^{-t}$$

as $\varepsilon \to 0$ *(i.e.,* $p_\varepsilon\tau_\varepsilon/\mu$ *converges in distribution to a unit exponential).*

THEOREM 2. *For any* $t > 0$, *as* $\varepsilon \to 0$,

$$P_{\pi_{A_\varepsilon}}\left[p_\varepsilon\tau_{A_\varepsilon}/\mu > t\right] - P_{\pi_{A_\varepsilon}}[\tau_{A_\varepsilon} > \rho_1]e^{-t} \to 0.$$

The factor $P_{\pi_{A_\varepsilon}}[\tau_{A_\varepsilon} > \rho_1]$ *can be interpreted as the probability of no early return to* A_n.

NOTE. In the process of revision the work of Korolyuk and Sil'vestrov (1984) came to our attention. They obtain a result in some respects more general than Theorem 1, showing that, if $P(x, A_\varepsilon) \to 0$ as $\varepsilon \to 0$ for *every* x, then the conclusion of Theorem 1 holds for $\phi = \delta_x$ for every x (and it follows readily that the theorem holds for arbitrary ϕ, not necessarily π continuous). While our Theorem 1 is not directly a consequence of their result, it is essentially a simpler result. It can be proven by methods similar to those of Korolyuk and Sil'vestrov, or by using renewal theory in a way similar to our proof of Theorem 2. For these reasons we omit the proof of Theorem 1.

PROOF OF THEOREM 2. As noted in the Introduction, it suffices to prove the theorem for an arbitrary sequence $\{A_n\}$ with $0 < \pi(A_n) \to 0$ as $n \to \infty$. To simplify notation, we let τ_n denote the first $k \geq 1$ such that $X_k \in A_n$, $\pi_n = \pi_{A_n}$, and $p_n = P_{\pi_C}[\tau_n \leq \rho_1]$. Let $\mathbb{X}^{Z^+}$ be the space of sequences $(X_0, X_1, \dots)$ in $\mathbb{X}$ and $\theta_j(X_0, X_1, \dots) = (X_j, X_{j+1}, \dots)$. For $f\colon \mathbb{X}^{Z^+} \to \mathbb{R}^1$ we write $E_\phi f$ for $E_\phi f(X_0, X_1, \dots)$. The following lemma is a variation on a standard result.

LEMMA 1. *Let $f\colon \mathbb{X}^{Z^+} \to \mathbb{R}^1$ be measurable and either bounded below or bounded above. Then*

$$E_\pi f = \mu^{-1} E_{\pi_C}\left(\sum_{k=1}^{\rho_1} f \circ \theta_k \right).$$

PROOF. Without loss of generality assume $f \geq 0$. By the ergodic theorem $\sum_{k=1}^{n} f \circ \theta_k / n \to E_\pi f$ a.s. $-P_\pi$. Since $\|\pi_C P^n - \pi\| \to 0$ as $n \to \infty$, $P_{\pi_C} = P_\pi$ on the tail σ field [see Orey (1971)], hence $\sum_{k=1}^{\rho_n} f \circ \theta_k / \rho_n \to E_\pi f$ a.s. $-P_{\pi_C}$. Since the blocks between ρ_n are i.i.d., the strong law of large numbers implies

$$\sum_{k=1}^{\rho_n} f \circ \theta_k / n \to E_{\pi_C}\left(\sum_{k=1}^{\rho_1} f \circ \theta_k \right) \text{ a.s. } -P_{\pi_C}$$

and $\rho_n / n \to E_{\pi_C} \rho_1 = \mu$ a.s. $-P_C$ as $n \to \infty$. Hence

$$E_\pi f =_{\text{a.s.}} \lim_{n \to \infty} \frac{n}{\rho_n} \frac{1}{n} \sum_{k=1}^{\rho_n} f \circ \theta_k =_{\text{a.s.}} E_{\pi_C}\left(\sum_{k=1}^{\rho_n} f \circ \theta_k \right). \square$$

LEMMA 2. $p_n \leq \mu \pi(A_n)$.

PROOF. Let $f_n(X_0, X_1, \dots) = I_{A_n}(X_0)$ in Lemma 1 to get

$$\pi(A_n) = E_\pi f_n = \mu^{-1} E_{\pi_C}\left(\sum_{k=1}^{\rho_1} I_{A_n}(X_k) \right) \geq \mu^{-1} P_{\pi_C}[\tau_n \leq \rho_1] = \mu^{-1} p_n. \square$$

LEMMA 3. *For any $\delta > 0$, as $n \to \infty$*

$$P_{\pi_n}[\rho_1 \geq \delta / p_n \textit{ and } \tau_n \geq \delta / p_n] \to 0.$$

PROOF. Let $f_n(X_0, X_1, \dots) = I_{A_n}(X_0) I_{[\rho_1 \geq \delta/p_n]} I_{[\tau_n \geq \delta/p_n]}$. Note that $f_n \circ \theta_j = 0$ or 1, and $f_n \circ \theta_j = 1$ for some $1 \leq j \leq \rho_1$ implies $\tau_n < \rho_1$, $\rho_1 \geq j + \delta/p_n$, and $f_n \circ \theta_k = 0$ for $j < k \leq j + \delta/p_n$, while $f \circ \theta_{\rho_1} = 1$ implies $\rho_2 \geq \delta/p_n$. Hence

$$\sum_{k=1}^{\rho_1} f_n \circ \theta_k \leq (p_n/\delta) \rho_1 I_{[\tau_n < \rho_1]} + I_{[\tau_n = \rho_1]} I_{[\rho_2 \geq \delta/p_n]},$$

and by Lemma 1 and the renewal property of ρ_1

$$\mu E_\pi f_n = E_{\pi_C}\left(\sum_{k=1}^{\rho_1} f_n \circ \theta_k\right)$$

$$\leq (p_n/\delta) E_{\pi_C}\left(\rho_1 I_{[\tau_n < \rho_1]}\right) + P_{\pi_C}[\tau_n = \rho_1] P_{\pi_C}[\rho_1 \geq \delta/p_n].$$

The first term on the right is $o(p_n)$ by the dominated convergence theorem, since $P_{\pi_C}[\tau_n < \rho_1] \leq P_{\pi_C}[\tau_n \leq \rho_1] = p_n \to 0$, while the second term is $o(p_n^2)$, hence $E_\pi f_n = o(p_n)$. Since $\pi(A_n)^{-1} \leq \mu p_n^{-1}$ by Lemma 2, we have

$$P_{\pi_n}[\rho_1 \geq \delta/p_n \text{ and } \tau_n \geq \delta/p_n] = \pi(A_n)^{-1} E_\pi f_n = o(1). \square$$

PROOF OF THEOREM 2. We have

$$P_{\pi_n}[p_n\tau_n/\mu > t] - P_{\pi_n}[\tau_n > \rho_1 \text{ and } p_n\tau_n/\mu > t]$$

$$\leq P_{\pi_n}[\rho_1 \geq \mu t/p_n \text{ and } \tau_n > \mu t/p_n] = o(1)$$

by Lemma 3. Use the renewal property of ρ_1 to get

$$P_{\pi_n}[\tau_n > \rho_1 \text{ and } \tau_n > \mu t/p_n] = E_{\pi_n}\left(\sum_{k=1}^{\infty} I_{[\tau_n > \rho_1 = k]} P_{\pi_C}[\tau_n + k > \mu t/p_n]\right)$$

$$\geq E_{\pi_n}\left(\sum_{k=1}^{\infty} I_{[\tau_n > \rho_1 = k]} P_{\pi_C}[\tau_n > \mu t/p_n]\right)$$

$$= P_{\pi_n}[\tau_n > \rho_1] e^{-t} + o(1),$$

using Theorem 1 at the last step. On the other hand, for any $0 < \delta < t$,

$$P_{\pi_n}[\tau_n > \rho_1 \text{ and } \tau_n > \mu t/p_n]$$

$$\leq P_{\pi_n}[\tau_n > \rho_1 > \delta\mu/p_n] + E_{\pi_n}\left(\sum_{k=1}^{[\delta\mu/p_n]} I_{[\tau_n > \rho_1 = k]} P_{\pi_C}[\tau_n > \mu(t-\delta)/p_n]\right)$$

$$= P_{\pi_n}[\rho_1 \leq \delta\mu/p_n \text{ and } \tau_n > p_1] e^{-(t-\delta)} + o(1)$$

by Lemma 3 and Theorem 1. Moreover,

$$P_{\pi_n}[\tau_n > \rho_1] - P_{\pi_n}[\rho_1 \leq \delta\mu/p_n \text{ and } \tau_n > \rho_1]$$

$$\leq P_{\pi_n}[\rho_1 > \delta\mu/p_n \text{ and } \tau_n > \delta\mu/p_n] = o(1)$$

by Lemma 3 again. Since $\delta > 0$ is arbitrary we have

$$P_{\pi_n}[\tau_n > \rho_1 \text{ and } p_n\tau_n/\mu > t] = P_{\pi_n}[\tau_n > \rho_1] e^{-t} + o(1),$$

and the theorem follows.

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