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SOME STRUCTURE RESULTS FOR MARTINGALES IN THE LIMIT AND PRAMARTS

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We show that an L^1-bounded Banach-space-valued martingale in the limit (X_n) can be written $X_n = Y_n + Z_n$, where (Y_n) is an L^1-bounded martingale and where (Z_n) is a martingale in the limit that goes to zero a.s. in norm. This theorem still holds for a new class that generalizes martingales in the limit. We show that a real-valued L^1-bounded pramart (X_n) can be written $X_n = Y_n + Z_n$, where Y_n is an L^1-bounded martingale, and Z_n has the following property: For each $\varepsilon > 0$, there is an m, an Σ_m-measurable subset A of Ω, and a supermartingale $(T_n)_{n \geq m}$ on A such that $\int_A T_n \, dP \leq \varepsilon$ and $|Z_n| \leq T_n$ on A for $n \geq m$.

1. Introduction and results. Let (Ω, Σ, P) be a complete probability space, (Σ_n) an increasing sequence of σ algebras. A stopping time is a random variable τ assuming positive integer values and such that for each n, $\{\tau = n\} \in \Sigma_n$. The collection of bounded stopping times is denoted by T. Let F be a Banach space. A sequence (X_n) of F-valued random variables is called adapted if X_n is Σ_n-measurable. For an integrable F-valued random variable Y, we write $E^n(Y) = E(Y|\Sigma_n)$. If τ is a bounded stopping time, we define X_τ by $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$. A *martingale in the limit* is an adapted sequence of integrable F-valued random variables such that there is a sequence (h_n) of measurable functions, h_n valued in $[0, \infty]$, $h_n \to 0$ with

$$\forall\, m \geq n, \qquad \|E^n(X_m) - X_n\| \leq h_n \quad \text{a.s.}$$

If moreover we have

$$\forall\, \tau \in T,\ \tau \geq n, \qquad \|E^n(X_\tau) - X_n\| \leq h_n \quad \text{a.s.}$$

we say that (X_n) is a *pramart.*

Martingales in the limit have been introduced by A. G. Mucci [8], who proved that real-valued L^1-bounded martingales in the limit converge. Pramarts have been introduced and extensively studied by A. Millet and L. Sucheston [7]. Although a more restrictive class than martingales in the limit, they enjoy much better probabilistic properties, e.g., the optional sampling property.

The main theorem we shall obtain still holds for a class more general than martingales in the limit that we now introduce.

DEFINITION 1. We say that an adapted sequence (X_n) of Banach-valued random variables is a *mil* if for each $\varepsilon > 0$, there exists p such that for each

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$n \geq p$ we have

$$P\Big(\sup_q \big\{\|X_q - E^q(X_n)\|;\ p \leq q \leq n\big\} > \varepsilon\Big) \leq \varepsilon.$$

In [7], A. Millet and L. Sucheston use the name mil for martingales in the limit. However, the notion introduced in Definition 1 is more general than the notion of martingales in the limit, while its convergence properties seem to be as good. Since Professor Sucheston himself suggested that the best notion should have the best name, we shall henceforth use the name mil as in Definition 1.

If (X_n) is a martingale in the limit, for each ε there is a p, and a measurable function h_p, with $P(h_p > \varepsilon) < \varepsilon$, such that $\|X_q - E^q(X_n)\| \leq h_p$ whenever $p \leq q \leq n$. In particular, a martingale in the limit is a mil.

Mils constitute a large class, as the following show.

THEOREM 2. *An adapted Banach-space-valued sequence* (X_n) *that converges a.s. and in* L^1 *is a mil.*

EXAMPLE 3. There exists a real-valued L^1-bounded mil that is not a martingale in the limit.

THEOREM 4. *A real-valued mil* (X_n) *such that* $\liminf E|X_n| < \infty$ *converges a.s.*

It should be mentioned that the proof of this theorem is not a stopping time proof.

COROLLARY 5. *An adapted equi-integrable real valued sequence* (X_n) *converges a.s. if and only if it is a mil.*

A classical result states that an L^1-bounded F-valued martingale (X_n) that goes scalarly to zero [i.e., $x^*(X_n) \to 0$ a.s. for each $x^* \in F^*$] goes to zero a.s. in norm. The proof is based on a lemma of Neveu [9]. Using an extension of Neveu's lemma, Egghe could prove that an L^1-bounded F-valued pramart that goes to zero scalarly goes to zero in norm [4]. It is not possible to extend this method of proof to martingales in the limit, because for a martingale in the limit (X_n) the sequence $(\|X_n\|)$ does not seem to have any useful property (even in the real-valued case, as was already shown in [2]). Nevertheless, we shall prove:

THEOREM 6. *Let* (X_n) *be an F-valued mil such that* $\liminf \int \|X_n\|\, dP < +\infty$. *If* $x^* \circ X_n \to 0$ a.s. *for each* x^* *in* E^*, *then* $\|X_n\| \to 0$ a.s.

It would be useful to have a simple description of F-valued L^1-bounded mils that converge to zero. The following example shows that even for pramarts, the Banach-space-valued situation is more complicated than the real-valued situation (that will be described in Theorem 10.)

EXAMPLE 7. There exists an l^2-valued L^1-bounded pramart (X_n) with $\|X_n\| \to 0$ a.s. and $(\|X_n\|)$ is not a mil.

The convergence of vector valued mils is best handled by the following structure theorem. (Special cases were obtained in [5] and [10].)

THEOREM 8. *Let (X_n) be an F-valued mil such that* $\liminf \int \|X_n\|\, dP < \infty$. *Then there is a unique decomposition $X_n = Y_n + Z_n$, where Y_n is an L^1-bounded martingale, and Z_n is a mil with $Z_n \to 0$ a.s.*

COROLLARY 9. *All F-valued L^1-bounded mils converge if (and only if) F has the Radon–Nikodym property.*

We now turn to a characterization of real-valued pramarts that is similar to the Ghoussoub–Sucheston characterization of amarts [6].

THEOREM 10. *Let (X_n) be a real-valued pramart such that* $\liminf \int |X_n|\, dP < \infty$. *Then there is a unique decomposition $X_n = Y_n + Z_n$, where Y_n is an equi-integrable L^1-bounded martingale, and $T_n = |Z_n|$ is a positive pramart that goes to zero. Conversely, if X_n can be written as $Y_n + Z_n$ where Y_n is a martingale and where $|Z_n| \le T_n$, T_n being a positive pramart that goes to zero a.s., then X_n is a pramart.*

The following result characterizes positive pramarts that go to zero a.s.

THEOREM 11. *An adapted sequence of positive random variables (T_n) is a pramart that goes to zero a.s. if and only if for each $\varepsilon > 0$ there is a $p > 0$, a set $A \in \Sigma_p$ with $P(A) \ge 1 - \varepsilon$, and a positive supermartingale $(S_n)_{n \ge p}$ on the space $(A, \Sigma|A, P|A)$ such that*

$$\int_A S_p\, dP \le \varepsilon \quad \text{and} \quad T_n \le S_n \quad \text{on } A \text{ for } n \ge p.$$

A consequence of Theorems 10 and 11 is that if (X_n), (Y_n) are L^1-bounded pramarts, $(X_n \vee Y_n)$ is a L^1-bounded pramart.

EXAMPLE 12. There exists an L^1-bounded martingale in the limit (X_n) with $X_n \to 0$ a.s. such that $|X_n|$ is not a mil.

2. Proof of Theorem 2 and construction of Example 3. If (X_n) converges in L^1, let Y be its limit. Let $Y_n = E^n(Y)$. Since both X_n and Y_n converge a.s. to Y, $Z_n = X_n - Y_n$ goes to zero in L^1 and a.s. Let $\varepsilon > 0$. Since (Z_n) goes to zero a.s., there exists p_1 such that

$$P\Big(\sup_{q \ge p_1} \|Z_q\| \ge \varepsilon\Big) \le \varepsilon.$$

Since (Z_n) goes to zero in L^1, there exist $p \geq p_1$ such that $\|Z_n\|_1 \leq \varepsilon^2$ for $n \geq p$. The maximal inequality shows that

$$P\Big(\sup_{q \leq n} \|E^q(Z_n)\| \geq \varepsilon\Big) \leq \varepsilon.$$

So, for any $n \geq p$, we have

$$P\Big(\sup_{p \leq q \leq n} \big\|E^q(Z_n) - Z_q\big\| \geq 2\varepsilon\Big) \leq 2\varepsilon.$$

This shows that (Z_n) is a mil, whence it follows that $X_n = Y_n + Z_n$ is a mil. Theorem 2 is proved.

We now produce an example of a positive adapted sequence (X_n) such that X_n goes to zero in L^1 and a.s., but is not a martingale in the limit. For $n \geq 1$, let $a_n = \prod_{i \leq n} 2^i = 2^{n(n+1)/2}$, and let $a_0 = 1$. For $n \geq 0$, let Q_n be the partition of $[0,1]$ in a_n intervals of equal length. Let $b_n = \sum_{i \leq n} a_i$. For $n \geq 1$, let Σ_n be the σ algebra of $[0,1]$ generated by Q_{m+1} where m is the unique integer such that $b_{m-1} < n \leq b_m$. (For $n = 1$, we take $m = 0$.) We let $X_1 = 1$ and for $n > 1$ we let $X_n = 0$ everywhere except on the first interval of Q_{m+1} that is contained in $[(n - b_{m-1} - 1)/a_m, (n - b_{m-1})/a_m]$. In this interval we take X_n constant, equal to a_{m+1}/m. We have that (X_n) goes to zero in L^1. Moreover, for each m,

$$P\Big(\sup_{b_{m-1} < n \leq b_m} \|X_n\| > 0\Big) \leq 2^{-m-1},$$

so we also have that (X_n) goes to zero a.s. However if $q = b_{m-1}$, we have

$$\sup_{b_{m-1} < n \leq b_m} E^q(X_n) = a_m/m,$$

since Σ_q is the algebra generated by Q_m. This shows that (X_n) fails to be a martingale in the limit.

3. Proof of Theorem 4. Let (X_n) be a mil that does not converge a.s. We shall prove that $\lim_n E|X_n| = \infty$, proving Theorem 4. We can find $a < b$ and a set $A \in \Sigma$ with $P(A) > 0$ such that $\limsup X_n(\omega) > b$, $\liminf X_n(\omega) < a$ for $\omega \in A$. By adding a constant, we can suppose $a \geq 0$. The main point of the proof is the following.

Claim. Let $n_1 \in \mathbb{N}$, $\varepsilon > 0$. Then there exist $n_2 \geq n_1$ such that for each $D \in \Sigma_{n_1}$, $P(D) \leq P(A)/2$, and each $n \geq n_2$ there exists $E \in \Sigma_{n_2}$ with $P(E) < \varepsilon$, $E \cap D = \varnothing$ such that $\int_E |X_n|\, dP \geq (b-a)P(A)/12$.

Proof. We first find $k \geq n_1$ and $A_1 \in \Sigma_k$ such that $P(A \triangle A_1) < \varepsilon/2$. We can assume $\varepsilon < P(A)/4$, and assume that k is so large that for $n \geq k$

$$P\Big(\sup_{k \leq p \leq n} \big|E^p(X_n) - X_p\big| > (b-a)/3\Big) < \varepsilon/2. \tag{1}$$

We then construct a sequence $k < p_1 < \cdots < p_l$ such that if for $i \le l$

$$B_i = \left(\{X_{p_i} > b\} \cap A_1\right) \setminus \bigcup_{j<i} \{X_{p_j} > b\}$$

we have $\sum_{i \le l} P(B_i) \ge 7P(A)/8$. Note that $B_i \in \Sigma_{p_i}$. Let $B = \bigcup_{i \le l} B_i$. Since $B \subset A_1$, we have $P(B \setminus A) < \varepsilon/2$. Since $\liminf X_n < a$ on A, we can find a sequence $p_l < q_1 < \cdots < q_m$ such that if for $j \le m$

$$C_j = \left(\{X_{q_j} < a\} \cap B_i\right) \setminus \bigcup_{i<j} \{X_{q_i} < a\}$$

we get $\sum_{j \le l} P(C_j) > P(B) - \varepsilon/2$.

We set $n_2 = q_m$. Let $n \ge n_2$, and let $D \in \Sigma_{n_1}$ with $P(D) < P(A)/2$. For $i \le l$, let

$$H_i = \left\{|E^{p_i}(X_n) - X_{p_i}| > (b-a)/3\right\}.$$

It follows from (1) that $P(\bigcup_{i \le l} H_i) < \varepsilon/2 < P(A)/8$. If we set $B_i^1 = B_i \setminus (H_i \cup D)$ we have $B_i^1 \in \Sigma_{p_i}$ and $\sum_{i \le l} P(B_i^1) \ge P(A)/4$. Let $B^1 = \bigcup_{i \le l} B_i^1$. We have $B_i^1 \in \Sigma_{p_l}$. For $j \le m$, let

$$K_j = \left\{|E^{q_j}(X_n) - X_{q_j}| > (b-a)/3\right\}.$$

It follows from (1) that $P(\bigcup_{j \le m} K_j) < \varepsilon/2$. Let $C_j^1 = B^1 \cap (C_j \cup H_j)$. Then $C_j^1 \in \Sigma_{q_j}$, and we have $\sum_{j \le m} P(C_j^1) \ge P(B^1) - \varepsilon$.

Let us fix $i \le l$. We have

$$\int_{B_i^1} X_n \, dP = \int_{B_i^1} E^{p_i}(X_n) \, dP \ge (2b+a) P(B_i^1)/3 \tag{2}$$

since $E^{p_i}(X_n) \ge X_{p_i} - (b-a)/3 \ge (2b+a)/3$ on B_i^1. For $j \le m$, let $L_{i,j} = B_i^1 \cap C_j^1$, and let $M_i = B_i^1 \setminus \bigcup_{j \le m} L_{i,j}$. We have, since $L_{i,j} \in \Sigma_{q_j}$:

$$\int_{L_{i,j}} X_n \, dP = \int_{L_{i,j}} E^{q_j}(X_n) \, dP \le (b+2a) P(L_{i,j})/3$$

since $E^{q_j}(X_n) \le X_{q_j} + (b-a)/3 \le (b+2a)/3$ on $L_{i,j}$. Summation over $j \le m$ gives

$$\int_{B_i^1 \setminus M_i} X_n \, dP \le (b+2a) P(B_i^1 \setminus M_i)/3 \le (b+2a) P(B_i^1)/3. \tag{3}$$

Together with (2), we get

$$\int_{M_i} X_n \, dP \ge (b-a) P(B_i^1)/3. \tag{4}$$

Let $E = \bigcup_{i \le l} M_i$. Since $M_i \in \Sigma_{q_l}$, we have $E \in \Sigma_{q_l} = \Sigma_{n_2}$ and of course $E \cap D = \varnothing$. Moreover $P(E) = P(B^1 \setminus \bigcup_{j \le m} C_j^1) \le \varepsilon$. Finally, summation of (4) over $i \le l$ gives

$$\int_E X_n \, dP \ge (b-a) P(A)/12,$$

and that proves the claim.

To prove the theorem, we now construct by induction an increasing sequence n_p with the following property: whenever $D \in \Sigma_{n_p}$, $P(D) \leq P(A)/2$, and $n \geq n_{p+1}$, there exists $E \in \Sigma_{n_{p+1}}$ with $P(E) < 2^{-p}P(A)$, $E \cap D = \varnothing$, and $\int_E |X_n|\,dP \geq (b-a)P(A)/12$. Now let $n \geq n_p$. We construct by induction for $i \leq p$ disjoint sets $D_i \in \Sigma_{n_i}$, with $D_1 = \varnothing$, $P(D_i) \leq 2^{-i}P(A)$, and $\int_{D_i}|X_n|\,dP \geq (b-a)P(A)/12$. It follows that $E|X_n| \geq (p-1)(b-a)P(A)/12$. Hence $\lim_n E|X_n| = \infty$. The theorem is proved. □

4. Proof of Theorem 6 and construction of Example 7. The proof will be very similar to that of Theorem 4. Let (X_n) be an F-valued mil that converges scalarly to zero, but that does not converge a.s. to zero. We shall prove that $\lim_n E\|X_n\| = \infty$, and that will prove Theorem 6. We can find $a > 0$ and a set $A \in \Sigma$ with $P(A) > 0$ such that $\limsup\|X_n(\omega)\| > a$ for ω in A. The main point of the proof is the following:

CLAIM. Let $n_1 \in \mathbb{N}$, $\varepsilon > 0$. Then there exists $n_2 \geq n_1$ such that for each $D \in \Sigma_{n_1}$ with $P(D) \leq P(A)/2$, and each $n \geq n_2$, there exists $E \in \Sigma_{n_2}$ with $P(E) < \varepsilon$, $E \cap D = \varnothing$ such that $\int_E \|X_n\|\,dP \geq P(A)/16$.

PROOF. We first pick $k \geq n_1$ so large that for $n \geq k$ we have

$$P\Big(\sup_{k \leq p \leq n} \big\|E^p(X_n) - X_p\big\| > a/4\Big) < \varepsilon/2. \tag{5}$$

We can of course assume $\varepsilon < P(A)/4$. We then construct $k < p_1 < p_2 < \cdots < p_l$ such that if for $i \leq l$

$$B_i = \{\|X_{p_i}\| > a\} \setminus \bigcup_{j<i} \{\|X_{p_j}\| > a\}$$

we have $\sum_{i \leq l} P(B_i) > 7P(A)/8$. We note that $B_i \in \Sigma_{p_i}$. We can find a finite subset G of the unit ball of F^* such that if we set

$$B_i^1 = \{\exists\, x^* \in G;\ x^*(X_{p_i}) > a\} \cap B_i$$

we have $\sum_{i \leq l} P(B_i^1) > 7P(A)/8$. Note that $B_i^1 \in \Sigma_{p_i}$. Let n_2 be large enough that $P(C) < \varepsilon/2$ where

$$C = \{\exists\, x^* \in G;\ x^*(X_{n_2}) > a/4\}.$$

The existence of n_2 follows from the fact that we assume that (X_n) goes scalarly to zero. Now let $n \geq n_2$ and $D \in \Sigma_{n_1}$ with $P(D) < P(A)/2$. For $i \leq l$, let

$$H_i = \{\|E^{p_i}(X_n) - X_{p_i}\| > a/4\}.$$

It follows from (5) that $P(\bigcup_{i \leq l} H_i) < \varepsilon/2 < P(A)/8$. For $i \leq l$, we set $B_i^3 = B_i \setminus (H_i \cup D)$. We note that $B_i^3 \in \Sigma_{p_i}$ and that $\sum_{i \leq l} P(B_i^3) \geq P(A)/4$. Let

$$K' = \{\|E^{n_2}(X_n) - X_{n_2}\| > a/4\}.$$

If follows from (5) that $P(K') < \varepsilon/2$. Let $K = K' \cup C$. We have $K \in \Sigma_{n_2}$ and $P(K) < \varepsilon$. Let us enumerate the elements of G as $x_1^*, \ldots, x_m^*$. For $j \leq m$, let

$$S_{i,j} = B_i^3 \cap \{x_j^*(X_{p_i}) > a, \forall\, s < j, x_s^*(X_{p_i}) \leq a\}.$$

We have $S_{i,j} \in \Sigma_{p_i}$, and the sets $(S_{i,j})_j$ form a partition of B_i^3. We have, since $S_{i,j} \cap H_i = \varnothing$:

$$\int_{S_{i,j}} x_j^*(X_n)\,dP = \int_{S_{i,j}} x_j^*(E^{p_i}(X_n))\,dP > \int_{S_{i,j}} \left(x_i^*(X_{p_i}) - a/4\right) dP \tag{6}$$
$$> 3aP(S_{i,j})/4.$$

Since $S_{i,j} \setminus K \in \Sigma_{n_2}$, we have

$$\int_{S_{i,j}\setminus K} x_j^*(X_n)\,dP = \int_{S_{i,j}\setminus K} x_j^*(E^{n_2}(X_n))\,dP.$$

Since $\|E^{n_2}(X_n) - X_{n_2}\| \le a/4$ and $x_j^*(X_{n_2}) < a/4$ outside K, we get

$$\int_{S_{i,j}\setminus K} x_j^*(X_n)\,dP \le aP(S_{i,j})/2.$$

Together with (6) we get

$$\int_{S_{i,j}\cap K} \|X_n\|\,dP \ge \int_{S_{i,j}\cap K} x_j^*(X_n)\,dP \ge aP(S_{i,j})/4.$$

and summation over i and j gives

$$\int_E \|X_n\|\,dP \ge aP(A)/16,$$

where $E = K \cap \bigcup_i B_i^3$. And we have $P(E) < \varepsilon$ and $E \in \Sigma_{n_2}$, $E \cap D = \varnothing$. The claim is proved. The rest of the proof of Theorem 6 is identical to the end of the proof of Theorem 4. □

We now go to the construction of Example 7.

Using Dvoretzky's theorem, the following construction could be performed in any infinite dimensional Banach space; but, for simplicity we shall perform it only in l^2.

Let $a_n = \prod_{i \le n} 4^i$, Q_n be the partition of $[0,1]$ in a_n equal intervals, and Σ_n be the algebra generated by Q_n. We denote by (e_n) the canonical basis of l^2. We define X_n in the following way: For any interval I of Q_{n-1}, and for $j \le 2^n$, X_n is constant equal to $a_{n-2}2^n e_{2n+j}$ on the jth interval of Q_n that is contained in I and X_n is zero on the other intervals. Since $P(X_n \neq 0) = 2^{-n}$, we see that $\|X_n\| \to 0$. Also, since $E^n(\|X_{n+1}\|) = 1$, we see that $(\|X_n\|)$ is not a mil.

Now, if τ is a stopping time with $\tau \ge n$, we have

$$E^n(X_\tau) = X_n\chi_{\{\tau=n\}} + \sum_{q>n} E^n(X_q\chi_{\{\tau=q\}}).$$

It is clear that

$$\left\|\sum_{q>n} E^n(X_q\chi_{\{\tau=q\}})\right\| \le \sum_{q>n} \|E^n(X_q)\|$$

and direct computation gives $\|E^n(X_q)\| = 2^{-q/2}$. It follows that

$$\|E^n(X_\tau) - X_n\| < \sup_{q \geq n} \|X_q\| + 5 \cdot 2^{-n/2}$$

and this shows that (X_n) is a pramart since $X_n \to 0$.

5. Proof of Theorem 8. Let (X_n) be an F-valued mil such that $\liminf E\|X_n\| < \infty$. From Theorem 4 we see that for each x^* in F^*, the limit $\lim_p x^*(X_p)$ exists. Fatou's lemma shows that it is an integrable random variable. Without loss of generality, we can assume F separable. Let (x_n) be a dense sequence in F. For each n, let

$$V_n = \{x^* \in F^*;\ \|x^*\| \leq 1, \forall\, i \leq n,\ |x^*(x_i)| \leq 1/n\}.$$

For each x in F, write

$$\|x\|_n = \sup\{x^*(x);\ x^* \in V_n\}.$$

This is a norm on E, and for x in E, we have $\lim \|x\|_n = 0$. We now fix q in $\mathbb{N}$. Let $h = \liminf \|X_n\|$. By Fatou's lemma, h is integrable. Let $g = E^q h$. We note that for x^* in F^*, $\|x^*\| \leq 1$, we have $\lim_s x^*(X_s) \leq h$. We can define g_n by

$$g_n = \operatorname{ess\,sup}\left\{E^q \lim_s x^*(X_s);\ x^* \in V_n\right\}.$$

We have $g_n \leq g$. The sequence g_n is decreasing.

Main Fact. $\lim_n g_n = 0$ a.s.

The proof of the main fact follows the pattern of the proof of Theorems 4 and 6. If the Main Fact is not true, there is $a > 0$ and $A \in \Sigma$ with $P(A) > 0$ such that for each n, $g_n > a$ on A. Let $\eta > 0$ be such that for $D \in \Sigma$ with $P(D) < 3\eta$ we have $\int_D g\, dP < aP(A)/8$. We can assume $\eta < P(A)/2$.

Claim. Let $n_1 \in \mathbb{N}$ and $\varepsilon > 0$. Then there is $n_2 \geq n_1$ such that for $n \geq n_2$ and each $D \in \Sigma_{n_1}$, with $P(D) < \eta$ there exists $E \in \Sigma_{n_2}$, with $P(E) < \varepsilon$, $E \cap D = \varnothing$, and $\int_E \|X_n\|\, dP \geq aP(A)/8$.

Proof. We can assume $\varepsilon < \eta$. We pick $k \geq n_1$ so large that for $n \geq k$ we have

$$P\left(\sup_{k \leq p \leq n} \|E^p(X_n) - X_p\| > a/8\right) < \varepsilon/4. \tag{7}$$

Since for x in F we have $\lim_n \|x\|_n = 0$, there exists $t \geq k$ such that

$$P(\|X_k\|_t > a/6) < \varepsilon/4. \tag{8}$$

We know that $g_k > a$ on A, so there is a sequence $x_1^*, \ldots, x_l^*$ of V_t such that if for $i \leq l$ we set

$$B_i = \left\{E^q \lim_s x_i^*(X_s) > a;\ \forall\, j \leq i,\ E^q \lim_s x_j^*(X_s) \leq a\right\}$$

we have $\sum_{i \le l} P(B_i) > 3P(A)/4$. Note that $B_i \in \Sigma_q$. For $i \le l$, let $f_i = \lim_s x_i^*(X_s)$. We have $\lim_n E^n(f_i) = f_i$ a.s., so for n_2 large enough we have $P(K_1) < \varepsilon/2$, where

$$K_1 = \left\{ \sup_{i \le l} |x_i^*(X_{n_2}) - E^{n_2}(f_i)| > a/6 \right\}.$$

Now let $n \ge n_2$ and let $D \in \Sigma_{n_1}$ with $P(D) < P(A)/2$. Let

$$K_2 = \left\{ \|X_{n_2} - E^{n_2}(X_n)\| > a/6 \right\}$$

and let $K = K_1 \cup K_2$. Note that $K \in \Sigma_{n_2}$ and that (7) implies that $P(K) < \varepsilon$. Let

$$L = \left\{ \|E^k(X_n)\|_t > a/3 \right\}.$$

We have $L \in \Sigma_k$. Since $\|\cdot\|_t \le \|\cdot\|$, it follows from (7) and (8) that $P(L) < \varepsilon/2 < \eta$. Let $M = L \cup D$, so $M \in \Sigma_k$ and $P(M) < 2\eta$. For $i \le l$, let $B_i' = B_i \setminus M$. We have

$$\int_{B_i} E^q(f_i)\, dP = \int_{B_i} f_i\, dP \ge aP(B_i).$$

Since $|f_i| \le g$ a.s., we have, if $N = K \cup M = K \cup L \cup D$

$$\int_{B_i'} f_i\, dP \ge aP(B_i) - \int_{N \cap B_i} g\, dP.$$

On $B_i' \setminus K$, we have $|x_i^*(X_{n_2}) - E^{n_2}(f_i)| < a/6$ and $|x_i^*(X_{n_2}) - x_i^*(E^{n_2}(X_n))| < a/6$, so we have $|E^{n_2}(f_i) - x_i^*(E^{n_2}(X_n))| < a/3$. It follows that

$$\int_{B_i' \setminus K} x_i^*(E^{n_2}(X_n))\, dP > \int_{B_i' \setminus K} (E^{n_2}(f_i) - a/3)\, dP$$
$$= \int_{B_i' \setminus K} f_i\, dP - aP(B_i')/3$$

so

$$\int_{B_i' \setminus K} x_i^*(E^{n_2}(X_n))\, dP \ge 2aP(B_i)/3 - \int_{N \cap B_i} g\, dP.$$

On the other hand, since $B_i' \in \Sigma_k$, we have

$$\int_{B_i'} x_i^*(E^{n_2}(X_n))\, dP = \int_{B_i'} x_i^*(E^k(X_n))\, dP < aP(B_i)/3$$

so it follows that

$$\int_{K \cap B_i'} \|X_n\|\, dP \ge \int_{K \cap B_i'} x_i^*(E^{n_2}(X_n))\, dP \ge aP(B_i)/3 - \int_{N \cap B_i} g\, dP.$$

Since $P(N) < \varepsilon < \eta$, we get $\int_N g\, dP < aP(A)/8$. Let $E = \bigcup_{i \le l} K \cap B_i'$. We have $E \in \Sigma_{n_2}$, $P(E) < \varepsilon$, $E \cap D = \varnothing$, and by summation $\int_E \|X_n\|\, dP \ge aP(A)/8$. The claim is proved.

To show that the claim implies the Main Fact, one proceeds as in the proof of Theorem 4, to show that the claim implies $\liminf E\|X_n\| = \infty$, a contradiction. The Main Fact is proved.

Let us fix $k > 0$. Egoroff's theorem shows that there is $B_k \in \Sigma_q$ with $P(B_k) > 1 - 1/k$ such that g_n goes to zero uniformly on B_k. Consider now the operator T from F^* to $L^\infty = L^\infty(B_k, \Sigma_q|B_k, P|B_k)$ given by $T(x^*) = E^q(\lim_s x^*(X_s))$. The restriction of T to the unit ball of F^* is weak* to norm continuous. It follows that there is a compact operator V from $L^1(B_k, \Sigma_q|B_k, P|B_k)$ to F such that $T = V^*$. Since V is compact, it is representable [1], that is there is a bounded Bochner measurable W^k: $B_k \to F$ such that $V(\phi) = \int \phi W^k dP$ for $\phi \in L^1$. It follows that $x^* \circ W^k = T(x^*)$ for $x^* \in F^*$. Patching the pieces we get a Σ_q-measurable map Y_q such that $\|Y_q\| \le g$ and $x^* \circ Y_q = E^q \lim_s x^*(X_s)$ for $x^* \in F^*$. It is obvious that (Y_k) is a martingale, that is L^1-bounded. If we set $Z_n = X_n - Y_n$, we see that Z_n is a mil with $\liminf E\|Z_n\| < \infty$, and that Z_n goes scalarly to zero. So Theorem 6 shows that (Z_n) goes in norm to zero, and Theorem 8 is proved. □

6. Proof of Theorem 10. It follows from Theorem 4 that (X_n) converges a.s. Let X be its limit. Let $Y_n = E^n(X)$. Then Y_n is an equi-integrable martingale, and $Z_n = X_n - Y_n$ is a pramart such that $\liminf E|Z_n| < +\infty$ and Z_n goes to zero a.s.

MAIN FACT. For each $\varepsilon > 0$, there is p such that

$$\forall\, p \le q \le n, \qquad P\big(E^q(|Z_n|) > \varepsilon\big) < \varepsilon.$$

Otherwise, there is $a > 0$ such that for each p, there exists $p \le q \le n$ with $P(E^q(|Z_n|) > 2a) > 2a$.

CLAIM. Let $n_1 \in \mathbb{N}$ and $D \in \Sigma_{n_1}$ with $P(D) < a/2$, and let $\varepsilon > 0$. Then there exists n_2 and $E \in \Sigma_{n_2}$ with $P(E) < \varepsilon$, $E \cap D = \varnothing$ such that for each $n \ge n_2$ we have $\int_E |Z_n|\, dP \ge a^2/16$.

PROOF. We can assume $\varepsilon < a/4$. Let $p \ge n_1$ be such that we have

$$P\Big(\sup_{p \le q \le \tau} |E^q(Z_\tau)| > a/4\Big) < \varepsilon.$$

Let $n_2 \ge q \ge p$ with $P(E^q(|Z_{n_2}|) > 2a) > 2a$. For definiteness, assume $P(A) > a$, where $A = \{E^q(Z_{n_2}^+) > a\}$. Let

$$E_1 = \{Z_{n_2} < -a/4\},\ H = \Big\{\sup_{q \le \tau} |E^q(Z_\tau)| \ge a/4\Big\}$$

and let $E = (A \cap E_1) \setminus (H \cup D)$. Then $P(E) < \varepsilon$, $E \cap D = \varnothing$, $E \in \Sigma_{n_2}$. Fix $n \ge n_2$, and let τ be the stopping time given by $\tau = n_2$ on $X \setminus E_1$ and $\tau = n$ on E_1. Let $A' = A \setminus (H \cup D)$, so $P(A') \ge a/4$. Let $B = \{-a/4 < Z_{n_2} < 0\}$. We get

$$E^q(Z_\tau) = E^q\big(Z_{n_2}^+\big) + E^q\big(Z_{n_2}\chi_B\big) + E^q\big(Z_n \chi_{E_1}\big).$$

On A' we have $|E^q(Z_\tau)| \le a/4$, $E^q(Z_{n_2}^+) > a$, and $|E^q(Z_{n_2}\chi_B)| < a/4$, so integration over A' yields

$$\int_{A'} E^q(Z_{n_2}\chi_{E_1}) \geq aP(A')/4 \geq a^2/16,$$

so

$$\int_E Z_{n_2}\,dP = \int_{A'} Z_{n_2}\chi_{E_1}\,dP \geq a^2/16,$$

and this proves the claim. The Main Fact follows from the claim and the hypothesis that $\liminf E(X_n) < \infty$.

We now complete the proof of Theorem 10. If τ is a stopping time greater than q, let $A = \{Z_\tau > 0\}$, let $n \geq \tau$, and let τ' be the stopping time given by $\tau' = \tau\chi_A + n\chi_{X\setminus A}$. We have

$$E^q(Z_\tau^+) = E^q(Z_\tau\chi_A) = E^q(Z_{\tau'}) - E^q(Z_n\chi_{X\setminus A})$$

so

$$|E^q(Z_\tau^+)| \leq |E^q(Z_{\tau'})| + E^q(|Z_n|).$$

A similar argument with $\tau'' = \tau\chi_{X\setminus A} + n\chi_A$ shows

$$E^q(|Z_\tau|) \leq |E^q(Z_{\tau'})| + |E^q(Z_{\tau''})| + 2E^q(|Z_n|).$$

Observe that the set of $E^q(|Z_\tau|)$ for $\tau \geq q$, $\tau \in T$ is filtering increasing. Let $h_q = \operatorname{ess\,sup}_{\tau \geq q} E^q(|Z_\tau|)$. Then the main fact shows that the h_q goes to zero in measure. Given $\varepsilon > 0$, there is q such that $P(A_q) \geq 1 - \varepsilon$, where $A_q = \{h_q < 1\}$. But it is obvious that on A_q, the sequence $(h_n)_{n \geq q}$ is a submartingale with respect to the restriction of Σ_n to A_q. So it converges a.s. on A_q. This shows that (h_q) converges a.s. so it must converge to zero. Hence $(|Z_n|)$ is a pramart, and the result is proved. □

7. Proof of Theorem 11 and construction of Example 12. We first prove the necessity. Let (T_n) be a positive pramart that goes to zero. For each n, notice that the set of functions $E^n(X_\tau)$ for $\tau \geq n$ is upward directed. Denote h_n its essential supremum. Then $h_n \to 0$ a.s. Given $\varepsilon > 0$, there is $p > 0$ such that $P(A) > 1 - \varepsilon$, where $A = \{h_p \leq \varepsilon\}$. For $m \geq n \geq p$, it is clear that $E^n(h_m\chi_A) \leq h_n$, that is $(h_n)_{n \geq p}$ is a supermartingale on the space $(A, \Sigma|A, P|A)$, and $h_n \geq T_n$ on A.

We now prove the sufficiency. Let ε, A, p be as is in the statement of the theorem. For $\tau \geq n \geq p$ we have $T_\tau \leq S_\tau$ on A, so $E^n(T_\tau) \leq E^n(S_\tau) \leq S_n$. For each n, let $h_n = \operatorname{ess\,sup}_{\tau \geq n} E^n(T_\tau)$. For $n \geq p$, we have $h_n \leq S_n$ on A, so $\limsup_n S_n \leq \limsup_n S_n$ on A. It follows that

$$\int_A \limsup h_n\,dP \leq \varepsilon.$$

As ε is arbitrary, we have $\lim h_n = 0$ a.s. Since $T_n \to 0$ it follows that (T_n) is a pramart, and the theorem is proved.

We now construct Example 12. Let $a_n = \prod_{i \leq n} 2^i$, and let Q_n be the partition of $[0,1]$ in a_n intervals of equal length. Let Σ_n be the algebra generated by Q_n.

For an interval I of Σ_{n-1}, we define $X_n = 2^{n-1}$ on the first interval of Q_n that is contained in I, $X_n = -2^{n-1}$ on the second interval of Q_n that is contained in I, and $X_n = 0$ elsewhere. We have $P(X_n \neq 0) = 2^{-n+1}$, so $X_n \to 0$ a.s. We have $E^n(X_m) = 0$ for $n < m$, so (X_n) is a martingale in the limit. However, $E^n(|X_{n+1}|) = 1$, so $|X_n|$ is not a mil.

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