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A MARTINGALE APPROACH TO SUPERCRITICAL (CMJ) BRANCHING PROCESSES

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A new method of tackling convergence properties of random processes turns out to be applicable to finite mean supercritical age-dependent branching processes. If $\{Z_t^\phi\}$ is a Crump–Mode–Jagers process counted with general characteristics ϕ, convergence in probability of $\{e^{-\alpha t}Z_t^\phi\}$ follows from convergence in distribution. Under some mild restrictions on ϕ, norming constants $\{C(t)\}$ are identified such that $\{C^{-1}(t)Z_t^\phi\}$ converges almost surely to a nondegenerate limit.

1. Introduction. Let us consider the age-dependent model defined by Jagers in Chapter 6 of [11], which is now known as the (CMJ) process, where (CMJ) stands for Crump–Mode–Jagers. The data of the model consist of a random point process ξ on $[0,\infty)$ ruling the reproduction ages of an individual, the life-length variable λ, and a random characteristic process $\{\phi(t)\}$. Write G for the distribution function of the life-length, i.e., $G(u) = P(\lambda \le u)$ and $\xi(t)$ for the ξ measure of $[0,t]$, i.e., $\xi(t) = \xi([0,t])$. Further $\mu = E(\xi)$ is to denote the intensity measure of ξ and $\mu(t) = E(\xi(t))$ is the so-called reproduction function, which we assume to be nonlattice. Suppose $P(\xi(\infty) < \infty) = 1$.

We shall throughout assume the following conditions on $\xi(t)$:

(i) There exists a Malthusian parameter $\alpha \in (0,\infty)$, i.e., a finite positive solution of the equation

$$\int_0^\infty e^{-\alpha t}\mu(dt) = 1.$$

(ii) The first moment of $e^{-\alpha t}\mu(dt)$ is finite, i.e.,

$$\int_0^\infty u e^{-\alpha u}\mu(du) < \infty.$$

We shall require that $\{\phi(t)\}$ be a product-measurable separable nonnegative random process and, define a (CMJ) process by

$$Z_t^\phi = \sum_{i=0}^{T_t} \phi_i(t - \sigma_{(i)}), \tag{1.1}$$

where $\sigma_{(0)} = 0$ is the birth time of the ancestor and $\sigma_{(i)}$, $i \ge 1$, is the birth time of the ith of its T_t descendants that have been born up to and including t. The $\{\phi_i(t), \lambda_i, \xi_i(t)\}$ are i.i.d. copies of $\{\phi(t), \lambda, \xi(t)\}$. Write $\mathscr{A}_{T_t}$ for the σ field generated by the biographies of the ancestor and its first T_t descendants. An

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ordinary Crump–Mode–Jagers process $\{Z_t\}$, introduced independently by Crump and Mode [7] and Jagers [10], is obtained by specializing $\phi(t)$ to $\phi(t) = 1$ if $t \leq \lambda$ and 0 otherwise. In this case Z_t^ϕ counts the number of individuals alive at time t. If reproduction is allowed only at the time of death of an individual, the resulting model is the so-called Sevastyanov process. If in addition λ and $\xi(\infty)$ are independent, one obtains the Bellman–Harris process.

We shall now assume that $\{\phi(t)\}$ satisfies the following:

CONDITION 1.1. The function $E(\phi(t))$ is continuous a.e. with respect to Lebesgue measure and

$$\sum_{k=0}^{\infty} \sup_{k \leq t < k+1} e^{-\alpha t} E(\phi(t)) < \infty. \tag{1.2}$$

It is known that (1.2) permits application of a renewal argument (see [11] and [14]) that yields

$$\lim_{t \to \infty} m_t^\phi = m_\infty^\phi = \frac{\int_0^\infty e^{-\alpha t} E(\phi(t))\, dt}{\int_0^\infty t \mu_\alpha(dt)}, \tag{1.3}$$

where $m_t^\phi = E(e^{-\alpha t} Z_t^\phi)$ and $\mu_\alpha(t) = \int_0^t e^{-\alpha s} \mu(ds)$.

We shall be concerned here with the limit behaviour of $\{C^{-1}(t) Z_t^\phi\}$ for suitably chosen constants $\{C(t)\}$. Problems of this kind have been studied by Jagers [11] under second moment assumptions, and by Nerman [14] under rather general assumptions. For an updated account and more results see Asmussen and Hering [3].

We shall show that the Laplace transform Φ of the limit distribution of $\{C^{-1}(t) Z_t^\phi\}$, or a subsequence thereof, satisfies the functional equation

$$\Phi(u) = E\left[\prod_{i=1}^{\infty} \Phi(u e^{-\alpha \sigma_{[i]}})\right], \tag{1.4}$$

where $\sigma_{[i]}$ is the birth time of the ith child of the ancestor.

This is the equation satisfied by the limit variable in an ordinary Crump–Mode–Jagers process [8]. A useful consequence of (1.4) is that any nondegenerate weak limit of $\{C^{-1}(t) Z_t^\phi\}$ must be a proper distribution.

Convergence in distribution turns out to be equivalent with convergence in probability for $\{e^{-\alpha t} Z_t^\phi\}$. Nondegenerate limits of this kind occur only if

$$E\left[{}_\alpha\xi(\infty) \log_\alpha^+ \xi(\infty)\right] < \infty, \tag{1.5}$$

where

$${}_\alpha\xi(\infty) = \int_0^\infty e^{-\alpha u} \xi(du).$$

We further derive a.s. convergence for $\{C^{-1}(t) Z_t^\phi\}$ for suitably chosen constants $\{C(t)\}$ and a slightly stronger condition on ϕ in a case that includes

$$E\left[{}_\alpha\xi(\infty) \log_\alpha^+ \xi(\infty)\right] = \infty. \tag{1.6}$$

The result under (1.6) parallels the one solved for the Bellman–Harris process in [4]. For an alternative proof and further results see [15].

The key idea of the approach rests upon the identification of a martingale derived from a (weakly) convergent subsequence of the process to be shown to converge. In the case (1.6) our proof seems simpler than those given in [4] and [5] for the less complex model of a Bellman–Harris process.

2. Outline of the proof. Suppose that $\{X_t\}$ with $t \in [0, \infty)$ is a random process that we would like to show to converge in probability or a.s. Assume that X_t is $\mathscr{F}_t$-measurable for some nondecreasing σ fields $\{\mathscr{F}_t\}$ and that we can choose x and $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} P(X_{t_n} \le x) = \gamma$, $0 < \gamma < 1$, and $\eta_t = \lim_{n\to\infty} P(X_{t_n} \le x | \mathscr{F}_t)$ a.s. exists for all t. Then $\{\eta_t\}$ is a martingale. If the limit of $\{\eta_t\}$ is identified to be 1_{Λ_x} for some event Λ_x, where 1_{Λ_x} denotes the indicator function of Λ_x, then we shall prove that $\lim_{n\to\infty} P(\{X_{t_n} \le x\} \Delta \Lambda_x) = 0$, Δ being the symbol of symmetric difference of two sets. Such a property for every x and $\{t_n\}$ entails convergence in probability for $\{X_t\}$.

Assuming that 1_{Λ_x} is the a.s. limit of $\{\eta_t\}$, the martingale property of $\{\eta_t\}$ yields $\lim_{t\to\infty} 1_{\{\eta_t > \delta\}} = 1_{\Lambda_x}$ a.s. for any constant δ with $0 < \delta < 1$, whereas if we show that for any x $P(\limsup_{t\to\infty}\{\eta_t > \delta\}\Delta\{X_t \le x\}) = 0$, then $\lim_{t\to\infty} 1_{\{X_t \le x\}} = 1_{\Lambda_x}$ a.s., which turns out to imply a.s. convergence for $\{X_t\}$.

In the case under discussion $X_t = C^{-1}(t) Z_t^{\phi}$ and $\{X_{t_n}\}$ is chosen to converge in distribution to a nondegenerate limit F. We also show that

$$\eta_t = P\left(\sum_{j \in \mathscr{I}(t)} W_j e^{-\alpha(\sigma_j - t)} \le x e^{\alpha t} \middle| \mathscr{A}_{T_t} \right) \quad \text{a.s.,} \tag{2.1}$$

where $\{W_j\}$ are i.i.d. copies of a random variable distributed according to F and independent of $\mathscr{A}_{T_t}$, $\mathscr{I}(t)$ is the set of individuals to be born after t whose mothers are born before or at t, and σ_i is the birth time of the ith individual of $\mathscr{I}(t)$. Thus, proving convergence in probability or a.s. convergence will boil down to dealing with (2.1) as $t \to \infty$, and this, in turn, will involve the limit properties of $\sum_{j\in\mathscr{I}(t)} e^{-\alpha(\sigma_j - t)}$ and $\sum_{j \in \mathscr{I}(t)} e^{-\alpha t} W_j$.

3. A key martingale. The results to be further derived rely heavily on the following:

THEOREM 3.1. *Suppose that $\{X_t\}$ is a random process with $t \in [0,\infty)$, X_t is $\mathscr{F}_t$-measurable, and $\mathscr{F}_t \subseteq \mathscr{F}_s$ for $t < s$. Assume further that there exists a real x and a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $\lim_{n\to\infty} P(X_{t_n} \le x|\mathscr{F}_t) = \eta_t$ (say) a.s. exists for any $t \in [0,\infty)$. Then*

(i) *$\{\eta_t\}$ is a martingale with respect to $\{\mathscr{F}_t\}$;*
(ii) *if $\lim_{t\to\infty} \eta_t = 1_\Lambda$ a.s. for some event Λ, then $\lim_{n\to\infty} P(\{X_{t_n} \le x\}\Delta\Lambda) = 0$ and $\eta_t = P(\Lambda|\mathscr{F}_t)$ a.s. for any $t \in [0,\infty)$.*

PROOF. Choose $t_n > s > t \ge 0$. Then $\mathscr{F}_t \subseteq \mathscr{F}_s$ in conjunction with an elementary property of conditional expectations yields

$$E\left[P(X_{t_n} \le x|\mathscr{F}_s)|\mathscr{F}_t\right] = P(X_{t_n} \le x|\mathscr{F}_t) \quad \text{a.s.} \tag{3.1}$$

Taking $n \to \infty$ in (3.1) gives

$$E(\eta_s|\mathscr{F}_t) = \eta_t \quad \text{a.s. for } s > t \tag{3.2}$$

and (i) is proved.

To prove (ii) notice that $\lim_{t\to\infty}\eta_t = 1_\Lambda$ a.s. implies

$$\lim_{t\to\infty}\int_{\{\eta_t>\delta\}}\eta_t\,dP = P(\Lambda) \tag{3.3}$$

for any constant δ with $0 < \delta < 1$.

On the other hand

$$\begin{aligned}\int_{\{\eta_t>\delta\}}\eta_t\,dP &= \int_{\{\eta_t>\delta\}}\lim_{n\to\infty}P(X_{t_n}\le x|\mathscr{F}_t)\,dP\\ &= \lim_{n\to\infty}\int_{\{\eta_t>\delta\}}P(X_{t_n}\le x|\mathscr{F}_t)\,dP\\ &= \lim_{n\to\infty}P(\{X_{t_n}\le x\}\cap\{\eta_t>\delta\}).\end{aligned} \tag{3.4}$$

Further, $\lim_{t\to\infty}1_{\{\eta_t>\delta\}} = 1_\Lambda$ a.s., (3.3), and (3.4) together yield

$$\begin{aligned}&\lim_{t\to\infty}\lim_{n\to\infty}P(\{X_{t_n}\le x\}\cap\{\eta_t>\delta\})\\ &\quad= \lim_{n\to\infty}P(\{X_{t_n}\le x\}\cap\Lambda) = P(\Lambda).\end{aligned} \tag{3.5}$$

Since $\{\eta_t\}$ is a martingale, $E(\eta_t) = \lim_{n\to\infty}P(X_{t_n}\le x) = P(\Lambda)$ which combined with (3.5) leads to $\lim_{n\to\infty}P(X_{t_n}\le x\}\Delta\Lambda) = 0$. The latter equality is easily seen to be equivalent with the convergence in probability of $\{1_{\{X_{t_n}\le x\}}\}$ to 1_Λ as $n\to\infty$, and invoking now the dominated convergence theorem for conditional expectations we get

$$\lim_{n\to\infty}P(X_{t_n}\le x|\mathscr{F}_t) = \lim_{n\to\infty}E(1_{\{X_{t_n}\le x\}}|\mathscr{F}_t) = E(1_\Lambda|\mathscr{F}_t) = P(\Lambda|\mathscr{F}_t) \quad \text{a.s.}$$

Thus $\eta_t = P(\Lambda|\mathscr{F}_t)$ a.s. and the proof is complete. □

We shall next identify $\{\eta_t\}$ in the case of a (CMJ) process. We need consider two conditions on $\{Z_t^\phi\}$.

Condition 3.1. The random process $\{e^{-\alpha t}Z_t^\phi\}$ converges weakly to a limit F with $F(0) < 1$.

Condition 3.2. There are some constants $\{C(t_n)\}$ with $\lim_{n\to\infty}C(t_n) = \infty$ such that $\{C^{-1}(t_n)Z_{t_n}^\phi\}$ converges weakly to a limit F with $F(0) < 1$, $\{(Z_t^\phi)^{-1}Z_{t+s}^\phi\}$ converges in probability to $e^{\alpha s}$ as $t\to\infty$ on $\{T_t\to\infty\}$ for any $s > 0$, and $\{C^{-1}(t_n)\phi(t_n)\}$ converges in probability to 0 as $n\to\infty$.

Proposition 3.1. *Suppose that either Condition* 3.1 *or Condition* 3.2 *holds, and write* $\eta_t = \lim_{s\to\infty}P(e^{-\alpha s}Z_s^\phi \le x|\mathscr{A}_{T_t})$ *if Condition* 3.1 *holds or* $\eta_t =$

$\lim_{n\to\infty} P(C^{-1}(t_n)Z_{t_n}^{\phi} \le x|\mathscr{A}_{T_t})$ *if Condition* 3.2 *holds, x being a continuity point of F. Then the limit defining η_t exists a.s. with*

$$\eta_t = P\Bigg(\sum_{j\in\mathscr{I}(t)} W_j e^{-\alpha\sigma_j} \le x|\mathscr{A}_{T_t}\Bigg) \quad a.s., \tag{3.6}$$

where $\mathscr{I}(t)$ is the set of individuals to be born after t whose mothers are born before or at t, σ_j the birth time of the jth individual of $\mathscr{I}(t)$, and $\{W_j\}$ some i.i.d. copies of a random variable distributed according to F and independent of $\mathscr{A}_{T_t}$.

REMARK. The formula (3.6) follows from a convergence in distribution property, and $\{W_j\}$ are *any* i.i.d. random variables with distribution function F, independent of $\mathscr{A}_{T_t}$. We can always extend, if necessary, the initial probability space on which $\{Z_t^{\phi}\}$ is defined to make it accommodate such a sequence of random variables. Thus, $\{\sum_{j\in\mathscr{I}(t)} W_j e^{-\alpha\sigma_j} \le x\}$ may be considered an event in (3.6).

PROOF. Let $s > t \ge 0$ and notice that if $\{{}_{(j)}Z_t^{\phi}\}$ are i.i.d. copies of $\{Z_t^{\phi}\}$, independent of $\mathscr{A}_{T_t}$, then

$$\begin{aligned} &P\big(e^{-\alpha s}Z_s^{\phi} \le x|\mathscr{A}_{T_t}\big) \\ &\quad = P\Bigg(\sum_{i=0}^{T_t} e^{-\alpha s}\phi_i(s-\sigma_{(i)}) + \sum_{j\in\mathscr{I}(t)} e^{-\alpha s}{}_{(j)}Z_{s-\sigma_j}^{\phi} \le x|\mathscr{A}_{T_t}\Bigg). \end{aligned} \tag{3.7}$$

Notice now that Condition 1.1, in conjunction with the Markov inequality, implies convergence in probability to 0 of $\{e^{-\alpha t}\phi_0(t)\}$ as $t\to\infty$, which shows that $\sum_{i=0}^{T_t} e^{-\alpha s}\phi_i(s-\sigma_{(i)})$ has no contribution to (3.7) as $s\to\infty$.

Letting $s\to\infty$ in (3.7) yields

$$\begin{aligned} &\lim_{s\to\infty} P\big(e^{-\alpha s}Z_s^{\phi} \le x|\mathscr{A}_{T_t}\big) \\ &\quad = \lim_{s\to\infty} P\Bigg(\sum_{j\in\mathscr{I}(t)} e^{-\alpha(s-\sigma_j)}{}_{(j)}Z_{s-\sigma_j}^{\phi} e^{-\alpha\sigma_j} \le x|\mathscr{A}_{T_t}\Bigg) \quad \text{a.s.} \\ &\quad = P\Bigg(\sum_{j\in\mathscr{F}(t)} W_j e^{-\alpha\sigma_j} \le x|\mathscr{A}_{T_t}\Bigg) \quad \text{a.s.} \end{aligned}$$

and (3.6) is proved under Condition 3.1.

The proof under Conditions 3.2 is similar and will be left to the reader. □

4. A functional equation. The variable Z_t^{ϕ} can be expressed as

$$Z_t^{\phi} = \phi_0(t) + \sum_{i=1}^{\xi_0(t)} {}_{(i)}Z_{t-\sigma_{[i]}}^{\phi} \tag{4.1}$$

where $\{{}_{(i)}Z_t^{\phi}\}$ is the ϕ-counted process of i descendants initiated under the assumption that i is born at 0. The processes $\{{}_{(i)}Z_t^{\phi}\}$, $i = 1, 2, \dots$ are independent copies of $\{Z_t^{\phi}\}$, independent of ϕ_0 and ξ_0.

Denote by Φ the Laplace transform of F.

THEOREM 4.1. *Suppose that either Condition* 3.1 *or Condition* 3.2 *holds. Then*

$$\Phi(u) = E\left[\prod_{i=1}^{\xi(\infty)} \Phi\left(ue^{-\alpha\sigma_{[i]}}\right)\right]. \tag{4.2}$$

PROOF. Suppose that Condition 3.1 holds. Then by (4.1)

$$e^{-\alpha t}Z_t^{\phi} = e^{-\alpha t}\phi(t) + \sum_{i=1}^{\xi_0(t)} e^{-\alpha(t-\sigma_{[i]})}{}_{(i)}Z^{\phi}_{t-\sigma_{[i]}}e^{-\alpha\sigma_{[i]}}. \tag{4.3}$$

As we have noticed in the course of the proof of Proposition 3.1, $\{e^{-\alpha t}\phi(t)\}$ converges in probability to 0 as $t \to \infty$, and (4.2) obtains on letting $t \to \infty$ in $E[\exp(-ue^{-\alpha t}Z_t^{\phi})]$.

The proof under Condition 3.2 is similar and will be omitted. □

COROLLARY 4.1. *If either Condition* 3.1 *or Condition* 3.2 *holds, then F is a proper distribution function, and $F(0) = q$, where q is the extinction probability of $\{Z_t\}$.*

PROOF. Letting $u \downarrow 0$ and $u \uparrow \infty$ in (4.2) yields

$$F(0) = E\left[F^{\xi(\infty)}(0)\right] \quad \text{and} \quad F(\infty) = E\left[F^{\xi(\infty)}(\infty)\right]. \tag{4.5}$$

Further, $F(0) < 1$ and Fatou's lemma applied to $\{\exp(-ue^{-\alpha t}Z_t^{\phi})\}$ imply $F(\infty) = 1 > F(0)$. However, in view of (4.5), both $F(0)$ and $F(\infty)$ are solutions to the equation $s = f(s)$, where $f(s) = \sum_{n=0}^{\infty} P(\xi(\infty) = n)s^n$. Since $E(\xi(\infty)) > 1$, by a well-known result in the branching processes theory, $s = f(s)$ has exactly one solution q in $(0, 1)$. It follows that $F(0) = q$ and $F(\infty) = 1$, where q was shown (p. 140 of [11]) to be the extinction probability of $\{Z_t\}$. □

Doney [8] proved that if (1.5) holds, then any nondegenerate solution to (4.2) is absolutely continuous on $(0, \infty)$, admits finite expectation, and is unique among the distributions with a given expectation.

5. Convergence in probability for normed $\{Z_t^{\phi}\}$.

The object of this section is to show that convergence in distribution for $\{e^{-\alpha t}Z_t^{\phi}\}$ implies convergence in probability.

According to Theorem 4.1 and the already mentioned result by Doney [8], $\{e^{-\alpha t}Z_t^{\phi}\}$ may admit a nondegenerate limit distribution only if (1.5) holds. An important ingredient in what follows is the martingale $\{Y_t\}$ with $Y_t = \sum_{j \in \mathscr{I}(t)} e^{-\alpha\sigma_j}$ identified by Nerman (Corollary 2.5 of [14]). We state next Nerman's result for further reference.

LEMMA 5.1. *$\{Y_t\}$ is a martingale with $E(Y_t) = 1$, and $Y_\infty = \lim_{t\to\infty} Y_t$ a.s. exists.*

We shall need the following result on weighted sums of independent random variables.

Lemma 5.2. *Suppose that* $\{W_j\}$ *is a sequence of independent identically distributed random variables,* $E(W_1) < \infty$, *and* $\{c_i^{(n)}\}$ *are some nonnegative constants with* $c_i^{(n)} \le 1$ *for all* n *and* i. *If* $\{C_n\}$ *are some constants with* $\lim_{n\to\infty} C_n = \infty$ *and* $\lim_{n\to\infty} C_n^{-1}(c_1^{(n)} + \cdots + c_n^{(n)}) = c$ *for some* $c \ge 0$, *then* $\{C_n^{-1}(c_1^{(n)}W_1 + \cdots + c_n^{(n)}W_n)\}$ *converges in probability to* $cE(W)$ *as* $n \to \infty$.

The proof of Lemma 5.2 can be carried out as in [13] (Theorem 1).

Theorem 5.1. *Suppose that* $\{e^{-\alpha t}Z_t^{\phi}\}$ *converges in distribution to a nondegenerate limit* F *as* $t \to \infty$. *Then* $\{e^{-\alpha t}Z_t^{\phi}\}$ *converges in probability to* $E(W)Y_\infty$ *as* $t \to \infty$, *where*

$$E(W) = \int_0^\infty xF(dx).$$

Proof. By Proposition 3.1

$$\eta_t = P\Big(\sum_{j\in\mathscr{I}(t)} W_j e^{-\alpha\sigma_j} \le x \,|\, \mathscr{A}_{T_t}\Big) = P\Big(e^{-\alpha t}\sum_{j\in\mathscr{I}(t)} W_j e^{-\alpha(\sigma_j - t)} \le x \,|\, \mathscr{A}_{T_t}\Big) \tag{5.2}$$

and applying Lemmas 5.2 and 5.1 to (5.2) yields

$$\lim_{t\to\infty} \eta_t = 1_{\{E(W)Y_\infty \le x\}} \quad \text{a.s.} \tag{5.3}$$

Further, Theorem 3.1(ii), in view of (5.3), applies and yields $\lim_{t\to\infty} P(\{e^{-\alpha t}Z_t^{\phi} \le x\}\Delta\{E(W)Y_\infty \le x\}) = 0$ for any $x > 0$, which is tantamount to the convergence in probability of $\{e^{-\alpha t}Z_t^{\phi}\}$ to $E(W)Y_\infty$ as $t \to \infty$. □

Another Proof. The a.s. convergence of $\{Y_t\}$ to Y_∞ with $E(Y_\infty) < \infty$ suffices for the proof of Theorem 5.1, and may be seen to follow from Theorem 3.1(i) without any appeal to Lemma 5.1. Indeed, choose a subsequence of $\{Y_t\}$, say $\{Y_{t_n}\}$, converging for a given ω to a limit Y_∞, where $\lim_{n\to\infty} t_n = \infty$. Combining (5.2) and Lemma 5.2 yields $\lim_{n\to\infty}\eta_{t_n} = 1_{\{E(W)Y_\infty \le x\}}$. According to Theorem 3.1 (i), the martingale $\{\eta_t\}$ converges as $t \to \infty$ for almost all ω, which boils down to (5.3). Hereafter the proof may be continued as before.

Remarks. Nerman [14] has given some conditions on $\{\phi(t)\}$ guaranteeing convergence in probability of $\{e^{-\alpha t}Z_t^{\phi}\}$ to $E(W)Y_\infty$ as $t \to \infty$. To show that $P(Y_\infty > 0) > 0$ it suffices to produce one random characteristic process $\{\phi(t)\}$ such that $\{e^{-\alpha t}Z_t^{\phi}\}$ converges in distribution to a nondegenerate limit F. Such a case is the ordinary Crump–Mode–Jagers process mentioned in the Introduction (see Doney [8]). This argument was invoked by Nerman [14] and goes back to Athreya and Kaplan [2] who used Athreya's result of [1] when proving a.s. convergence in the Bellman–Harris case. Asmussen and Hering [3] used a different approach based on a Kesten–Stigum-type result (Chapter X, Theorem 4.1). For another proof see Jagers and Nerman [12]. In a remark at the end of Section 6, we shall indicate yet another way of showing that $P(Y_\infty > 0) = 1 - q > 0$.

6. Almost sure convergence for normed $\{Z_t^\phi\}$. We shall identify a class of random characteristic processes $\{\phi(t)\}$ and norming constants $\{C(t)\}$ such that $\{C^{-1}(t)Z_t^\phi\}$ converges a.s. to a nondegenerate limit, under the assumption $E[{}_\alpha\xi(\infty)\log_\alpha^+\xi(\infty)] \leq \infty$. For the proof we shall need some results of [14] and [4]. The following proposition is due to Nerman [14] (see also [3] p. 375).

LEMMA 6.1. *Let ϕ, ψ be characteristics with sample functions which are right-continuous and satisfy $E(\sup_t e^{-\beta t}\phi(t)) < \infty$ and $E(\sup_t e^{-\beta t}\psi(t)) < \infty$ for some $0 \leq \beta < \alpha$ and suppose that $\mu_\gamma(\infty) < \infty$ for some $0 \leq \gamma < \alpha$. Then, on $\{T_t \to \infty\}$*

$$\frac{Z_t^\phi}{Z_t^\psi} \to \frac{m_\infty^\phi}{m_\infty^\psi} \quad \text{a.s.} \quad \text{as } t \to \infty.$$

The following result is contained in Lemma 7 of [4]; see also [6].

LEMMA 6.2. *Suppose that $\{c_i^{(n)}; i = 1, \ldots, n\}$ are some nonnegative constants with $c_i^{(n)} \leq 1$ for all n and i, and $\{W_j\}$ are independent and identically distributed random variables. Write $S_n = \sum_{i=1}^n W_i$, $T_n = \sum_{i=1}^n c_i^{(n)} W_i$, and $V_n = \sum_{i=1}^n c_i^{(n)}$. Assume that $\{V_{n_k}/n_k\}$ converges to c as $k \to \infty$ with $c \geq 0$, where $\{n_k\}$ is a sequence of integers with $\lim_{k\to\infty} n_k = \infty$, and $\{S_{n_k}/b_k\}$ converges in probability to m as $k \to \infty$ for some constants $\{b_k\}$ and m. Then $\{T_{n_k}/b_k\}$ converges in probability to cm as $k \to \infty$.*

Lemma 6.2 expresses a result similar to Lemma 5.2, but unlike Lemma 5.2 it may accommodate the case $E(W) = \infty$ which appears when $E[{}_\alpha\xi(\infty)\log_\alpha^+\xi(\infty)] = \infty$. As in [4] we shall define the norming constants $\{C(t)\}$ to be the γ quantiles of $\{Z_t^\phi\}$, i.e., $C(t)$ is the positive integer with the property

$$P\big(Z_t^\phi \leq C(t)\big) \leq \gamma < P\big(Z_t^\phi \leq C(t) + 1\big) \tag{6.1}$$

with $\gamma \in (q, 1)$, q being the extinction probability of $\{Z_t\}$.

The following results are the objects of this section.

THEOREM 6.1. *Let $\{\phi(t)\}$ be a right-continuous random characteristic process with $E(\sup_t e^{-\beta t}\phi(t)) < \infty$ for some $0 \leq \beta < \alpha$ and $\mu(\infty) < \infty$. Then, with $C(t)$ defined by (6.1), $\lim_{t\to\infty} C^{-1}(t)Z_t^\phi = W$ a.s., where W is a nondegenerate random variable. If F denotes the distribution function of W, then F is continuous on $(0, \infty)$, $F(0) = q$, and Φ, the Laplace transform of F, satisfies the function equation*

$$\Phi(u) = E\left[\prod_{i=1}^{\xi(\infty)} \Phi\left(ue^{-\alpha\sigma_{[i]}}\right)\right].$$

Before giving the next result we need consider the notion of slow variation: A function L will be said to be *slowly varying* if $\lim_{x\to\infty} L(\delta x)/L(x) = 1$ for any $\delta > 0$.

THEOREM 6.2. *Suppose that the assumptions of Theorem* 6.1 *are satisfied. Then* $L(x) = \int_0^x P(W > u)\,du$ *is a slowly varying function,* $E(W^u) < \infty$ *for* $u < 1$ *and* $C(t) \sim e^{\alpha t}L(e^{\alpha t})$.

We need more lemmas for the proofs. It will be convenient to work first with a truncated ϕ, i.e., to assume that

$$\phi(t) = 0 \quad \text{for } t > \text{some } v > 0. \tag{6.2}$$

Throughout this section we shall assume that the conditions of Theorem 6.1 are in force.

LEMMA 6.3. *Suppose that* (6.2) *holds. Then, on* $\{T_t \to \infty\}$,

$$\lim_{t\to\infty}\left(Z_t^\phi\right)^{-1}Z_{t+s}^\phi = e^{\alpha s} \text{ a.s. for any } s > 0.$$

PROOF. Take $\psi(t) = \phi(t - s)$ and apply Lemma 6.1 to ϕ and ψ. □

LEMMA 6.4. *Suppose that* (6.2) *holds and choose any weakly convergent subsequence of* $\{C^{-1}(t)Z_t^\phi\}$, *say* $\{C^{-1}(t_n)Z_{t_n}^\phi\}$, *with* $\lim_{n\to\infty}t_n = \infty$. *If* F *denotes the limit distribution of* $\{C^{-1}(t_n)Z_{t_n}^\phi\}$, *then* F *is a proper distribution and* $F(0) = q$.

PROOF. Notice that (6.2) makes the convergence of $\{C^{-1}(t)\phi(t)\}$ to 0 as $t \to \infty$ obvious. Thus in view of Lemma 6.3, Condition 3.2 is satisfied and the result stated now follows on applying Theorem 4.1 and Corollary 4.2. □

LEMMA 6.5. *F is continuous on* $(0, \infty)$.

PROOF. It follows from Proposition 3.1 as in the proof of Lemma 2 of [4] by noticing that the concentration function of such sums of independent random variables tends to 0 as $t \to \infty$. For an alternative proof see [15]. □

PROOF OF THEOREM 6.1.

STEP 1. In the light of Lemma 6.1 it suffices to prove the theorem for $\{C^{-1}(t)N_t\}$, where N_t is the cardinality of $\mathscr{I}(t)$. Indeed, if $\hat{\phi}(t)$ is the truncate of $\phi(t)$ that satisfies (6.2), then Lemma 6.1 implies the existence of a constant $\hat{k}$ with $0 < \hat{k} < \infty$ such that $\lim_{t\to\infty} Z_t^{\hat{\phi}}/Z_t^\phi = \hat{k}$ a.s. on $T_t \to \infty$. Thus, a.s. convergence for $\{C^{-1}(t)Z_t^{\hat{\phi}}\}$ would imply a.s. convergence for $\{C^{-1}(t)Z_t^\phi\}$. Since the limit variable of $\{C^{-1}(t)Z_t^{\hat{\phi}}\}$ or converging subsequence thereof differs from the corresponding one of $\{C^{-1}(t)Z_t^\phi\}$ only by a multiplicative constant, the previous lemmas apply to ϕ as well. This reasoning may also be used to show that it suffices to prove the theorem for a particular ϕ satisfying the conditions of Lemma 6.1. It will be convenient to take $\phi(t) = \xi(\infty) - \xi(t)$, in which case $Z_t^\phi = N_t$.

STEP 2. Suppose now and hereafter that a sequence $\{t_n\}$ satisfying the condition of Lemma 6.4 is chosen and kept fixed, and W is a random variable distributed according to F, which until proven otherwise depends on $\{t_n\}$. Then we shall see that from any sequence $\{u_n\}$ with $\lim_{n\to\infty} u_n = \infty$ one can extract a subsequence $\{u'_n\}$ such that

$$G_x(y) = \lim_{n\to\infty} P\left(e^{-\alpha u'_n} \sum_{j=1}^{[C(u'_n)y]} W_j \le x \right) \tag{6.3}$$

exists for all x and y nonnegative, $\{G_.(y)\}$ are nondegenerate distributions for any $y > 0$, and $\lim_{y\to 0} G_x(y) = 1$ for any $x > 0$.

To prove this we shall start off, as before, with the martingale

$$\eta_t = P\left(e^{-\alpha t} \sum_{j\in\mathscr{I}(t)} W_j e^{-\alpha(\sigma_j - t)} \le x | \mathscr{A}_{T_t} \right) \quad \text{a.s.} \tag{6.4}$$

Notice that if we take $\phi(t) = \xi(\infty) - \xi(t)$, yielding $Z_t^\phi = N_t$, and $\psi(t) = e^{\alpha t}\int_t^\infty e^{-\alpha s}\xi(ds)$, yielding $Z_t^\psi = \sum_{j\in\mathscr{I}(t)} e^{-\alpha(\sigma_j - t)}$, then by Lemma 6.1

$$\lim_{t\to\infty} \frac{\sum_{j\in\mathscr{I}(t)} e^{-\alpha(\sigma_j - t)}}{N_t} = k \quad \text{a.s.} \tag{6.5}$$

on $\{T_t \to \infty\}$, where k is a constant with $0 < k \le 1$. Choose now a constant u with $0 < u < k$ and let $\Gamma_t = \{j \in \mathscr{I}(t)\colon e^{-\alpha(\sigma_j - t)} > u\}$ and β_t be the number of elements in Γ_t. Then

$$\sum_{j\in\mathscr{I}(t)} W_j e^{-\alpha(\sigma_j - t)} > u \sum_{j\in\Gamma_t} W_j. \tag{6.6}$$

It is easy to see that $\sum_{j\in\mathscr{I}(t)} e^{-\alpha(\sigma_j - t)} \le u(N_t - \beta_t) + \beta_t$, so that

$$\beta_t \ge \frac{N_t(k_t - u)}{1-u}, \tag{6.7}$$

where $k_t = \sum_{j\in\mathscr{I}(t)} e^{-\alpha(\sigma_j - t)}/N_t$. By (6.5) $\lim_{t\to\infty} k_t = k$ a.s. on $\{T_t \to \infty\}$, which in conjunction with (6.7) implies that $\lim_{t\to\infty} 1_{\{\beta_t > \beta N_t\}} = 1_{\{T_t\to\infty\}}$ a.s., where β is a constant and $0 < \beta < (k-u)/(1-u)$. Further (6.4), (6.6), and (6.7) entail

$$\eta_t \le P\left(e^{-\alpha t} \sum_{j=1}^{\beta N_t} W_j \le u^{-1}x | \mathscr{A}_{T_t} \right) \quad \text{a.s. on } \{\beta_t > \beta N_t\}. \tag{6.8}$$

On the other hand, $e^{-\alpha(\sigma_j - t)} < 1$ implies

$$\eta_t \ge P\left(e^{-\alpha t} \sum_{j=1}^{N_t} W_j \le x | \mathscr{A}_{T_t} \right) \quad \text{a.s.} \tag{6.9}$$

It is easy to see that (6.8) leads to

(6.10)

$$\eta_{u'_n} \le P\left(e^{-\alpha u'_n} \sum_{j=1}^{[\beta C(u'_n)y]} W_j \le u^{-1}x \right) \quad \text{a.s. on } \{N_{u'_n} > C(u'_n)y\} \cap \{\beta_{u'_n} > \beta N_{u'_n}\}$$

and (6.9) gives

$$(6.11)\qquad \eta_{u_n'} \geq P\left(e^{-\alpha u_n'} \sum_{j=1}^{[C(u_n')y]} W_j \leq x \right) \quad \text{a.s. on } \{N_{u_n'} < C(u_n')y\},$$

where $\{u_n'\}$ may be any sequence, but in what follows will be assumed to be a subsequence of $\{u_n\}$.

Since $E(\eta_t) = F(x)$, by Lemma 6.4 $\{u_n'\}$ can be chosen such that $\{C^{-1}(u_n')N_{u_n'}\}$ converges in distribution to a nondegenerate limit, (6.10) and (6.11) boil down to the existence of a subsequence of $\{u_n'\}$, denoted also $\{u_n'\}$, such that

$$(6.12)\qquad G_x(y_0) = \lim_{n\to\infty} P\left(e^{-\alpha u_n'} \sum_{j=1}^{[C(u_n')y_0]} W_j \leq x \right)$$

exists for a certain y_0 and all $x, G(y_0)$ being a nondegenerate vague limit. Since $\{W_n\}$ are i.i.d. we deduce that $\{G_x(y)\}$ exists for all x and y nonnegative. If we notice that $F(x) = E(\eta_t)$ may be made as close as desired to 1 by choosing x large enough, and take (6.10) into account, we deduce that $\lim_{x\to\infty} G_x(y) = 1$ for any $y > 0$. Thus $G(y)$ are proper distributions and since by a classical result they are infinitely divisible, it follows that $\lim_{y\to\infty} G_x(y) = 1$ for any $x > 0$.

STEP 3. Write $\eta_\infty = \lim_{t\to\infty} \eta_t$ a.s. We next prove that η_∞ is not a.s. constant on $\{T_t \to \infty\}$. Indeed, choose a subsequence of $\{t_n\}$, say $\{t_n'\}$, for which $G_x(y)$, defined with $\{t_n'\}$ replacing $\{u_n'\}$, exists. Since $\lim_{n\to\infty} P(N_{t_n'} < C(t_n')y) = F(y)$ and by Step 2, $\lim_{y\to 0} G_x(y) = 1$ we may invoke (6.11) for $\{t_n'\}$ to claim that $P(\eta_\infty > 1 - \varepsilon) > q$ for any $\varepsilon > 0$ provided that we prove that $\inf\{x\colon F(x) > q\} = 0$. Such a property in conjunction with $E(\eta_\infty) = F(x) < 1$ would lead to the existence of a constant δ and a set Λ with $1_\Lambda = \lim_{t\to\infty} 1_{\{\eta_t > \delta\}}$ a.s. where $q < P(\Lambda) < 1$, proving Step 3. To complete the proof write $m(X) = \inf\{x\colon F(x) > F(0)\}$ for a nonnegative random variable X with distribution function F which may have an atom at 0. If X and Y are independent variables of this kind, then $m(X + Y) \leq m(X)m(Y)$ and $m(XY) = m(X)m(Y)$, whereas if $H(x, \omega)$ is a variant of $P(X \leq x | \mathscr{A})$, $\mathscr{A}$ being an arbitrary σ field, them $m(X) = \operatorname{ess\,inf}_\omega \inf_x\{x\colon H(x, \omega) > F(0)\}$. Using these properties in (3.6) we get $m(W) \leq m(W)Y_t$ a.s. on $\{Y_t > 0\}$. Then, if $q = 0$ $m(W) = 0$ follows from $E(Y_t) = 1$ and the fact that $\{Y_t\}$ being not a.s. constant must admit values smaller than 1 with positive probability. If $q > 0$ we get $m(W) \leq e^{-\alpha\sigma_i} m(W)$ a.s. for any $i \in \mathscr{I}(t)$ and $m(W) = 0$ obtains in either case.

STEP 4. We shall show that there exists an event Λ_x such that $\eta_\infty = 1_{\Lambda_x}$ a.s. Indeed, choose x' and y' such that $G_{u_x^{-1\prime}}(\beta y') = \delta$ for the δ defined in Step 3, and take (6.10) into account to conclude that Λ defined in Step 3 has the property

$$(6.13)\qquad 1_\Lambda \leq 1_{\liminf_{n\to\infty}\{N_{t_n'} \leq C(t_n')y'\}} \quad \text{a.s.}$$

If we write $\eta_{t_n'}' = \lim_{m\to\infty} P(N_{t_m'} \leq C(t_m')y' | \mathscr{A}_{t_n'})$ a.s. and take the martingale convergence theorem into account we get from (6.13) that $\lim_{n\to\infty}\eta'_{t'_n} \geq 1_\Lambda$ a.s. and (6.10) leads to the conclusion that there must be some x and y with $G_x(y) = 1$. We notice that $G_x(y)$ considered here, unlike in Step 3, is not restricted to the case $\{u'_n\} \equiv \{t'_n\}$. Indeed, since $\{\eta'_t\}$ is a martingale we get $\lim_{n\to\infty}\eta'_{u'_n} \geq 1_\Lambda$ a.s. as a consequence, and (6.10) may be applied to $\{u'_n\}$; thus $\{e^{-\alpha u'_n}\sum_{j=1}^{[C(u'_n)y]} W_j\}$ turns out to converge in distribution to a limit distribution with bounded support. According to a well-known result for infinitely divisible distributions (see, e.g., [9], p. 177) such a limit must be degenerate. Thus we can assume that for some $y > 0$

$$\left\{e^{-\alpha u'_n}\sum_{j=1}^{[C(u'_n)y]} W_j\right\} \to_P cy \quad \text{as } n \to \infty, \tag{6.14}$$

where c is a constant and $\to_P$ denotes convergence in probability. Since $\{W_n\}$ are i.i.d., (6.14) must hold for any $y > 0$.

Notice further that (6.14), (6.5), and Lemma 6.2 imply that for any constants ε and η with $\varepsilon > 0$ and $0 < \eta < 1$

$$\left\{N_{u'_n} < C(u'_n)k^{-1}c^{-1}(x-\varepsilon)\right\} \setminus \{\eta_{u'_n} > \eta\} \text{ i.o.} \subset \Gamma \tag{6.15}$$

and

$$\left\{N_{u'_n} > C(u'_n)k^{-1}c^{-1}(x+\varepsilon)\right\} \setminus \{\eta_{u'_n} < \eta\} \text{ i.o.} \subset \Gamma, \tag{6.16}$$

where $A \setminus B$ denotes the difference of sets A and B and Γ is the set of probability 0 on which (6.5) fails. Since η is arbitrary, (6.15) and (6.16) boil down to

$$\left\{N_{u'_n} < C(u'_n)k^{-1}c^{-1}(x-\varepsilon)\right\} \setminus \{\eta_\infty = 1\} \text{ i.o.} \subset \Gamma \tag{6.17}$$

and

$$\left\{N_{u'_n} > C(u'_n)k^{-1}c^{-1}(x+\varepsilon)\right\} \setminus \{\eta_\infty = 0\} \text{ i.o.} \subset \Gamma. \tag{6.18}$$

Since by Lemma 6.4 the limit distribution of $\{C^{-1}(u'_n)N_{u'_n}\}$ is continuous we deduce that $\eta_\infty = 1_{\Lambda_x}$ a.o. for some event Λ_x.

STEP 5. Next we shall prove that

$$\left\{e^{-\alpha t}\sum_{j=1}^{[C(t)y]} W_j\right\} \to_P k^{-1}y \quad \text{as } t \to \infty. \tag{6.19}$$

Indeed, if we write F' for the limit distribution of $\{C^{-1}(u'_n)N_{u'_n}\}$ then (6.17) and (6.18) lead to $F(x) = F'(k^{-1}c^{-1}x)$. Since F is continuous and the γ in (6.1) is at our disposal we can argue as in the proof of Lemma 3 of [4] to get that for some γ, $(F(1) - F(1-\varepsilon))(F(1+\varepsilon) - F(1)) > 0$ for all $\varepsilon > 0$. Because (6.1) implies $F(1) = \lim_{t\to\infty}(N_t \leq C(t))$ we get $F(1) = F'(k^{-1}c^{-1})$ which entails $c = k^{-1}$. Thus c does not depend on the choice of $\{u'_n\}$ and since $\{u'_n\}$ was extracted from an arbitrary sequence $\{u_n\}$ with $\lim_{n\to\infty} u_n = \infty$ we get (6.19).

STEP 6. We are now in a position to finish the proof by showing that $\{C^{-1}(t)N_t\}$ converges a.s. as $t \to \infty$. Indeed, having established (6.19) we may remove $k^{-1}c^{-1}$ from (6.17) and (6.18) and let $\varepsilon \to 0$ to deduce that $\lim_{n\to\infty} 1_{\{C^{-1}(u_n)N_{u_n} \leq x\}} = 1_{\{\eta_\infty = 1\}}$ for $\omega \notin \Gamma$ and any sequence $\{u_n\}$ with

$\lim_{n\to\infty} u_n = \infty$. Because Γ does not depend on the choice of $\{u_n\}$ we conclude that $\lim_{t\to\infty} 1_{\{C^{-1}(t)N_t \le x\}} = 1_{\{\eta_\infty = 1\}}$ for $\omega \notin \Gamma$, which is tantamount with a.s. convergence for $\{C^{-1}(t)N_t\}$. □

The proof of Theorem 6.2 may be carried out as in [4] or [15] and will be omitted.

REMARK. If Theorem 6.1 is applied to the case $E[_\alpha\xi(\infty)\log_\alpha^+\xi(\infty)] < \infty$, then Theorem 4.1 in conjunction with Theorem A of Doney [8] on the solution to (1.4) yields $E(W) < \infty$. In this case $C(t) \sim e^{\alpha t}$ follows from (6.19). In particular, if we choose $\phi(t) = e^{\alpha t}\int_t^\infty e^{-\alpha s}\xi(ds)$, then $e^{-\alpha t}Z_t^\phi = Y_t$ and Corollary 4.1 implies $P(Y_\infty > 0) = 1 - q$, the result referred to in the remark following the proof of Theorem 5.1.

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