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THE COUPLED BRANCHING PROCESS IN RANDOM ENVIRONMENT

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We consider a Markov process $(\eta_t^\mu)_{t \in \mathbb{R}^+}$ on $(\mathbb{N})^S$ $(S = \mathbb{Z}^d)$ with initial distribution μ and the following time evolution: At rate $b\sum_y q(y,x)\eta(y)$ a particle is born at site x; at rate $\tilde{d}\eta(x)$ a particle dies at site x. All particles perform independent from each other continuous time random walk with kernel $p(x, y)$ and rate m. All particles at a site x die at rate $D(x)$. Here $D(x)$ are random variables taking the values D_1, D_2 $(D_2 \geq D_1 \geq 0)$. We assume $\{D(x)\}_{x \in S}$ to be stationary and ergodic. This paper studies the features of the model for $p(x, y)$, $q(x, y)$ symmetric.

We calculate the exponential growth rate λ of $\tilde{E}(\eta_t^\mu(x))$ (with $\tilde{E}$ denoting conditional expectation with respect to the environment) and show that λ is nonrandom and strictly bigger than $b - \tilde{d} - E(D(x))$, if $D_2 > D_1$. We have $\lambda = b - \tilde{d} - D_1$.

Introduce the process $(\hat{\eta}_t^\mu)_{t \in \mathbb{R}^+}$ by setting $\hat{\eta}_t^\mu(x) = (\tilde{E}(\eta_t^\mu(x)))^{-1}\eta_t^\mu(x)$. A critical phenomenon with respect to the parameter $p := D_1(\tilde{d} + ED(x))^{-1}$ occurs in the sense that for $p > p^{(2)}$ the quantity $\tilde{E}(\hat{\eta}_t^\mu(x))^2$ grows exponentially fast, while for $p \leq p^{(2)}$, $\lambda > 0$ the exponential growth rate of $\tilde{E}(\hat{\eta}_t^\mu(x))^2$ is 0. $p^{(2)}$ is the same as for a system with $D(x) \equiv D_1$ and can be calculated explicitly.

0. Introduction.

0.1. The model and its main features. We consider the Markov process $(\eta_t^\mu)_{t \in \mathbb{R}^+}$ with state space $(\mathbb{N})^S$ $(S = \mathbb{Z}^d)$, initial distribution μ, and the following time evolution:

(i) Birth of a particle at site x occurs at rate $b\sum_{y \in S} q(y,x)\eta(y)$.

(ii) Death of one particle at site x occurs at rate $\tilde{d}\eta(x)$.

(iii) All particles perform independent from each other a continuous time random walk with rate m and transition kernel $p(x, y)$.

(iv) All particles at site x die at rate $D(x)$. Here $\{D(x)\}_{x \in S}$ is a collection of random variables with values $D_1, D_2 \in \mathbb{R}^+$ $(D_2 \geq D_1)$.

For the construction of processes of this type we refer the reader to [4], [9]. Other models of branching processes in random environment are studied in [2], [5], and [6].

It is not essential that $D(x)$ assume only two values D_1, D_2. We could generalize our results easily to the case where $D(x)$ takes values in a finite interval; then our D_1 and D_2 are replaced by the essential infimum respectively

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supremum of the random variables $D(x)$. The notation would become quite complicated though so we focus here on the case where all the weight is concentrated on the extreme points.

There are two main reasons why such an evolution seems to be of interest:

(i) In many infinite particle systems that have been studied the assumption that the underlying mechanism is translation invariant is crucial for the methods of analysis to work [7], [8], [9]. In many of these systems though it would be interesting to know if essential features of the ergodic theory of the process (for example critical phenomena) are also present if the process lives in a random environment. For example, how does the basic contact process behave for recovering rates that are site (hence object) dependent and given at random at time 0? Can blocks of quickly recovering objects dominate the behavior of the process as far as the question of the persistence of infection is concerned? We will show that in our model such a phenomenon of domination occurs. This can also be viewed as a stability property of the process (see remarks below), the question being if small changes in the evolution mechanism will or will not affect certain features of the process.

(ii) Another reason to study the coupled branching process is in our view that it would be nice to have a population growth model that takes into account some interaction between the population and the environment. We started in [4] to study a branching random walk with additional (constant) site killing rates. It is of course much more natural to consider these rates as having a spatial structure and being assigned at random. Other efforts in this direction were done by D. Dawson [2]. He studied a model with a site dependent branching mechanism: the branching distribution at a site is assigned at random at time 0 and mean 1. The case of varying mean has been treated by Greven [5], [6].

In this paper we will consider first the mean particle density at a site and show that its exponential growth rate is no longer $b - (\tilde{d} + E(D(x)))$ as in classical models [1]. Such a phenomenon is common for evolution in random environments. For a simple random walk in random environment $E \log[\alpha/(1-\alpha)] = 0$ is the criterion for recurrence in one dimension, replacing the classical E(increments) $= 0$ [10].

In our model the *exponential growth rate of the mean particle density is completely determined by the lowest value D_1 of the site killing rate as long as the kernels $p(x, y)$ and $q(x, y)$ are symmetric. In asymmetric cases this is, in general, not true.*

The behavior of the second moments of the particle density at a site parallels that in the "deterministic case" $[D(x) \equiv D_1]$, $\tilde{E}$ denoting conditional expectation with respect to the environment: the exponential growth rate of $\tilde{E}(\eta_t^\mu(x)(\tilde{E}\eta_t^\mu(x))^{-1})^2$ is 0 if the quantity $p = D_1(\tilde{d} + E(D(x)))^{-1}$ is smaller than $p^{(2)}$, and bigger than 0 if $p > p^{(2)}$. *The behavior of the variance of the particle density is governed by the value D_1 alone.* This remains true if we allow $D(x)$ to range in a finite interval and replace D_1 by the essential infimum of $D(x)$ in the above statement.

These two facts can be viewed as a stability result: Disturb a given symmetric system by *adding* some additional site killing and the behavior of the first and second moments will not change very much, provided the system was not stationary at the beginning. In the latter case it will now die out instead of being stable. We also have the fact that systems with symmetric kernels $p(x, y)$, $q(x, y)$ are "stable" as opposed to the ones with asymmetric kernels, where the growth rate decreases if we add more site killing at random to the original system. In the other direction *leaving out* site killing at certain sites will always have immense effects.

The following assumptions will be made throughout this paper:

(a) $\{D(x)\}_{x \in S}$ is stationary and shift ergodic.

$$\begin{aligned} &\text{Prob}\big(D(x) = D_1, \text{ for all } x \in [-n, n]^d\big) > 0, \text{ for all } n \in \mathbb{N}, \\ &\text{Prob}\big(D(x) = D_2\big) > 0. \end{aligned} \tag{0.1}$$

[For example, (0.1) holds in the i.i.d. case.]

(b) The kernel $p(x, y) + q(x, y)$ is irreducible [if $m = 0$ or $b = 0$ assume that $q(x, y)$ respectively $p(x, y)$ are irreducible].
$p(x, y), q(x, y)$ are symmetric.

(c) The initial distribution μ is translation invariant, shift ergodic, and satisfies

$$E^{\mu}\big(\eta^2(x)\big) < \infty. \tag{0.2}$$

We will always denote the σ algebra generated by the random environment $\{D(x)\}_{x \in S}$ by $\mathscr{D}$. We write:

$$\tilde{E}(X) := E(X | \mathscr{D}). \tag{0.3}$$

0.2. The results in detail. Our main interest is focused on the behavior of the mean particle density $\tilde{E}(\eta_t^{\mu}(x))$ and of $\tilde{E}(\eta_t^{\mu}(x))^2$ for $t \to \infty$. We define the following characteristic parameters of our model:

$$\tilde{\tilde{d}} := E(D(x)), \tag{0.4}$$

$$d := \tilde{d} + \tilde{\tilde{d}}. \tag{0.5}$$

Under the assumptions summarized in part (0.1) we have:

THEOREM 1.

(a)

$$\lim_{t \to \infty} \left[\frac{1}{t} \log(\tilde{E}(\eta^{\mu_t}(x))) \right] = b - (D_1 + \tilde{d}) := \lambda \quad a.s. \tag{0.6}$$

In the case where $(b - (\tilde{d} + D_1)) = 0$ *we have*:

$$\lim_{t \to \infty} \tilde{E}(\eta_t^{\mu}(x)) = 0 \quad \textit{iff } D_2 > D_1. \tag{0.7}$$

(b)

$$\lim_{t \to \infty} \left(e^{-\lambda t} \tilde{E}(\eta_t^{\mu}(x))\right) = 0 \quad \textit{iff } D_2 > D_1. \tag{0.8}$$

COROLLARY.

$$\lim_{t\to\infty}\left[\frac{1}{t}\log(E(\eta_t^\mu(x)))\right] = b-(D_1+\tilde{d}) := \lambda.$$

Note that this says that, contrary to the deterministic case $[D(x)\equiv D_1]$ or the case of a branching random walk ($D_2 = D_1 = 0$), for no values of the parameters do we have an invariant measure.

More interesting is the fact that only the smallest value of $D(x)$ matters for the exponential growth rate of the mean particle density. Note that this result depends on the symmetry assumption on the kernels. In general it does not hold that $\lambda = b - \tilde{d} - D_1$. An example is the situation $S = \mathbb{Z}^1$, $p(x, y) = q(x, y)$, and $p(x, x+1) = 1$. In this case our methods show that $\lambda < b - \tilde{d} - D_1$.

In order to get a more detailed picture of the behavior of $\tilde{\mathscr{L}}(\eta_t^\mu)$ for $t\to\infty$ it is natural to introduce the following normalized process $(\hat{\eta}_t^\mu)_{t\in\mathbb{R}^+}$:

$$\hat{\eta}_t^\mu(x) := a_t^{-1}(x)\eta_t^\mu(x) \quad \text{with} \quad a_t(x) := \tilde{E}(\eta_t^\mu(x)). \tag{0.9}$$

The normalized process has, of course, the property

$$\tilde{E}(\hat{\eta}_t^\mu(x)) = 1 \quad \text{for all} \quad x\in\mathbb{Z}^d,\ t\in\mathbb{R}^+. \tag{0.10}$$

The following parameter will be important for the behavior of $E(\hat{\eta}_t^\mu(x))^2$:

$$p := D_1 d^{-1}. \tag{0.11}$$

THEOREM 2.

(a)

$$\lim_{t\to\infty}\inf\left(\frac{1}{t}\log\tilde{E}(\hat{\eta}_t^\mu(x))^2\right) > 0 \quad a.s., \quad \textit{if } \lambda\ge 0 \textit{ and } p > p^{(2)}, \textit{ or } \lambda < 0, \tag{0.12}$$

$$\lim_{t\to\infty}\sup\left(\frac{1}{t}\log\tilde{E}(\hat{\eta}_t^\mu(x))^2\right) = 0 \quad a.s., \quad \textit{if } \lambda\ge 0 \textit{ and } p \le p^{(2)}. \tag{0.13}$$

(b) *The critical value $p^{(2)}$ can be calculated as follows*:

$$p^{(2)} = \left(\frac{1}{2}\frac{d}{m+b}G_{\hat{r}}(0,0)\right)^{-1} \tag{0.14}$$

with

$$\begin{aligned} \hat{r}(x,y) &:= \tfrac{1}{2}(r(x,y)+r(y,x)),\\ r(x,y) &= \frac{b}{m+b}q(x,y)+\frac{m}{m+b}p(x,y),\\ G_{\hat{r}} &:= \sum_{n=0}^{\infty}\hat{r}^n. \end{aligned} \tag{0.15}$$

Note that

$$p \underset{(\ge)}{<} p^{(2)} \Leftrightarrow \left(\frac{1}{2}\frac{D_1}{m+b}G_{\hat{r}}(0,0)\right)^{-1} \underset{(\ge)}{<} 1. \tag{0.16}$$

Note that the critical value is the same as for the deterministic system $D(x) \equiv D_1$.

REMARK. If we consider $\tilde{E}(\eta_t(x))^k$, $k = 3, 4, \ldots$ we encounter the very same phenomena as in Theorem 2: There are critical values $p^{(k)}$ where exponential growth of these moments starts occurring and these critical values are the same as for the system with $D(x) \equiv D_1$. The methods of proof are very similar to the ones used for Theorems 1 and 2 here so we leave the details to the reader (compare [4], [5]).

1. Proof of Theorem 1.

1.1. A duality relation. In order to prove our Theorem 1 we will express the quantity $\tilde{E}^{\mu}(\eta_t(x))$ in terms of a dual process $(\xi_t^{\{x\}})_{t \in \mathbb{R}^+}$. Consider the Markov process $(\xi_t^{\{u\}})_{t \in \mathbb{R}^+}$ on $(\mathbb{N})^S$ with initial distribution $\delta_{\{\xi = 1_{\{u\}}\}}$ and the following time evolution [compare (0.15) for the definition of $r(x, y)$]:

(i) At rate $(m + b)\xi(x)r(y, x)$ the following transition occurs:

$$\xi \to \xi(-1_{\{x\}} + 1_{\{y\}}) + \xi \qquad \text{(motion from } x \text{ to } y).$$

(ii) At rate $(\tilde{\tilde{d}} - D(y))^+$: $\xi \to \xi + \xi(y)1_{\{y\}}$ (duplication).

(iii) At rate $(\tilde{\tilde{d}} - D(y))^-$: $\xi \to \xi - \xi(y)1_{\{y\}}$ (extinction).

We can define now the total population $N_t(x)$ as:

$$N_t(x) = \sum_{y \in S} \xi_t^{\{x\}}(y). \tag{1.1}$$

We will prove the following duality relation between $(\eta_t^{\mu})_{t \in \mathbb{R}^+}$ and $(\xi_t^{\{u\}})_{t \in \mathbb{R}^+}$.

PROPOSITION 1. *Define* $\rho := E^{\mu}(\eta(x))$. *Then*

$$\tilde{E}(N_t(x)) = (\rho e^{-(b-d)t})\tilde{E}(\eta_t^{\mu}(x)). \tag{1.2}$$

PROOF. Define

$$f_t(x) := \tilde{E}(\eta_t^{\mu}(x)), \qquad f(x, \eta) := \eta(x). \tag{1.3}$$

By G we denote the generator of the coupled branching process (η_t). We have, as a straightforward calculation shows,

$$\begin{aligned}(Gf(x, \cdot))_{(\eta)} &= (m + b)\Big(\sum_{y \in S} r(y, x)\eta(y) - \eta(x)\Big) \\ &\quad + \big(\tilde{\tilde{d}} - D(x)\big)\eta(x) + (b - d)\eta(x).\end{aligned} \tag{1.4}$$

From the construction of the process (compare [4], [7]) we obtain via (1.4) the following system of differential equations for $\{f_t(x)\}_{x\in\mathbb{Z}^d}$:

$$\frac{d}{dt}f_t(x) = (m+b)\Big(\sum_{y\in S} r(y,x)f_t(y) - f_t(x)\Big) + \big(\tilde{\tilde{d}} - D(x)\big)f_t(x) + (b-d)f_t(x). \tag{1.5}$$

Now introduce $\hat{f}_t(x) := e^{-(b-d)t}f_t(x)$. Then (1.5) can be written as

$$\frac{d}{dt}\hat{f}_t(x) = (m+b)\Big(\sum_{y\in S} r(y,x)\hat{f}_t(y) - \hat{f}_t(x)\Big) + \big(\tilde{\tilde{d}} - D(x)\big)^+\hat{f}_t(x) - \big(\tilde{\tilde{d}} - D(x)\big)^-\hat{f}_t(x). \tag{1.6}$$

A straightforward calculation shows that the function $\{t \to \tilde{E}(N_t(x))\}_{x\in S}$ also fulfills the system (1.6) of differential equations. Since the functions $t \to \{\hat{f}_t(x)\}_{x\in S}$, $t \to \{\tilde{E}(N_t(x))\}_{x\in S}$ take both values in $l^\infty(S)$ for $t\in\mathbb{R}^+$ and both fulfill (1.6) we immediately obtain the assertion of Proposition 1.

1.2. The evaluation of $\tilde{E}(N_t(u))$ as a large deviation problem. We want to calculate $\tilde{E}(N_t(u))$. We will need the following:

(1.7) $(M_t)_{t\in\mathbb{R}^+}$: *Poisson process with rate* $(m+b)$;

(1.8) $(X_k)_{k\in\mathbb{N}}$: *random walk with kernel* $\bar{r}(x,y)$ $[:= r(y,x)]$ *starting at the point* u.

Denote by $\mathscr{R}$ the σ algebra generated by $(M_t), (X_k)$. The following set is crucial for the evolution of $N_t(u)$:

$$A := \{x \mid D(x) = D_1\}. \tag{1.9}$$

Denote by n the number of jumps up to time t, by t_k the jump times of M_t, and by x_k the points reached. An elementary calculation now shows

$$\begin{aligned}\tilde{E}(N_t(u)\mid\mathscr{R}) = \exp\Big(&\sum_0^{n-1}\big(\tilde{\tilde{d}} - D_1\big)(t_{k+1}-t_k)1_A(x_k)\\ &-\sum_0^{n-1}\big(D_2 - \tilde{\tilde{d}}\big)(t_{k+1}-t_k)1_{CA}(x_k)\Big)\\ &\cdot\exp\Big((t-t_n)\big[1_A(x_n)\big(\tilde{\tilde{d}} - D_1\big) - 1_{CA}(x_n)\big(D_2 - \tilde{\tilde{d}}\big)\big]\Big).\end{aligned} \tag{1.10}$$

If we denote the number of jumps of $(M_t)_{t\in\mathbb{R}^+}$ before time t by $n(t)$ we obtain from (1.10) that for $u\in A$

$$\begin{aligned}&\tilde{E}(N_t(u)\mid n(t)=n;\ X_1 = x_1,\dots,X_n = x_n)\\ &\quad\ge \exp\big(t\big(\tilde{\tilde{d}} - D_1\big)\big)1_{\{x_i\in A,\, i=1,\dots,n\}}.\end{aligned} \tag{1.11}$$

Therefore

$$\tilde{E}(N_t(u)) \ge e^{t(\tilde{\tilde{d}} - D_1)}\sum_{n=0}^{\infty}\Big(e^{-\tilde{t}}\frac{\tilde{t}^n}{n!}\Big)\operatorname{prob}\big(X_i^{\{u\}}\in A, i\le n\big) \tag{1.12}$$

where $\tilde{t} = (m + b)t$. Now note that this implies that if we have for a sequence $a_k \searrow 0$; $M_k \subseteq A$:

$$\lim_{n\to\infty} \left| \frac{1}{n} \log\left(\text{prob}\left(X_i^{\{u\}} \in A, i \le n\right)\right) \right| \le a_k \quad \text{for all } u \in M_k, \tag{1.13}$$

$$\text{prob}\left(X_i^{\{u\}} \text{ hits } M_k \text{ for some } i \in \mathbb{N}\right) = 1 \quad \text{for all } u \in S. \tag{1.14}$$

Then we can conclude from (1.12) with straightforward analysis that

$$\lim_{t\to\infty} \left(\frac{1}{t} \ln \tilde{E}N_t(u) \right) = \tilde{\tilde{d}} - D_1 \quad \text{for all } u \in S, \tag{1.15}$$

which would prove together with Proposition 1 our Theorem 1(a). So it remains to prove the large deviation result (1.13) for symmetric random walks (irreducible) on $\mathbb{Z}^d$. However, since A by assumption (0.1) has the property that for every k the cubes $x + [-k, k]^d$ that lie in A have positive density, we only have to prove that, with the definition

$$q_N = \lim_{n\to\infty} \left(\frac{1}{n} \ln \text{prob}\left\{ X_k^{\{0\}} \in [-N, N]^d \text{ for all } k \le n \right\} \right), \tag{1.16}$$

we have

$$\lim_{N\to\infty} \sup q_N = 0 \tag{1.17}$$

since we then can choose $M_k = \{x | x + [-k, k]^d \in A\}$, $a_k = q_k$. The above relation holds for symmetric random walks as shown in Lemma 0. We mention first the brief idea: The random walk (suppose for a second we have simple random walk) can leave the cube only from the boundary points. The walk restricted to the cube spends only a with $N \nearrow \infty$ to 0 decreasing portion of time at the boundary. This "implies" (1.17). The reader will notice immediately that this need not hold in asymmetric situations. More precisely we proceed as follows:

Lemma 0. *For a symmetric random walk $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ starting at 0 (state space $\mathbb{Z}^d$) we have*

$$\lim_{N\to\infty} \left(\lim_{t\to\infty} \frac{1}{n} \text{prob}\left(X_i \in [-N, N]^d \text{ for } i = 1, \dots, n \right) \right) = 0. \tag{1.18}$$

Proof of Lemma 0: Denote by (X_k^B) the random walk obtained by restricting the walk with kernel $r(x, y)$ to the cube B [with kernel $r_B(x, y) = r(x, y)(\sum_{z\in B} r(x, z))^{-1}$]. Define $l_B(x) := \sum_{y\in B} r(x, y)$. Since $r(x, y) = r(y, x)$ we can assume without loss of generality that $e_B(x) > 0$ for $x \in B$. Now we estimate as follows:

$$\begin{aligned} \text{prob}^{\{x\}}(X_k \in B \text{ for all } k \le j) &= E^{\{x\}} \exp\left(\sum_{k=0}^{j} \log l_B\left(X_j^B\right) \right) \\ &\ge \exp\left(-E^{\{x\}} \sum_{k=0}^{j} \left| \log l_B\left(X_j^B\right) \right| \right). \end{aligned} \tag{1.19}$$

If we denote by π_B the invariant measure of the chain (X_k^B) (assume $|B| < \infty$; note that without loss of generality for our purposes we can assume this chain to be ergodic) we obtain:

$$\sum_{j=0}^{k} \frac{E^{\{x\}}\left|\log l_B\left(X_j^B\right)\right|}{k+1} \underset{k\to\infty}{\to} \sum_{x\in B} \pi_B(x)|\log l_B(x)|$$

(1.20)
$$= \frac{1}{\sum_{x\in B} e_B(x)} \cdot \sum e_B(x)|\log e_B(x)|$$

$$= \frac{1}{B} \cdot \sum_{x\in B} l_B(x)|\log l_B(x)| + o\left(\frac{1}{|B|}\right).$$

The first identity follows from the fact that since $r(x, y)$ is symmetric $l_B(x)$ fulfills the detailed balance conditions with respect to r_B as one easily checks. The second identity follows from the definition of $e_B(x)$.

If we choose $B = [-n, n]$, the right side in (1.20) will converge to 0 for $n \to \infty$, since the proportion of points with $l_B(x) \leq 1 - \varepsilon$ tends to zero for every positive ε. Then we can conclude from (1.19) and (1.20) that

(1.21)

$$\lim_{N\to\infty} \sup \lim_{n\to\infty} \sup\left(\frac{1}{n} \ln \operatorname{prob}\left\{X_k^{\{0\}} \in [-N, N]^d \text{ for all } k \leq n\right\}\right) = 0. \quad \square$$

1.3. Proof of Theorem 1(b). The statements (0.6), (0.7) of our Theorem 1 follow immediately from the fact that we conclude from (1.10) and Proposition 1: If $(X_s^{\{x\}})_{s\in\mathbb{R}^+}$ is a random walk with rate $(m + b)$ and kernel $\bar{r}(x, y)$, starting at point x_1 then

$$\tilde{E}(\eta_t^\mu(x)) \leq \rho \tilde{E} \exp\left(\lambda t - (D_2 - D_1)\int_0^t 1_{\{X_s^{\{x\}} \in CA\}}\, ds\right) \tag{1.22}$$

so that

$$e^{-\lambda t}\tilde{E}(\eta_t^\mu(x)) \leq \rho \tilde{E} \exp\left(-(D_2 - D_1)\int_0^t 1_{\{X_s^{\{x\}} \in CA\}}\, ds\right). \tag{1.23}$$

The right hand side of the above inequality tends to 0 since CA is under our assumptions that: $D_2 > D_1$ and $\{D(x)\}_{x\in S}$ stationary ergodic together with (0.1), of course visited infinitely often by the random walk. $\square$

2. Proof of Theorem 2.

2.1. The differential equations for $\tilde{E}(\eta_t^\mu(x)\eta_t^\mu(y))$: A duality relation. We define

$$f(x, y; \eta) := \eta(x)\eta(y) - \delta(x, y)\eta(x) \qquad f(x; \eta) := \eta(x), \tag{2.1}$$

$$f_t(x, y) := \tilde{E}^\mu(f(x, y; \eta_t)) \qquad f_t(x) := \tilde{E}^\mu f(x; \eta_t). \tag{2.2}$$

We denote by G the generator of our process (η_t).

LEMMA 1.

$$
\begin{aligned}
&[Gf(x, y; \cdot)](\eta) \\
&\quad = (m+b)\cdot\Big(\sum_{z\in S} r(z,x)f(z,y;\eta) + r(z,y)f(z,x;\eta) - 2f(x,y;\eta)\Big) \\
(2.3)\quad &\quad + \big[(\tilde{\tilde{d}} - D(x)) + (\tilde{\tilde{d}} - D(y))\big] f(x,y;\eta) \\
&\quad + \delta(x,y)D(x)f(x,y;\eta) \\
&\quad + 2(b-d)f(x,y;\eta) \\
&\quad + q(y,x)f(x,\eta) + q(x,y)f(y,\eta)
\end{aligned}
$$

In order to obtain a transparent system of differential equations from Lemma 1 we introduce

$$(2.4)\qquad \tilde{f}_t(x,y) := e^{-2(b-d+\tau)t} f_t(x,y) \quad \text{with } \tau := \lambda - (b-d).$$

From Lemma 1 we can obtain, using the methods in [4], [7], that

$$
\begin{aligned}
\frac{d}{dt}\tilde{f}_t(x,y) &= (m+b)\Big(\sum_{z\in S} r(z,x)\tilde{f}_t(z,y) + r(z,y)\tilde{f}_t(z,x) - 2\tilde{f}_t(x,y)\Big) \\
(2.5)\qquad &\quad + \big[(\tilde{\tilde{d}} - D(x)) + (\tilde{\tilde{d}} - D(y)) - 2\tau\big]\tilde{f}_t(x,y) \\
&\quad + \delta(x,y)D(x)\tilde{f}_t(x,y) \\
&\quad + e^{-2(b-d+\tau)t}\big(q(y,x)f_t(x) + q(x,y)f_t(y)\big).
\end{aligned}
$$

This system of differential equations is still too complicated to treat, so we study first solutions $\{\hat{f}_t(x,y)\}_{x,y\in S}$ of the system obtained by deleting in (2.5) the last row. For this system we can obtain now a duality relation. In the end we will prove that $\hat{f}_t(x,y)$ approximates $\tilde{f}_t(x,y)$ well enough for our purposes.

Consider the process $(\zeta_t^{\{x,y\}})_{t\in\mathbb{R}^+}$ with state space $\mathbb{N}^S \times \mathbb{N}^S$ $(S = \mathbb{Z}^d)$; initial distribution $\delta_{\{\zeta \equiv 1_{\{x,y\}}\}}$. The process will be defined in terms of the $(\xi_t)_{t\in\mathbb{R}}$ as introduced in Section 1.1 (environment fixed).

Take independent versions of the dual processes $(\xi_t^{\{x\}})_{t\in\mathbb{R}^+}, (\xi_t^{\{y\}})_{t\in\mathbb{R}^+}$ and define $W_t(x; y)$ as follows: Whenever $\xi_t^{\{x\}}$ and $\xi_t^{\{y\}}$ are concentrated on the same point say z, an exponential clock with rate $D(z)$ starts ticking, $W_t(x,y) := \#\{\text{the random clock rang for some } z \in S \text{ prior to time } t\}$.

$$(2.6)\qquad N_t^1(x) := \sum_{z\in S}\xi_t^{\{x\}}(z), \qquad N_t^2(y) := \sum_{z\in S}\xi_t^{\{y\}}(z).$$

Then define $\zeta_t^{\{x,y\}}$ by setting

$$(2.7)\qquad \zeta_{t(z_1,z_2)}^{\{x,y\}} := \xi_t^{\{x\}}(z_1)\xi_t^{\{y\}}(z_2)2^{W_t(x,y)}, \qquad \tilde{\zeta}_t^{\{x,y\}} := N_t^1(x)N_t^2(y)^{W_t(x,y)}$$

This process is well defined and a straightforward calculation shows that $\{\tilde{E}\zeta_{t(z_1,z_2)}^{\{x,y\}}\}_{x,y}$ fulfills the same system of differential equations as $\{e^{2\tau t}\hat{f}_t(x,y)\}_{x,y}$, so the following duality relation holds:

PROPOSITION 2.

$$\hat{f}_t(x, y)e^{2\tau t} = \tilde{E}\left(\sum_{z_1, z_2} \xi_t^{\{x\}}(z_1)\xi_t^{\{y\}}(z_2)2^{W_t(x,y)}f_0(z_1, z_2)\right). \tag{2.8}$$

This proposition allows us immediately to conclude:

COROLLARY. *Suppose* $\inf_{x,y}[\mu(\eta(x) > 0, \eta(y) > 0)] > 0$. *Then*

$$c_1\tilde{E}(\bar{\zeta}_t^{\{x,y\}}) \le e^{2\tau t}\hat{f}_t(x, y) \le c_2\tilde{E}(\bar{\zeta}_t^{x,y}). \tag{2.9}$$

For general μ (2.9) can be replaced by a more complicated statement. To keep the notation more concise we will always argue for the above situation though. This means that for our purposes it is sufficient to study $\tilde{E}(\bar{\zeta}_t^{\{x,y\}})$.

We proceed now by representing $\tilde{E}(\bar{\zeta}_t^{(u,v)})$ in the form $\tilde{E}(\exp(\cdot))$. We will need: $(X_k^1)_{k\in\mathbb{N}}, (X_k^2)_{k\in\mathbb{N}}$ are independent versions of a random walk with transition kernel $\bar{r}(x, y)$ $(= r(y, x))$, starting in the points u, v. $(M_t^1)_{t\in\mathbb{R}^+}$, $(M_t^2)_{t\in\mathbb{R}^+}$ are independent Poisson processes with rate $(m + b)$.

The two above ingredients allow us to construct two continuous time random walks $(X_s^1)_{s\in\mathbb{R}^+}, (X_s^2)_{s\in\mathbb{R}^+}$. An elementary calculation shows now, that:

PROPOSITION 3. $A := \{x|D(x) = D_1\}$,

$$\tilde{E}(\bar{\zeta}_t^{\{u,v\}}) = \tilde{E}\exp\left[\tau_1\int_0^t 1_{\{X_s^1\in A\}} + 1_{\{X_s^2\in A\}}\,ds + \tau_2\int_0^t 1_{\{X_s^1 = X_s^2\in CA\}}\,ds \right.$$
$$\left. + \tau_3\int_0^t 1_{\{X_s^1 = X_s^2\}}\,ds - \tau_4\int_0^t 1_{\{X_s^1\in CA\}} + 1_{\{X_s^2\in CA\}}\,ds\right],$$

$$\tau_1 = \tilde{\tilde{d}} - D_1, \qquad \tau_2 = D_2 - D_1, \qquad \tau_3 = D_1, \qquad \tau_4 = D_2 - \tilde{\tilde{d}}. \tag{2.10}$$

2.2. The asymptotic behavior of $\tilde{E}(\bar{\zeta}_t^{\{u,v\}})$.

PROPOSITION 4.

(i)

$$\lim_{t\to\infty} \sup\left(\frac{1}{t}\log\tilde{E}(\bar{\zeta}^{\{u,u\}t})\right) \le 2\tau \quad \textit{for } p \le p^{(2)}. \tag{2.11}$$

(ii)

$$\lim_{t\to\infty} \inf\left(\frac{1}{t}\log\tilde{E}(\bar{\zeta}_t^{\{u,u\}})\right) > 2\tau \quad \textit{for } p > p^{(2)}. \tag{2.12}$$

[*Here* $\tau = \tilde{\tilde{d}} - D_1 = \lambda - (b - d)$.]

Proof.

(i) From Proposition 3 we can conclude that

$$\tilde{E}\left(\bar{\zeta}_t^{\{u,v\}}\right) \le \tilde{E}\left(\exp\left(2\tau_1 t + \tau_3 \int_0^t 1_{\{X_s^1 = X_s^2\}}\, ds\right)\right) = e^{2\tau t} \tilde{E} \exp\left(\tau_3 \int_0^t 1_{\{X_s^1 = X_s^2\}}\, ds\right). \tag{2.13}$$

Now write for abbreviation $g_t(x,y) = E^{\{x,y\}} \exp(\tau_3 \int_0^t 1_{\{X_s^1 = X_s^2\}}\, ds)$. We obtain the following system of differential equations:

$$\begin{aligned} \frac{d}{dt} g_t(x,y) &= (m+b)\Big(\sum_z r(z,x) g_t(z,y) + \sum_z r(z,y) g_t(x,z) - 2 g_t(x,y)\Big) \\ &\quad + \tau_3 g_t(x,y)\,\delta(x,y), \\ g_0(x,y) &= 1. \end{aligned} \tag{2.14}$$

This system has been studied in [4] Section 1 and gives for our situation

$$\begin{aligned} &\sup_t \left(g_t(x,x)\right) < \infty \quad \text{if } p < p^{(2)}, \\ &g_t(x,x) \le K \cdot t \quad \text{if } p = p^{(2)}. \end{aligned} \tag{2.15}$$

(2.13) and (2.15) together allow us to conclude (2.11).

(ii) Here the main idea is again that large cubes where $D(x) = D_1$ will determine the behavior of the moments at least in the logarithmic scale. The first important step for a rigorous proof will be formulated in (2.20). First we need some more notation.

Consider the following functions:

$$P_{x,B}(t) := E^{\{x,x\}} 1_{\{X_s^1 \in B,\, X_s^2 \in B \text{ for } s \le t\}}. \tag{2.16}$$

$(X_s^i)_{s \in \mathbb{R}^+}$ are independent copies of the random walk with rate $(m+b)$ and kernel $\bar{r}(x,y)$.

$$d_{x,B}^{\tilde{B}}(t) := E\left(\exp\left(\tau_3 D_B^{\tilde{B}}(t)\right)\right), \qquad B \subseteq \tilde{B}, \tag{2.17}$$

$$\begin{aligned} D_B^{\tilde{B}}(t) = {} & \text{time the walk } \left(X_s^1, X_s^2\right) \text{ restricted to } \tilde{B} \times \tilde{B} \\ & \text{spends on the diagonal of } B \times B, \\ & \left(\text{restricted} \triangleq \text{with kernels: } \left(\sum_{y \in \tilde{B}} \bar{r}(x,y)\right)^{-1} \bar{r}(x,y)\right) \end{aligned} \tag{2.18}$$

For a set $\tilde{B} \subseteq A = [z | D(z) = D_1]$ and a point $x \in B$ we have by Proposition 3 that

$$\tilde{E} \bar{\zeta}_t^{\{x,x\}} \ge \tilde{E} \exp\left(\tau_3 D_B^{\tilde{B}}(t)\right) 1_{\{X_s^1 \in \tilde{B},\, X_s^2 \in \tilde{B},\, s \le t\}} e^{2\tau_1 t}; \tag{2.19}$$

this implies, since $x \in B \subseteq \tilde{B}$, that

$$\tilde{E}\left(\bar{\zeta}_t^{\{x,x\}}\right) \ge p_{x,\tilde{B}}(t)\, d_{x,B}^{\tilde{B}}(t)\, e^{2\tau t} \tag{2.20}$$

or as an immediate consequence

$$\frac{1}{t}\log \tilde{E}\left(\zeta_t^{\{x,x\}}\right) \geq \frac{1}{t}\log\left[d_{x,B}^{\tilde{B}}(t)\right] - \frac{1}{t}\left|\log\left[p_{x,\tilde{B}}(t)\right]\right| + 2\tau. \tag{2.21}$$

To finish our proof we need the following two lemmas:

LEMMA 2. *Define*

$$a_n := \liminf_{\tilde{B}\uparrow S}\lim_{t\to\infty}\left(\frac{1}{t}\log d_{x,B_n}^{\tilde{B}}(t)\right), \tag{2.22}$$

where $B_n = [-n, n]^d$. *Then* $\liminf_{n\to\infty} a_n \geq a > 0$ *where* a *is the solution of the equation*

$$\int e^{-at}\hat{r}(t,0,0)\,dt = \left(\frac{pd}{m+b}\right)^{-1}, \qquad a \in \mathbb{R}^+ \left(\textit{since}\, p > p^{(2)}\right) \tag{2.23}$$

with $r(t,\cdot,\cdot) = \sum_{n=0}^{\infty} e^{-t}\cdot(t^n/n!)\hat{r}^n$, $\hat{r}(x,y) = \frac{1}{2}(r(x,y) + r(y,x))$.

LEMMA 3.

$$\lim_{n\to\infty}\left(\limsup_{t\to\infty}\left(\frac{1}{t}\log p_{0,\tilde{B}_n}(t)\right)\right) = 0, \qquad \tilde{B}_n = [-n,n]^d. \tag{2.24}$$

If we apply these lemmas to (2.21) we have for $u \in [-n,n]^d + u \subseteq A$ and $n \geq n_0$:

$$\liminf_{t\to\infty}\frac{1}{t}\log \tilde{E}\left(\bar{\zeta}_t^{\{u,u\}}\right) > 2\tau \quad \text{for } p > p^{(2)}. \tag{2.25}$$

Since points u with the properties required above have positive density by assumption, we can reach from any given point (x,x) such a point (u,u) by the random walk (X_t^1, X_t^2) with positive probability. (Here we use that the random environment is stationary and ergodic.) Therefore (2.25) implies:

$$\bigwedge_{u\in S}\liminf_{t\to\infty}\left(\frac{1}{t}\log \tilde{E}\left(\bar{\zeta}_t^{\{u,u\}}\right)\right) > 2\tau \quad \text{for } p > p^{(2)} \qquad \square \tag{2.26}$$

It remains now to prove our Lemmas 2, 3.

2.3. Proof of Lemma 2. First we need to introduce the following notation: $L, L_{\tilde{B}}$ stands for the generator of the walk $(X_s^1, X_s^2)_{s\in\mathbb{R}^+}$ respectively the one restricted to $\tilde{B}\times\tilde{B}$. By $d_{x,B}(t)$ we abbreviate $d_{x,B}^S(t)$ (i.e., the quantity for the unrestricted walk). A theorem of Donsker and Varadhan [3] says now that

$$\lim_{t\to\infty}\left(\frac{1}{t}\log \tilde{E}\exp\left(\tau_3 D_{x,B}^{\tilde{B}}(t)\right)\right) = \sup_{\mu\in\mathscr{M}}\left(\tau_3\mu(B) - I_{\tilde{B}}(\mu)\right) \tag{2.27}$$

with $-I_{\tilde{B}}(\mu) = \inf_{u>0}\left(\int \frac{L_{\tilde{B}}(u)}{u}\,d\mu\right)$

To proceed further note that for our purposes we can assume that $r(0,x)$ has finite range (since the exponential decay rate for the probability that up to time t no jump of size bigger than n occurred can be made arbitrarily small by making n large). Now write in (2.27) $\sup_{\mathscr{M}}(\cdot) = \sup_n(\sup_{\mathscr{M}_n}(\cdot))$ (this is straightforward analysis using the special properties of $\mathscr{M}_n$) where $\mathscr{M}_n$ contains all probability measures concentrated on C_n with $C_n \uparrow S, |C_n| < \infty$. Then for each $\mathscr{M}_n$ the functionals $I_{\tilde{B}}(\mu)$, $I(\mu)$ agree for $\mu \in \mathscr{M}_n$ if $\tilde{B}$ is big enough. Therefore we have for a fixed set B:

(2.28)

$$\liminf_{\tilde{B}\uparrow S}\left(\lim \frac{1}{t}\log E\exp\left(\tau_3 D^{\tilde{B}}_{x,B}(t)\right)\right) \geq \lim_{t\to\infty}\left(\frac{1}{t}\log E\exp\left(\tau_3 D_{x,B}(t)\right)\right)$$

which shows (2.22). Now we go back to our special situation and define

$$a_n := \lim_{t\to\infty}\left(\frac{1}{t}\log E\exp\left(\tau_3 D_{x,B_n}(t)\right)\right). \tag{2.29}$$

Specialising (2.27) to our situation with sets $B_n = [-n,n]^d$ substituted for B we obtain:

$$a_n = \lim_{t\to\infty}\left(\frac{1}{t}\log d_{0,B_n}(t)\right) = \sup_{\mu\in\mathscr{M}}\left(\tau_3\mu(\operatorname{diag}(B_n\times B_n)) - I(\mu)\right),$$

so that

$$\liminf_{n\to\infty} a_n \geq \lim_{t\to\infty}\left(\frac{1}{t}\log d_{0,D}(t)\right), \qquad D = \operatorname{diag}(S\times S). \tag{2.30}$$

In order to evaluate the right hand side explicitly we note that the $\{h_t(x)\}_{x\in S}$, $h_t(x) := E^{\{0,x\}}\exp(\tau_3\int_0^t 1_{\{X_s^1 - X_s^2 = 0\}}\,ds)$ fulfill, by the Feynman-Kac formula, the following differential equations [L denotes the generator of $(X_s^1 - X_s^2)_{s\in\mathbb{R}^+}$]:

$$\frac{d}{dt}h_t(x) = (Lh_t)_{(x)} + \tau_3\delta_0(x). \tag{2.31}$$

In [4] Section 2 we studied this system using renewal theory to obtain that

$$h_t(0) \underset{t\to\infty}{\sim} c\cdot e^{+at} \quad \text{with } a \text{ being the solution of (2.23).} \tag{2.32}$$

Since $d_{0,D}(t) = h_t(0)$ this proves our assertion (2.23).

Proof of Lemma 3. This is a special case of Lemma 0 (Section 1) if we consider the random walk $(X_s^{(1)}, X_s^{(2)})$ as a $2d$-dimensional symmetric walk and apply Lemma 0 to $[-N,N]^{2d}$.

2.4. Proof of Theorem 2. From Proposition 4 and Proposition 2 and the Corollary we obtain

$$\begin{aligned}
&\liminf_{t\to\infty}\left(t^{-1}\log \hat{f}_t(x,x)\right) > 0 \quad \text{for } p > p^{(2)} \quad \text{a.s.,}\\
&\limsup_{t\to\infty}\left(t^{-1}\log \hat{f}_t(x,x)\right) \leq 0 \quad \text{for } p \leq p^{(2)} \quad \text{a.s.}
\end{aligned} \tag{2.33}$$

Lemma 4 below tells us that the above statement holds also for the function $\tilde{f}_t(x, x)$. Since the definition $f_t(x, x) = \tilde{E}(\eta_t^\mu(x))^2 - \tilde{E}(\eta_t^\mu(x))$ and therefore for $\lambda \geq 0$ by (0.6): $(1/t)\log \tilde{E}(\hat{\eta}_t^\mu(x))^2 \sim_{t\to\infty} (1/t)\log \tilde{f}_t(x, x)$. We are done in the case $\lambda \geq 0$. The case $\lambda < 0$ is a trivial consequence of Lemma 4(b) below.

LEMMA 4. *Denote* $\lambda = b - \tilde{d} - D_1$ *and* $h_t(x) = \tilde{E}(\eta_t^\mu(x))e^{-2\lambda t}$. *Then we have*:

(a)

$$\lambda \geq 0 \qquad p \underset{(=)}{<} p^{(2)}: \tilde{f}_t(x, y) \leq \hat{f}_t(x, y) + K \ (+Kt^2), \tag{2.34}$$

(b)

$$\lambda < 0 \qquad \tilde{f}_t(x, y) \geq \hat{f}_t(x, y) + ce^{(-\lambda t)}, \tag{2.35}$$

(c)

$$\tilde{f}_t(x, y) \geq \hat{f}_t(x, y). \tag{2.36}$$

To prove this lemma one uses the following well known fact: Let U_t, V_t be two semigroups of operators on some L^∞, with generators L_u, L_v. Then

$$V_t(f) = U_t(f) + \int_0^t U_{t-s}(L_v - L_u)V_s(f)\,ds. \tag{2.37}$$

If we apply this to our situation, i.e., we consider $L^\infty(S \times S) \oplus L^\infty(S)$ [denote an element of this space in the form (f, h) with $f \in L^\infty(S \times S)$, $h \in L^\infty(S)$] and define

$$U_t(f_0, h_0) = (\hat{f}_t, h_t), \qquad V_t(f_0, h_0) = (\tilde{f}_t, h_t). \tag{2.38}$$

Then we obtain after some calculations [note by (1.23) $h_t(x) \leq \rho e^{-\lambda t}$] that:

$$\begin{aligned} 0 &\leq \tilde{f}_t(x, y) - \hat{f}_t(x, y) \\ &\leq c \cdot \left(\int_0^t h_{t-s}(x, y)e^{-2s}\,ds + \int_0^t h_{t-s}(y, x)e^{-2s}\,ds \right), \end{aligned} \tag{2.39}$$

where $\{h_s(x, y)\}_{x, y \in S}$ is the solution of (2.5) deleting the last row, with initial conditions $h_0(x, y) = \hat{q}(x, y)$ $(= \frac{1}{2}(q(x, y) + q(y, x)))$.

If $p < p^{(2)}$ then $h_s(x, y) \leq C$ for all $s \in \mathbb{R}^+$ [by (2.8), (2.13), (2.15)], so that the right side of (2.39) is bounded by (for $\lambda > 0$):

$$\tilde{c}\int_0^t e^{-2s}\,ds \leq K, \quad \text{by } Kt^2 \text{ for } p = p.^{(2)} \tag{2.40}$$

This proves (a). Parts (b) and (c) are immediate consequences of formula (2.37).

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