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OCCUPATION TIMES FOR CRITICAL BRANCHING BROWNIAN MOTIONS

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We prove central limit theorems, strong laws, large deviation results, and a weak convergence theorem for suitably normalized occupation times of critical binary branching Brownian motions started from Poisson random fields on R^d, $d \geq 2$. The results are strongly dimension dependent. The main result (Theorem 2) asserts that in two dimensions, as opposed to all other dimensions, the average occupation time of a bounded set with positive measure converges in distribution to a nondegenerate limit.

1. Introduction. The infinite particle system known as critical branching Brownian motion has been widely studied. Sawyer [19] and Fleischman [10] connect the model with problems in mathematical genetics, and give a number of references to the early literature. Dawson and Ivanoff [7] provide a more recent survey with some additional references. Probably the simplest version of this system can be described as follows. Particles are initially situated in R^d according to a Poisson random field with uniform density one. Letting ξ_t denote the countable set of sites in R^d occupied by particles at time t, the particle at each $x \in \xi_t$ undergoes Brownian motion until it either splits into two particles at exponential rate one or disappears at exponential rate one. This process is density preserving and exhibits *clustering* in dimension one or two, but *stability* in three or more dimensions. The precise formulation of the dichotomy is

$$\xi_t \to^{v} 0 \quad \text{as} \quad t \to \infty, \qquad d = 1 \text{ or } 2, \tag{1.1}$$

$$\to^{v} \xi_\infty \quad \text{as} \quad t \to \infty, \qquad d \geq 3, \tag{1.1'}$$

where $\to^{v}$ denotes vague convergence in distribution (cf., [7], pp. 63–4), 0 is the "no particles" random field, and ξ_∞ is a nontrivial infinitely divisible field. Conceptually, (1.1) indicates that the extant particles pack into smaller and smaller regions of space while preserving the overall density; (1.1′) is the more familiar convergence to equilibrium. Proofs of (1.1) and (1.1′) are indicated in [7] and [6]. Letting $N_t(A)$ denote the number of particles in A at time t, (1.1)

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amounts to the assertion that

$$\lim_{t\to\infty} P(N_t(A) = 0) = 1, \qquad A \text{ bounded Borel,}$$

for $d = 1, 2$. Fleischman [10] has some interesting normalized limit laws for $N_t(A)$ in these cases of clustering. Additional results concerning $N_t(A)$ can be found in the nice paper by Holley and Stroock [12]. Rather similar models are the critical "shower processes" of [17] and [15], Durrett's $\beta = 1$ infinite particle system on R^d with additive interaction [8], and Dawson's critical measure diffusion process [6]. Somewhat less closely related is the interacting particle system known as the voter model [3], [11].

Our object here is to study the *occupation times* of critical binary branching Brownian motion:

$$T_t(A) = \int_0^t N_s(A)\, ds$$

$$= \text{the total time particles spend in } A \text{ up to time } t,$$

A (nonempty) bounded Borel. As far as we are aware, only two papers have considered such occupation times to date. The first, by Sawyer and Fleischman [20], deals primarily with dimension one. They prove that

$$\lim_{t\to\infty} T_t(A) < \infty \quad \text{a.s.}, \qquad d = 1, \tag{1.2}$$

$$= \infty \quad \text{a.s.}, \qquad A \text{ open}, \ d \geq 2. \tag{1.2'}$$

This intriguing result implies that in one dimension the clustering is so strong that any bounded set is eventually vacated with probability one. In two dimensions, on the other hand, even though the chance that a particle occupies a given bounded set at large times becomes arbitrarily small, still the set is visited recurrently with probability one. Of course for $d \geq 3$, (1.2′) is to be expected in light of (1.1′). The second paper, by Iscoe [13], deals with occupation times for Dawson's measure-valued branching diffusions. In the case analogous to ours ($\alpha = 2, \beta = 1$), Iscoe proves central limit theorems for dimensions $d \geq 3$. The normalization is standard for $d \geq 5$, but is larger if $d = 3$ or 4. He has no results when $d = 2$.

In a density preserving particle system one expects the average density of particles on A up to time t, $t^{-1}T_t(A)$, to converge almost surely except in rare cases. For instance (1.2) implies

$$\lim_{t\to\infty} t^{-1}T_t(A) = 0 \quad \text{a.s.}, \qquad d = 1, \tag{1.3}$$

and in light of (1.1′) we expect

$$\lim_{t\to\infty} t^{-1}T_t(A) = |A| \quad \text{a.s.}, \qquad d \geq 3. \tag{1.3'}$$

($|A|$ denotes the Lebesgue measure of A; *henceforth assume* $|A| > 0$.) As another example, consider the voter model with density $\theta \in (0,1)$. We showed in [4] that

$$\lim_{t\to\infty} t^{-1} T_t^{\text{voter}}(A) = \theta \# A \quad \text{a.s.}, \qquad d \ge 2$$

($A \subset Z^d$, $\#A$ = the cardinality of A). Thus the limit is constant *even in dimension two*, the case of critical clustering. For the *one-dimensional* voter model, on the other hand, it turns out that

$$t^{-1} T_t^{\text{voter}} \Rightarrow Y \quad \text{as} \quad t \to \infty, \qquad d = 1,$$

($\Rightarrow$ denotes convergence in distribution), where Y is a nontrivial random variable which can be represented in terms of coalescing Brownian motions, by means of Arratia's invariance principle [1]. Again, see [4].

Our primary goal in this paper is to fill the gap between (1.3) and (1.3′) by proving a limit theorem for $t^{-1}T_t(A)$ in dimension two. First though, to warm up, we derive central limit theorems analogous to Iscoe's, and the corresponding strong laws (1.3′), in dimensions $d \ge 3$. The preliminary results are as follows.

THEOREM 1. *For $d \ge 3$, A bounded Borel, as $t \to \infty$,*

$$\frac{T_t(A) - |A|t}{b_t} \Rightarrow \mathcal{N}(0, \sigma^2),$$

where

$$\begin{aligned} b_t &= t^{3/4}, && d = 3,\\ &= \sqrt{t \log t}, && d = 4,\\ &= t^{1/2}, && d \ge 5, \end{aligned}$$

$$\begin{aligned} \sigma^2 &= 8(\sqrt{2} - 1)|A|^2/(3\pi^{3/2}), && d = 3,\\ &= |A|^2/(2\pi^2), && d = 4,\\ &= 2\textstyle\int_0^\infty \int_A \int_A (1 + s) p_s(x, y)\, dx\, dy\, ds, && d \ge 5, \end{aligned}$$

and $p_s(x, y) = (2\pi s)^{-d/2} \exp\{-|x - y|^2/2s\}$. Also, (1.3′) holds.

We do not know whether Iscoe's methods can be modified to yield Theorem 1. Our proof is rather different from his, following instead the lead of [4], where more difficult results with the same scalings are proved for the voter model. (One could undoubtedly extend Theorem 1 to the normalized occupation time random field setting of [4], but we will not do so.)

Turning next to our main result, it will be shown that for critical branching Brownian motion in two dimensions $t^{-1}T_t(A)$ converges in distribution to a nontrivial limit, so the limit is *not* a.s. constant in this instance of critical clustering.

THEOREM 2. *Let $d = 2$. Then as $t \to \infty$,*

$$\frac{T_t(A)}{|A|t} \Rightarrow \mathscr{V},$$

where $\mathscr{V}$ is a nontrivial infinitely divisible random variable, independent of A, with $\mathrm{Var}(\mathscr{V}) = (\log 2)/\pi$, *and with moments of all orders.*

Unfortunately we are not able to identify the limit $\mathscr{V}$ in any satisfactory sense. It does not seem to have a standard distribution, nor can we give a functional representation in the spirit of Arratia's invariance principle for the one-dimensional voter model. In fact we cannot even resolve the simplest questions, e.g., whether $\mathscr{V} > 0$ a.s.

What we *will* do is give the *cumulants* of $\mathscr{V}$. The proof of Theorem 2 and indeed all the results of this paper are based on the so-called method of cumulants. At the crudest level this method can be used to prove central limit theorems and strong laws as in Theorem 1. A more refined application of the same method (cf., [18], [9]) gives large deviation theorems of the form

$$P\left(\frac{T_t(A)}{|A|t} > \alpha\right) \propto \exp\{-I(\alpha)a_t\} \quad \text{as} \quad t \to \infty,$$

or more precisely,

$$\lim_{t\to\infty} a_t^{-1}\log P\left(\frac{T_t(A)}{|A|t} > \alpha\right) = -I(\alpha) \in (-\infty, 0), \tag{1.4}$$

for $\alpha > 1$, with analogous statements for deviations below the mean. This program was carried out completely in [5] for Poisson systems of independent random walks. Our final theorems give similar results for critical branching Brownian motion in dimensions $d \geq 3$. The proper normalizations are

$$\begin{aligned} a_t &= \sqrt{t}, && d = 3, \\ &= t/\log t, && d = 4, \\ &= t, && d \geq 5. \end{aligned} \tag{1.5}$$

Moreover our main result, Theorem 2, is essentially a special case of (1.4) with

$$a_t = 1, \qquad d = 2.$$

For clarity and aesthetic reasons this special case will be presented first. The subsequent large deviation theorems are not entirely satisfactory because the method of cumulants only establishes (1.4) for levels α sufficiently close to 1. (We were able to overcome this obstacle in [5] by using transforms on the simpler systems.) To make matters worse, in dimension four we have not been able to carry out the exact asymptotics necessary to prove the existence of $I(\alpha)$, and so have to settle for a weaker result. These are undoubtedly technical shortcomings;

(1.4) and the corresponding result below the mean should hold for all $\alpha > 0$ in each $d \geq 3$, with a_t as in (1.5). The large deviation results we will prove are as follows. For brevity only results above the mean are stated; strictly analogous statements can be proved below the mean.

THEOREM 3. *Let $d = 3$. Then there is an $\alpha_+ \in (1, \infty)$ such that*

$$\lim_{t\to\infty} t^{-1/2} \log P\left(\frac{T_t(A)}{|A|t} > \alpha \right) = -I(\alpha) \in (-\infty, 0)$$

for $1 < \alpha < \alpha_+$, with α_+ and $I(\alpha)$ independent of A.

THEOREM 4. *Let $d = 4$. Then there is an $\alpha_+ \in (1, \infty)$ such that*

$$-\infty < \liminf_{t\to\infty} \frac{\log t}{t} \log P\left(\frac{T_t(A)}{|A|t} > \alpha \right)$$
$$\leq \limsup_{t\to\infty} \frac{\log t}{t} \log P\left(\frac{T_t(A)}{|A|t} > \alpha \right) < 0$$

for $1 < \alpha < \alpha_+$. (Our α_+ depends on $|A|$.)

THEOREM 5. *Let $d \geq 5$. Then there is an $\alpha_+ \in (1, \infty)$ such that*

$$\lim_{t\to\infty} t^{-1} \log P\left(\frac{T_t(A)}{|A|t} > \alpha \right) = -I_A(\alpha) \in (-\infty, 0)$$

for $1 < \alpha < \alpha_+$. (Our α_+ depends on A.)

The remainder of the paper is organized as follows. Section 2 contains a standard reduction of cumulants for the infinite system ξ_t to corresponding moments for finite systems ξ_t^x of critical branching Brownian motions starting with a single particle at x. The basic formulas we use to compute the cumulants of $T_t(A)$ are then derived. Next, in Section 3 we prove Theorem 1. The main result, Theorem 2, is proved in Section 4. This may be read independently of Section 3, although Theorem 1 is a good trial run. Finally, Section 5 contains the proofs of the large deviation Theorems 3, 4, and 5.

To keep matters as simple as possible, all our theorems are proved only for the basic critical binary branching Brownian motion described in the first paragraph above. Various extensions are doubtless possible without qualitative change, e.g.,

(i) to more general branching mechanisms,

(ii) to more general diffusions,

(iii) to more general initial states,

(iv) to branching random walks, shower processes, etc.,

(v) to measure-valued branching diffusions.

Many of the papers cited in the references study one or more of these generalizations. It would be interesting to see the extensions of some of our results worked out in detail.

2. Preliminaries. Let $T_t(A)$, $t \geq 0$, A bounded Borel with $|A| > 0$, be the occupation times of critical (binary) branching Brownian motion, as described above. The distribution of $T_t(A)$ is captured by its cumulant generating function

$$\mathscr{C}(\lambda) = \log E\left[\exp\{\lambda T_t(A)\}\right].$$

In this section we give an inductive procedure [formulas (2.2), (2.3), (2.5), and (2.6) below] to evaluate $\mathscr{C}(\lambda)$. The first ingredient is a familiar identity, sometimes called *Campbell's formula*:

$$\mathscr{C}(\lambda) = \int [M^x(\lambda) - 1]\, dx, \tag{2.1}$$

where $M^x(\lambda)$ are the one particle moment generating functions

$$M^x(\lambda) = E\left[\exp\{\lambda T_t^x(A)\}\right],$$
$$T_t^x(A) = \int_0^t \#\xi_s^x \cap A\, ds,$$

and

$$\xi_t^x = \text{critical (binary) Brownian motion starting with a single particle at } x \in R^d.$$

(Throughout the paper we will write $\int$ for $\int_{R^d}$.) The easy computation for (2.1) can be modelled on the proof of Lemma 2 in [5]. From (2.1) we see that when the expansion of $\mathscr{C}(\lambda)$ in terms of its cumulants

$$\mathscr{C}(\lambda) = \sum_{n=1}^{\infty} m_n(t) \frac{\lambda^n}{n!} \tag{2.2}$$

converges absolutely, then one has

$$m_n(t) = \int m_n(x, t)\, dx, \tag{2.3}$$

with

$$m_n(x, t) = E\left[\left(T_t^x(A)\right)^n\right] = \text{the } n\text{th moment of } T_t^x(A).$$

(Here and below we often regard the set A as fixed and suppress it from the notation.) So to get a handle on $T_t(A)$ one needs to compute the $m_n(x, t)$. A heuristic evaluation can be made using the time of the first split or disappearance to derive a simple differential equation. Namely, for $n \geq 1$ a little formal calculation shows that one should have

$$\frac{dm_n(x, t)}{dt} = \varphi_n(x, t) + \tfrac{1}{2}\Delta m_n(x, t), \tag{2.4}$$

where

$$\varphi_n(x,t) = nm_{n-1}(x,t)1_A(x) + \sum_{j=1}^{n-1}\binom{n}{j}m_j(x,t)m_{n-j}(x,t) \tag{2.5}$$

$[m_0(x,t) = 1$ and $\Delta =$ Laplacian]. The key observation is that (due to the criticality of the process) φ_n depends only on the m_j with $j < n$. Thus (2.4) can be solved by means of the well known representation

$$m_n(x,t) = \int_0^t \int p_{t-u}(x,y)\varphi_n(y,u)\,dy\,du, \tag{2.6}$$

$p_t(x,y)$ the Brownian transition kernel. Any reader willing to accept the validity of (2.6) may proceed directly to the next section. For the skeptics we now outline a rigorous derivation of (2.6), working directly from a paper by Kulperger [16]. In the notation of [16], it is shown that for $n \geq 1$, $0 < s_1 < \cdots < s_n$, $y_i \in R^d$, and $k_i = 0$ or 1 $(1 \leq i \leq n)$, there is a function

$$p^x_{(k_1,\dots,k_n)}(s_1,\dots,s_n; y_1,\dots,y_n)$$

giving the density for ξ^x to have particles at y_i at time s_i whenever $k_i = 1$. [Ignore (y_i, s_i) such that $k_i = 0$.] Moreover, using techniques from [14], he derives the recursion formulas

$$\begin{aligned}
&p^x_{(1,\dots,1)}(s_1,\dots,s_n; y_1,\dots,y_n)\\
&\quad = p_{s_1}(x,y_1)p^{y_1}_{(1,\dots,1)}(s_2 - s_1,\dots,s_n - s_1; y_2,\dots,y_n)\\
&\quad\quad + \sum_{j=1}^{n-1}\int_0^{s_1} ds \int dy\, p_s(x,y)\\
&\quad\quad \cdot \sum_{\substack{(k_1,\dots,k_n)\\ k_1+\cdots+k_n=j}} p^y_{(k_1,\dots,k_n)}(s_1 - s,\dots,s_n - s; y_1,\dots,y_n)\\
&\quad\quad\quad \cdot p^y_{(1-k_1,\dots,1-k_n)}(s_1 - s,\dots,s_n - s; y_1,\dots,y_n)
\end{aligned} \tag{2.7}$$

for $n \geq 2$, with $p^x_1(s_1; y_1) = p_{s_1}(x,y_1)$. This last shows that (2.6) holds for $n = 1$. To get (2.6) from (2.7) for $n \geq 2$, write

$$\begin{aligned}
m_n(x,t) = n!\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_A dy_1 \cdots \int_A dy_n\\
\cdot p^x_{(1,\dots,1)}(s_1,\dots,s_n; y_1,\dots,y_n).
\end{aligned} \tag{2.8}$$

Substituting the first term on the right side of (2.7) into (2.8) yields

$$\begin{aligned}
&n!\int_0^t ds_1 \int_A dy_1\, p_{s_1}(x,y_1)\int_0^{t-s_1} du_2 \int_{u_2}^{t-s_1} du_3 \cdots \int_{u_{n-1}}^{t-s_1} du_n\\
&\quad \cdot \int_A dy_2 \cdots \int_A dy_n\, p^{y_1}_{(1,\dots,1)}(u_2,\dots,u_n; y_2,\dots,y_n)\\
&\quad = n\int_0^t ds \int_A dy\, p_{t-s}(x,y)m_{n-1}(y,s),
\end{aligned}$$

$n \geq 2$. For fixed j, a typical $(k_1, \dots, k_n)$ in the sum from the second term of (2.7) is $k_1 = \cdots = k_j = 1$, $k_{j+1} = \cdots = k_n = 0$. Substituting into (2.8) yields

$$n! \int_0^t ds \int_s^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{R^d} dy \int_A dy_1 \cdots \int_A dy_n \, p_s(x, y)$$
$$\cdot p^y_{(1,\dots,1)}(s_1 - s, \dots, s_j - s; y_1, \dots, y_j)$$
$$\cdot p^y_{(1,\dots,1)}(s_{j+1} - s, \dots, s_n - s; y_{j+1}, \dots, y_n)$$
$$= n! \int_0^t ds \int_0^{t-s} du_1 \int_{u_1}^{t-s} du_2 \cdots \int_{u_{n-1}}^{t-s} du_n \int_{R^d} dy \int_A dy_1 \cdots \int_A dy_n \, p_s(x, y)$$
$$\cdot p^y_{(1,\dots,1)}(u_1, \dots, u_j; y_1, \dots, y_j)$$
$$\cdot p^y_{(1,\dots,1)}(u_{j+1}, \dots, u_n; y_{j+1}, \dots, y_n)$$
$$= \int_0^t ds \int_{R^d} dy p_s(x, y)$$
$$\cdot \left[\int_0^{t-s} du_1 \cdots \int_0^{t-s} du_j \int_A dy_1 \cdots \int_A dy_j \, p^y_{(1,\dots,1)}(u_1, \dots, u_j; y_1, \dots, y_j) \right]$$
$$\cdot \left[\int_0^{t-s} du_{j+1} \cdots \int_0^{t-s} du_n \int_A dy_{j+1} \cdots \int_A dy_n \cdot p^y_{(1,\dots,1)}(u_{j+1}, \dots, u_n; y_{j+1}, \dots, y_n) \right]$$
$$= \int_0^t ds \int_{R^d} dy p_{t-s}(x, y) m_j(y, s) m_{n-j}(y, s),$$

$n \geq 2$. Each $(k_1, \dots, k_n)$ with $\Sigma k_i = j$ yields the same result. There are $\binom{n}{j}$ such terms, so (2.6) follows for $n \geq 2$.

3. The proof of Theorem 1. To prove our central limit theorems by the method of cumulants it suffices to show that if $d \geq 3$, then

(a) $m_1(t) = |A|t$ for all t,
(b) $m_2(t) \sim \sigma^2 b_t^2$ as $t \to \infty$, and
(c) $m_n(t) = o(b_t^n)$ as $t \to \infty$ for each $n \geq 3$.

(This is a "combinatorial" variation on the usual method of moments; formally one is checking that

$$\lim_{t \to \infty} \log E\left[\exp\left\{\lambda\left(\frac{T_t(A) - |A|t}{b_t}\right)\right\}\right] = \frac{1}{2}\sigma^2\lambda^2.)$$

For the strong law it is enough to have

(d) $m_2(t) = O(t^{2-\varepsilon})$ as $t \to \infty$ for some $\varepsilon > 0$.

(Apply Chebyshev and Borel–Cantelli to a geometric subsequence; see Section 6 of [4] for a subtler version of the argument.) Let us now proceed to verify (a)–(d).

Claim (a) is immediate from (2.3), (2.5), and (2.6) with $n = 1$:

$$m_1(t) = \int m_1(x,t)\,dx = \int_0^t du \int dx \int_A dy\, p_{t-u}(x,y) = |A|t.$$

The variance computations for claim (b) are more involved. First, some notation. Set

$$G_u(A) = \int_0^u dv \int_A dy \int_A dz\, p_v(y,z), \qquad G(A) = G_\infty(A);$$

$$H_u(A) = \int_0^u dv \int_A dy \int_A dz\, v p_v(y,z), \qquad H(A) = H_\infty(A).$$

Observe that $G(A) < \infty$ for $d \ge 3$, $H(A) < \infty$ for $d \ge 5$ (A bounded Borel). From (2.3), (2.5), and (2.6),

$$\begin{aligned} m_2(t) &= 2\left[\int_0^t du \int_A dy\, m_1(y,u) + \int_0^t du \int dy\, m_1^2(y,u)\right] \\ &= 2\left[\int_0^t G_u(A)\,du + \int_0^t du \int dy \int_0^u dv_1 \int_0^u dv_2 \int_A dz_1 \int_A dz_2\, p_{v_1}(y,z_1) \cdot p_{v_2}(y,z_2)\right] \qquad (3.1) \\ &= 2\left[\int_0^t G_u(A)\,du + \int_0^t du \int_A dz_1 \int_A dz_2 \int_0^u dv_1 \int_0^u dv_2\, p_{v_1+v_2}(z_1,z_2)\right] \\ &= 2\int_0^t G_u(A)\,du + 4\int_0^t (t-u)(G_{2u}(A) - G_u(A))\,du, \end{aligned}$$

this last by a change of variables. Now suppose that $d \ge 5$. Then $G_u(A) \uparrow G(A) < \infty$ as $u \to \infty$ and

$$\begin{aligned} 4\int_0^\infty (G_{2u}(A) - G_u(A))\,du &= 4\int_0^\infty du \int_u^{2u} dv \int_A dy \int_A dz\, p_v(y,z) \\ &= 2H(A) < \infty. \end{aligned}$$

By monotone convergence we see that

$$\lim_{t\to\infty} t^{-1} m_2(t) = 2(G(A) + H(A)) = \sigma^2,$$

as desired. If $d = 3$ or 4 one has to compute a bit more. As $u \to \infty$,

$$\begin{aligned} u^{(d/2)-1}(G_{2u}(A) - G_u(A)) &= \int_1^2 \frac{dv}{(2\pi v)^{d/2}} \int_A dx \int_A dy\, e^{-|x-y|^2/2uv} \\ &\to (2\pi)^{-d/2}|A|^2 \int_1^2 \frac{dv}{v^{d/2}} \qquad (3.2) \\ &= \frac{2|A|^2}{d-2}(2\pi)^{-d/2}\left(1 - \left(\frac{1}{2}\right)^{(d/2)-1}\right). \end{aligned}$$

Thus

$$G_{2u}(A) - G_u(A) \sim Cu^{1-(d/2)} \quad \text{as} \quad u \to \infty,$$

with C given by (3.2). Substituting in (3.1) one easily verifies claim (b) for $d = 3$

and $d = 4$. [The first integral of (3.1) is of lower order than the second in these cases.] In passing we alert the reader that the manipulations leading to (3.1) constitute the *basic trick* of this paper. Many variations on that calculation will appear below.

Claim (d) follows immediately from (b), so to finish the proof of Theorem 1 it remains to verify (c). Let us write

$$\overline{G}(t) = \int_1^t p_u(0,0)\,du, \qquad \overline{H}(t) = \int_1^t u p_u(0,0)\,du.$$

Note that $\overline{G}(t) \le 2$ for all $t \ge 1$, $d \ge 3$, and as $t \to \infty$,

$$\begin{aligned} \overline{H}(t) &\sim 2^{-1/2}\pi^{-3/2}t^{1/2}, && d = 3,\\ &\sim \frac{1}{4\pi^2}\log t, && d = 4,\\ &\sim \frac{2}{d-4}(2\pi)^{-d/2}, && d \ge 5. \end{aligned}$$

We will prove the following *key estimate*: for $d \ge 3$

$$m_n(t) \le C(n)t\tilde{H}(t)^{n-1}, \qquad t \ge 1, \tag{3.3}$$

where

$$C(n) = n!4^{n-1}(|A| \vee 1)^n$$

and

$$\tilde{H}(t) = 12 + 2\overline{H}(2t).$$

Since $\tilde{H}(t)$ has the same growth rate as $\overline{H}(t)$, some easy calculations show that (3.3) implies (c). Actually, (3.3) is a somewhat more careful estimate than one needs for Theorem 1, but its full strength will be used in Section 5 for our large deviation results. Write

$$\varphi_n(t) = \int \varphi_n(x,t)\,dx, \qquad n \ge 1, t \ge 0.$$

It is easy to verify by induction that $\varphi_n(x,t)$, $\varphi_n(t)$, $m_n(x,t)$, and $m_n(t)$ are all nondecreasing functions of t for each $n \ge 1$. From (2.3) and (2.6),

$$m_n(t) = \int_0^t \varphi_n(u)\,du.$$

So to get (3.3) it suffices to show by induction that for all $k \ge 1$

$$\sup_x \varphi_k(x,t) \vee \varphi_k(t) \le D(k)(|A| \vee 1)^k \tilde{H}(t)^{k-1}, \tag{3.4}$$

where $D(k) \le k!4^{k-1}$. Let $D(1) = 1$ and define $D(k)$, $k \ge 2$, recursively by

$$D(k) = \sum_{l=1}^{k-1} \binom{k}{l} D(l)D(k-l). \tag{3.5}$$

Check that in closed form,

$$D(k) = \frac{(2k-2)!}{(k-1)!} \le k!\binom{2(k-1)}{k-1} \le k!4^{k-1}.$$

We will demonstrate (3.4) for these $D(k)$. Inequality (3.4) is obvious for $k = 1$. Assume $n \geq 2$, and that (3.4) holds for $1 \leq k < n$. Integrating separately over $(0, 1)$ and $[1, t]$ in (2.6), and using monotonicity, one has

$$\begin{aligned} m_k(x,t) &\leq \sup_y \varphi_k(y,t) + \overline{G}(t)\varphi_k(t) \\ &\leq 3D(k)(|A| \vee 1)^k \tilde{H}(t)^{k-1}. \end{aligned} \tag{3.6}$$

Use this bound in (2.5) to find that

$$\begin{aligned} \varphi_n(x,t) &\leq 3nD(n-1)(|A| \vee 1)^{n-1}\tilde{H}(t)^{n-2} \\ &\quad + 9\sum_{k=1}^{n-1}\binom{n}{k}D(k)D(n-k)(|A| \vee 1)^n \tilde{H}(t)^{n-2} \\ &\leq D(n)(|A| \vee 1)^n\left[12\tilde{H}(t)^{n-2}\right] \\ &\leq D(n)(|A| \vee 1)^n \tilde{H}(t)^{n-1}, \end{aligned}$$

proving the first part of (3.4). For the second part write

$$\begin{aligned} \varphi_n(t) &= n\int_A m_{n-1}(x,t)\,dx + \int \sum_{k=1}^{n-1}\binom{n}{k} m_k(x,t)m_{n-k}(x,t)\,dx \\ &= \qquad I_1(t) \qquad + \qquad I_2(t). \end{aligned}$$

By (3.6),

$$I_1(t) \leq 3nD(n-1)(|A| \vee 1)^n \tilde{H}(t)^{n-2}.$$

To estimate $I_2(t)$, first substitute for m_k and m_{n-k} using (2.6), then use monotonicity and the basic trick to see that

$$I_2(t) \leq \sum_{k=1}^{n-1}\binom{n}{k}\int_0^t du \int_0^t dv \int dy \int dz\, p_{u+v}(y,z)\varphi_k(y,t)\varphi_{n-k}(z,t).$$

The right side is majorized by $I_2^1(t) + 2I_2^2(t)$, where

$$\begin{aligned} I_2^1(t) &= \sum_{k=1}^{n-1}\binom{n}{k}\int_0^1 du \int_0^1 dv \int dy \int dz\, p_{u+v}(y,z)\varphi_k(y,t)\varphi_{n-k}(z,t) \\ &\leq \sum_{k=1}^{n-1}\binom{n}{k}\sup_x \varphi_k(x,t)\cdot\varphi_{n-k}(t) \\ &\leq D(n)(|A| \vee 1)^n \tilde{H}(t)^{n-2} \end{aligned}$$

and

$$\begin{aligned} I_2^2(t) &= \sum_{k=1}^{n-1}\binom{n}{k}\int_1^t du \int_0^t dv \int dy \int dz\, p_{u+v}(y,z)\varphi_k(y,t)\varphi_{n-k}(z,t) \\ &\leq \sum_{k=1}^{n-1}\binom{n}{k}\varphi_k(t)\varphi_{n-k}(t)\int_1^t du \int_0^t dv\, p_{u+v}(0,0) \\ &\leq \sum_{k=1}^{n-1}\binom{n}{k}D(k)D(n-k)(|A| \vee 1)^n \tilde{H}(t)^{n-2}\int_1^{2t} u p_u(0,0)\,du \\ &\leq D(n)(|A| \vee 1)^n \tilde{H}(t)^{n-2}\overline{H}(2t). \end{aligned}$$

Putting the pieces together, we conclude that

$$\varphi_n(t) \le D(n)(|A| \vee 1)^n \left[4\tilde{H}(t)^{n-2} + 2\tilde{H}(t)^{n-2}\overline{H}(2t)\right]$$
$$\le D(n)(|A| \vee 1)^n \tilde{H}(t)^{n-1},$$

as desired. The proof of Theorem 1 is finished. □

4. The proof of Theorem 2. Throughout this section $d = 2$. To prove Theorem 2 by the method of cumulants, one needs to show the existence of finite constants $\overline{m}_n$ such that

$$\lim_{t\to\infty} \frac{m_n(t)}{t^n} = \overline{m}_n \quad \text{for all} \quad n \ge 1 \tag{4.1}$$

and

(4.2) the sequence $\overline{m}_n$ is distribution determining.

Again, this is a variation on the method of moments; cf., for example, Section 8.12 of [2]. A sufficient condition for (4.2) is

$$|\overline{m}_n| \le n! M^n \quad \text{for some} \quad M < \infty \qquad n \ge 1. \tag{4.3}$$

(If X has cumulants $\overline{m}_n$ and (4.3) holds, then

$$\log E[e^{\lambda X}] = \sum \frac{\overline{m}_n}{n!}\lambda^n < \infty$$

for small positive λ, and so $E[e^{\lambda X}] < \infty$ for small positive λ. Hence the law of X is uniquely determined by Proposition 8.49 of [2].)

We will prove (4.1) and (4.3). The idea for (4.1) is to use the self-similarity property of Brownian motion,

$$p_{st}(x\sqrt{t}, y\sqrt{t}) = t^{-1}p_s(x, y),$$

by setting

$$\varphi_n^t(x,s) = \frac{\varphi_n(x\sqrt{t}, st)}{t^{n-2}}, \qquad m_n^t(x,s) = \frac{m_n(x\sqrt{t}, st)}{t^{n-1}}, \qquad m_n^t(s) = \int m_n^t(x,s)\,dx,$$

$n \ge 1, t \ge 0, x \in R^2 - \{0\}, 0 \le s \le 1$. The equations (2.5) and (2.6) become

$$\varphi_n^t(x,s) = n 1_{A/\sqrt{t}}(x) m_{n-1}^t(x,s) + \sum_{k=1}^{n-1} \binom{n}{k} m_k^t(x,s) m_{n-k}^t(x,s) \tag{4.4}$$

and

$$m_n^t(x,s) = \int_0^s du \int dy\, p_{s-u}(x,y)\varphi_n^t(y,u),$$

respectively ($x \ne 0, m_0^t(x,s) = t$). Consequently (4.1) is equivalent to the existence of $\overline{m}_n$ such that

$$\lim_{t\to\infty} m_n^t(1) = \overline{m}_n \quad \text{for all} \quad n \ge 1. \tag{4.5}$$

As $t \to \infty$, the first term in (4.4) concentrates near the origin, so one is led to the following recursive scheme for evaluating the $\overline{m}_n$. Let

$$m_1^\infty(x,s) = |A| \int_0^s p_u(x,0)\,du$$

and for $n \geq 2$,

$$\varphi_n^\infty(x,s) = \sum_{k=1}^{n-1} \binom{n}{k} m_k^\infty(x,s) m_{n-k}^\infty(x,s),$$

$$m_n^\infty(x,s) = \int_0^s du \int dy\, p_{s-u}(x,y) \varphi_n^\infty(y,u),$$

$x \in R^2 - \{0\}, 0 \leq s \leq 1$. We will show below that (4.5) holds with

$$\bar{m}_n = \int m_n^\infty(x,1)\, dx. \tag{4.6}$$

First, though, let us dispense with (4.3) by proving the following result.

LEMMA 1. *For $h > 0$, set $\bar{m}_n(h) = \sup_{|x| \geq h} m_n^\infty(x,1)$, and define $\bar{m}_n$ as above. Then for all $n \geq 1$,*

$$0 \leq \bar{m}_n(h) < \infty \quad \textit{for each} \quad h > 0$$

and

$$0 \leq \bar{m}_n \leq (n-1)!|A|^n [4(\log 2)/\pi]^{n-1}.$$

PROOF OF LEMMA 1. This parallels the proof of the key estimate (3.3). The calculations here are cleaner, but a little more delicate. Clearly $\bar{m}_1(h) < \infty$ for each $h > 0$, and $\bar{m}_1 = |A|$. It is also easy to see by induction that $m_n^\infty(x,\cdot)$ and $\varphi_n^\infty(x,\cdot)$ are nonnegative nondecreasing. So, writing $\bar{\varphi}_1 = |A|$ and

$$\bar{\varphi}_n(h) = \sup_{|x| \geq h} \varphi_n^\infty(x,1), \qquad \bar{\varphi}_n = \int \varphi_n^\infty(x,1)\, dx$$

for $n \geq 2$ we have

$$\begin{aligned}
\bar{m}_n(h) &\leq \sup_{|x|\geq h} \int_0^1 du \int_{|y-x| \geq |x|/2} dy\, p_{s-u}(x,y) \varphi_n^\infty(y,1) \\
&\quad + \sup_{|x|\geq h} \int_0^1 du \int_{|y-x| < |x|/2} dy\, p_{s-u}(x,y) \varphi_n^\infty(y,1) \\
&\leq \left[\sup_{|x| \geq h, v \leq 1} p_v(0, x/2) \right] \bar{\varphi}_n \\
&\quad + \sup_{|x| \geq h} \int_0^1 du \int_{|y| \geq h/2} dy\, p_u(x,y) \varphi_n^\infty(y,1) \\
&\leq C(h) \bar{\varphi}_n + \bar{\varphi}_n(h/2),
\end{aligned}$$

with $C(h) < \infty$ for each $h \geq 0$. Also,

$$\bar{m}_n \leq \int dx \int_0^1 du \int dy\, p_{1-u}(x,y) \varphi_n^\infty(y,1) = \bar{\varphi}_n$$

and

$$\bar{\varphi}_n(h) \le \sup_{|x| \ge h} \sum_{k=1}^{n-1} \binom{n}{k} m_k^{\infty}(x,1) m_{n-k}^{\infty}(x,1)$$

$$\le \sum_{k=1}^{n-1} \binom{n}{k} \bar{m}_k(h) \bar{m}_{n-k}(h).$$

So to prove the lemma it suffices to show by induction that

$$\bar{\varphi}_n \le D(n)|A|^n \left(\frac{\log 2}{\pi} \right)^{n-1}, \qquad n \ge 1, \tag{4.7}$$

with $D(n)$ as in (3.5). This is trivial if $n = 1$. For $n = 2$,

$$\bar{\varphi}_2 = 2 \int dx \left[|A| \int_0^1 p_u(x,0)\, du \right] \left[|A| \int_0^1 p_v(x,0)\, dv \right]$$

$$= 2|A|^2 \int_0^1 \int_0^1 p_{u+v}(0,0)\, du\, dv = \frac{2|A|^2}{\pi} \log 2,$$

this last by a little calculus, and (4.7) holds. For $n \ge 3$,

$$\bar{\varphi}_n = 2n \int dx \left[\int_0^1 du\, p_u(x,0) \right] \left[\int_0^1 dv \int dz\, p_{1-v}(x,z) \varphi_{n-1}^{\infty}(z,v) \right]$$

$$+ \sum_{k=2}^{n-2} \binom{n}{k} \int dx \left[\int_0^1 du \int dy\, p_{1-u}(x,y) \varphi_k^{\infty}(y,u) \right]$$

$$\cdot \left[\int_0^1 dv \int dz\, p_{1-v}(x,y) \varphi_{n-k}^{\infty}(z,v) \right]$$

$$\le \sum_{k=1}^{n-1} \binom{n}{k} \bar{\varphi}_k \bar{\varphi}_{n-k} \int_0^1 \int_0^1 p_{u+v}(0,0)\, du\, dv$$

$$\le D(n)|A|^n \left(\frac{\log 2}{\pi} \right)^{n-1},$$

by induction. Lemma 1 is proved. □

To complete the proof of Theorem 2 we need to show (4.5). Consider the *error terms*:

$$\varepsilon_n^t(x,s) = m_n^t(x,s) - m_n^{\infty}(x,s), \qquad x \ne 0$$

$$= \int_0^s \int (p_u(x,y) - p_u(x,0)) t 1_{A/\sqrt{t}}(y)\, dy\, du, \qquad n = 1,$$

$$= \int_0^s \int p_{s-u}(x,y) \delta_n^t(y,u)\, dy\, du, \qquad n \ge 2,$$

where

$$\delta_n^t(x,s) = nm_{n-1}^t(x,s)1_{A/\sqrt{t}}(x) + 2\sum_{k=1}^{n-1}\binom{n}{k}\varepsilon_k^t(x,s)m_{n-k}^\infty(x,s) + \sum_{k=1}^{n-1}\binom{n}{k}\varepsilon_k^t(x,s)\varepsilon_{n-k}^t(x,s), \qquad n \geq 2.$$

Then (4.5) is equivalent to

$$\lim_{t\to\infty}\int \varepsilon_n^t(x,1)\,dx = 0, \qquad n \geq 1. \tag{4.8}$$

In analogy with the $\overline{m}$ and $\overline{\varphi}$ above, we introduce

$$\bar{\varepsilon}_n^t(h) = \sup_{s\leq 1, |x|\geq h} |\varepsilon_n^t(x,s)|, \qquad \bar{\varepsilon}_n^t = \int \sup_{s\leq 1}|\varepsilon_n^t(x,s)|\,dx, \qquad (h > 0, n \geq 1),$$

$$\bar{\delta}_n^t(h) = \sup_{s\leq 1, |x|\geq h} |\delta_n^t(x,s)|, \qquad \bar{\delta}_n^t + \int \sup_{s\leq 1}|\delta_n^t(x,s)|\,dx, \qquad (h > 0, n \geq 2).$$

By convention $\bar{\delta}_1^t \equiv 0$. The convergence (4.8) is an immediate consequence of the following result.

LEMMA 2. *For all* $n \geq 1$,

$$\lim_{t\to\infty} \bar{\varepsilon}_n^t(h) = 0 \quad \textit{for each} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\varepsilon}_n^t = 0.$$

The proof of Lemma 2 follows the same outline as the proof of Lemma 1, but is more involved. For clarity we divide it into three parts.

LEMMA 3.

$$\lim_{t\to\infty} \bar{\varepsilon}_1^t(h) = 0 \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\varepsilon}_1^t = 0.$$

LEMMA 4. *If* $n \geq 2$, *and for every* $k \leq n - 1$

$$\lim_{t\to\infty} \bar{\varepsilon}_k^t(h) = 0 \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\varepsilon}_k^t = 0$$

and

$$\lim_{t\to\infty} \bar{\delta}_k^t(h) = 0 \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\delta}_k^t = 0,$$

then

$$\lim_{t\to\infty} \bar{\delta}_n^t(h) = 0 \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\delta}_n^t = 0.$$

LEMMA 5. *For $n \geq 2$, if*

$$\lim_{t\to\infty} \bar{\delta}_n^t(h) = 0 \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\delta}_n^t = 0,$$

then

$$\lim_{t\to\infty} \bar{\varepsilon}_n^t(h) = 0 \quad \textit{for all} \quad h > 0 \quad \textit{and} \quad \lim_{t\to\infty} \bar{\varepsilon}_n^t = 0.$$

Clearly Lemmas 3 through 5 will prove Lemma 2 by induction.

PROOF OF LEMMA 3.

$$\bar{\varepsilon}_1^t(h) \leq \sup_{|x|\geq h} \int_0^1 du \int_{y\in A/\sqrt{t}} dy\,|p_u(x,y) - p_u(x,0)|t.$$

Divide $(0,1)$ into $(0,q)$ and $[q,1)$ to get

$$\bar{\varepsilon}_1^t(h) \leq |A| \sup_{\substack{|x|\geq h,\, y\in A/\sqrt{t} \\ q\leq u\leq 1}} |p_u(x,y) - p_u(x,0)|$$
$$+|A|q \sup_{\substack{|x|\geq h,\, y\in A/\sqrt{t} \\ u\leq 1}} [p_u(x,y) + p_u(x,0)].$$

For fixed q the first term on the right tends to 0 and the second is at most $C(h)q$ as $t \to \infty$, where $C(h) < \infty$. Let $q \to 0$ to find that $\lim_t \bar{\varepsilon}_1^t(h) = 0$. Similarly,

$$\bar{\varepsilon}_1^t \leq |A| \int dx \sup_{\substack{y\in A/\sqrt{t} \\ q\leq u\leq 1}} |p_u(x,y) - p_u(x,0)| + 2|A|q.$$

The first term on the right tends to 0 as $t \to \infty$ by dominated convergence. Let $q \to 0$ to get $\lim_t \bar{\varepsilon}_1^t = 0$. □

PROOF OF LEMMA 4. Write

$$|\delta_n^t(x,s)| \leq n\alpha_n^t(x,s) + \sum_{k=1}^{n-1} \binom{n}{k} \left(2\beta_{n,k}^t(x,s) + \gamma_{n,k}^t(x,s)\right),$$

where

$$\alpha_n^t(x,s) = m_{n-1}^t(x,s) 1_{A/\sqrt{t}}(x),$$
$$\beta_{n,k}^t(x,s) = |\varepsilon_k^t(x,s)| m_{n-k}^\infty(x,s),$$
$$\gamma_{n,k}^t(x,s) = |\varepsilon_k^t(x,s)\varepsilon_{n-k}^t(x,s)|.$$

Define

$$\bar{\alpha}_n^t(h) = \sup_{s \le 1, |x| \ge h} \alpha_n^t(x,s), \qquad \bar{\alpha}_n^t = \int \sup_{s \le 1} \alpha_n^t(x,s)\,dx$$

and define $\bar{\beta}_{n,k}^t(h)$, $\bar{\beta}_{n,k}^t$, $\bar{\gamma}_{n,k}^t(h)$, and $\bar{\gamma}_{n,k}^t$ analogously. It suffices to show that these six types of terms tend to 0 as $t \to \infty$. We will use the fact that $\varphi_n^t(x,\cdot)$ and $m_n^t(x,\cdot)$ are monotone, which is easily checked by induction.

$$\alpha) \qquad \bar{\alpha}_n^t(h) = 0 \quad \text{for } t \text{ large enough that } \frac{A}{\sqrt{t}} \subset \{|z| \le h\}.$$

$$\bar{\alpha}_n^t = \int_{x \in A/\sqrt{t}} dx \int dy \int_0^1 du\, p_u(x,y) \varphi_{n-1}^t(y, 1-u).$$

Integrate u separately over $(0, t^{-1})$ and $[t^{-1}, 1)$ to get

$$\bar{\alpha}_n^t \le \left(\frac{1}{t} + \frac{|A| \log t}{2\pi t} \right) \int \varphi_{n-1}^t(y,1)\,dy.$$

The last integral is bounded uniformly in t for each n, as a consequence of (3.4). So $\bar{\alpha}_n^t \to 0$ as $t \to \infty$ as desired.

$$\beta) \qquad \bar{\beta}_{n,k}^t(h) \le \bar{\varepsilon}_k^t(h) \bar{m}_{n-k}(h) \to 0 \quad \text{as} \quad t \to \infty$$

by Lemma 1 and the induction hypothesis. Next suppose $2 \le k < n-1$. Then

$$\begin{aligned}
\bar{\beta}_{n,k}^t &= \int dx \sup_{s \le 1} \left| \int_0^s \int p_{s-u}(x,y) \delta_k^t(y,u)\,dy\,du \right| \int_0^s \int p_{s-v}(x,z) \varphi_{n-k}^\infty(z,v)\,dz\,dv \\
&\le \int dx \iint dy\,dz \sup_{\tau \le 1} |\delta_k^t(y,\tau)| \varphi_{n-k}^\infty(z,1) \int_0^1 \int_0^1 p_{1-u}(x,y) p_{1-v}(x,z)\,du\,dv \\
&= \iint dy\,dz \sup_{\tau \le 1} |\delta_k^t(y,\tau)| \varphi_{n-k}^\infty(z,1) \int_0^1 \int_0^1 p_{u+v}(y,z)\,du\,dv \\
&\le \iint dy\,dz \sup_{\tau \le 1} |\delta_k^t(y,\tau)| \varphi_{n-k}^\infty(z,1) \int_0^1 \int_0^1 \frac{1}{2\pi(u+v)}\,du\,dv \\
&= \frac{\log 2}{\pi} \bar{\delta}_k^t \bar{\varphi}_{n-k} \to 0 \quad \text{as} \quad t \to \infty
\end{aligned}$$

by induction. If $2 \le k = n-1$ a similar estimate gives

$$\begin{aligned}
\bar{\beta}_{n,n-1}^t &\le |A| \int dx \int dy \sup_{\tau \le 1} |\delta_{n-1}^t(y,\tau)| \int_0^1 \int_0^1 p_u(x,y) p_v(x,0)\,du\,dv \\
&\le |A| \frac{\log 2}{\pi} \bar{\delta}_{n-1}^t \to 0 \quad \text{as} \quad t \to \infty.
\end{aligned}$$

It remains to check the case $k = 1, n \geq 2$. If $n \geq 3$,

$$\bar{\beta}^t_{n,1} \leq \int dx \int_0^1 du \int dy \, |p_u(x, y) - p_u(x,0)| t 1_{A/\sqrt{t}}(y)$$

$$\cdot \int_0^1 dv \int dz \, p_{1-v}(x, z) \varphi^\infty_{n-1}(z, v)$$

$$\leq |A| \int dx \int dz \, \varphi^\infty_{n-1}(z,1) \int_q^1 \int_q^1 \sup_{y \in A/\sqrt{t}} |p_u(x, y) - p_u(x,0)| p_v(x, z) \, du \, dv$$

$$+ \int dy \int dz \, \varphi^\infty_{n-1}(z,1)$$

$$\cdot \iint_{\{u<q \text{ or } v<q\}} [p_{u+v}(y, z) + p_{u+v}(0, z)] t 1_{A/\sqrt{t}}(y) \, du \, dv$$

$$\leq \frac{|A|}{2\pi q} \bar{\varphi}_{n-1} \int dx \sup_{\substack{y \in A/\sqrt{t} \\ q \leq u \leq 1}} |p_u(x, y) - p_u(x,0)|$$

$$+ \frac{2|A|}{2\pi} \bar{\varphi}_{n-1} \iint_{\{u<q \text{ or } v<q\}} \frac{1}{u+v} \, du \, du.$$

As in the proof of Lemma 3, induction and dominated convergence show that the first term on the right tends to 0 as $t \to \infty$ for fixed $q > 0$, and that the second tends to 0 as $q \to 0$. Thus $\lim_{t\to\infty} \bar{\beta}^t_{n,1} = 0$. A similar argument yields the last inequality when $n = 2$ as well (with $\bar{\varphi}_1 = |A|$), so $\lim_{t\to\infty} \bar{\beta}^t_{2,1} = 0$.

γ)
$$\bar{\gamma}^t_{n,k}(h) \leq \bar{\varepsilon}^t_k(h) \bar{\varepsilon}^t_{n-k}(h) \to 0 \quad \text{as} \quad t \to \infty$$

by induction. Supposing $n \geq 2$ and $2 \leq k < n - 1$, and proceeding as in β),

$$\bar{\gamma}^t_{n,k} \leq \frac{\log 2}{\pi} \bar{\delta}^t_k \bar{\delta}^t_{n-k} \to 0 \quad \text{as} \quad t \to \infty,$$

again by induction. Finally, as in β), to conclude that $\lim_{t\to\infty} \bar{\gamma}^t_{n,1} = 0$, it suffices to show that as $t \to \infty$,

$$|A| \int dx \int dz \sup_{\tau \leq 1} |\bar{\delta}^t_{n-1}(z, \tau)| \int_q^1 \int_q^1 \sup_{y \in A/\sqrt{t}} |p_u(x, y) - p_u(x,0)| p_v(x, z) \, du \, dv$$

$$+ \int dy \int dz \sup_{\tau \leq 1} |\bar{\delta}^t_{n-1}(z, \tau)|$$

$$\cdot \iint_{\{u<q \text{ or } v<q\}} [p_{u+v}(y; z) + p_{u+v}(0, z)] t 1_{A/\sqrt{t}}(y) \, du \, dv$$

$$\to 0 \quad \text{for} \quad n \geq 3$$

and

$$|A|^2\int dx \int_q^1\int_q^1 \sup_{y\in A/\sqrt{t}} |p_u(x,y)-p_u(x,0)| \sup_{z\in A/\sqrt{t}} |p_v(x,z)-p_v(x,0)|\,du\,dv$$

$$+4\int dy\int dz \iint_{\{u<q \text{ or } v<q\}} p_{u+v}(0,0)t1_{A/\sqrt{t}}(y)t1_{A/\sqrt{t}}(z)\,du\,dv \to 0.$$

The last two expressions are bounded by

$$\frac{\bar{\delta}_{n-1}^t}{2\pi q}|A|\int dx \sup_{\substack{y\in A/\sqrt{t}\\ q\le u\le 1}} |p_u(x,y)-p_u(x,0)| + \frac{2\bar{\delta}_{n-1}^t}{2\pi}|A|\iint_{\{u<q \text{ or } v<q\}}\frac{1}{u+v}\,du\,dv$$

and

$$\frac{|A|^2}{2\pi}\int dx \sup_{\substack{y\in A/\sqrt{t}\\ q\le u\le 1}} |p_u(x,y)-p_u(x,0)| + \frac{2|A|^2}{\pi}\iint_{\{u<q \text{ or } v<q\}}\frac{1}{u+v}\,du\,dv,$$

respectively. The desired conclusion follows. □

PROOF OF LEMMA 5. We easily obtain the estimates

$$\bar{\varepsilon}_n^t(h) \le C(h)\bar{\delta}_n^t + \bar{\delta}_n^t(h/2), \qquad \bar{\varepsilon}_n^t \le \bar{\delta}_n^t, \qquad C(h) < \infty. \qquad \square$$

The proof of Lemma 2 is finished. □

Two remarks and a computation will now complete the proof of Theorem 2. We have shown that $t^{-1}T_t(A)$ converges in distribution. The limit is infinitely divisible because the initial mean 1 Poisson field is an independent sum of n mean n^{-1} Poisson fields, which evolve independently and which each satisfy an analogous limit law. Also, an easy induction shows that the cumulants $\bar{m}_n$ in (4.6) are of the form $|A|^n\bar{\bar{m}}_n$, with $\bar{\bar{m}}_n$ independent of A. In other words,

$$\frac{T_t(A)}{t|A|} \Rightarrow \mathscr{V} \quad \text{independent of } A \quad \text{as} \quad t\to\infty.$$

Finally, we compute

$$\begin{aligned}\operatorname{Var}(\mathscr{V}) = \bar{\bar{m}}_2 &= 2\int dx\int_0^1 du\int dy\, p_{1-u}(x,y)\left[\int_0^u p_v(y,0)\,dv\right]^2\\ &= 2\int_0^1 du\int_0^u\int_0^u \frac{1}{2\pi(v_1+v_2)}\,dv_1\,dv_2\\ &= \int_0^1 \frac{2\log 2}{\pi}u\,du = \log 2/\pi. \qquad \square\end{aligned}$$

5. Large deviation theorems. This final section proves large deviation results in dimensions $d\ge 3$. Theorems 3, 4, and 5 constitute the second phase of a project initiated in [5] to study large deviations for occupation times of some simple infinite particle systems. The reader would do well to look at [5] before proceeding, because the results given there for Poisson systems of independent random walks are more complete and transparent. There the large deviation tails are "fat" in dimensions $d = 1, 2$, and "usual" for $d \geq 3$. Here the tails are fat if $d = 3, 4$, and usual for $d \geq 5$. (The third phase of our program, in a subsequent paper, will study an *interacting* particle system, the voter model, whose occupation time large deviations behave pretty much like those of critical branching.)

Theorems 3 and 5 are applications of a general large deviations result due to Plachky and Steinebach [18]. In our cases the normalized cumulants converge individually, and so the normalized cumulant generating function converges in a neighborhood of 0. We formulate this special case of the Plachky-Steinebach theorem as a lemma.

LEMMA 6. *Let Y_t, $t \geq 0$, be a sequence of random variables, and $a_t \to \infty$ a normalizing sequence, such that for some $\lambda_0 \in (0, \infty)$,*

$$\Psi_t(\lambda) = a_t^{-1} \log E\left[e^{\lambda Y}\right] < \infty, \qquad |\lambda| < \lambda_0,$$

and such that the power series expansion

$$\Psi_t(\lambda) = \sum_{n=1}^{\infty} a_t^{-1} c_n(t) \frac{\lambda^n}{n!}$$

converges absolutely for $|\lambda| < \lambda_0$. Suppose that

(i) $\lim_{t \to \infty} a_t^{-1} c_n(t) = c_n \in (-\infty, \infty)$ *for each* $n \geq 1$,

(ii) $c_2 > 0$, *and*

(iii) $\sup_t |c_n(t)/a_t| = O(n! \lambda_0^{-n})$ *as* $n \to \infty$.

Put

$$\Psi(\lambda) = \sum_{n=1}^{\infty} c_n \lambda^n / n!, \qquad \alpha_+ = \lim_{\lambda \uparrow \lambda_0} \Psi'(\lambda), \qquad \alpha_- = \lim_{\lambda \downarrow -\lambda_0} \Psi'(\lambda).$$

Then $c_1 \in (\alpha_-, \alpha_+)$, and

$$\lim_{t \to \infty} a_t^{-1} \log P\left(\frac{Y_t}{a_t} > \alpha\right) = -I(\alpha) \in (-\infty, 0), \qquad \alpha \in (c_1, \alpha_+),$$

$$\lim_{t \to \infty} a_t^{-1} \log P\left(\frac{Y_t}{a_t} < \alpha\right) = -I(\alpha) \in (-\infty, 0), \qquad \alpha \in (\alpha_-, c_1),$$

where

$$I(\alpha) = \alpha \lambda_\alpha - \Psi(\lambda_\alpha),$$

and $\lambda_\alpha \in (-\lambda_0, \lambda_0)$ is the unique solution of $\Psi'(\lambda) = \alpha$.

PROOF. Conditions (i)–(iii) ensure that the hypotheses of the main result in [18] apply. [(iii) implies that $\Psi(\lambda)$ is real analytic and (ii) gives strict convexity on $(-\lambda_0, \lambda_0)$.] See [5], [9], and [18] for more details. □

PROOF OF THEOREM 3. We simply mimic the proof of Theorem 2, but now with $d = 3$. Define $a_t = \sqrt{t}, Y_t = T_t(A)/\sqrt{t}$. Then $c_n(t) = m_n(t)/t^{n/2}$, so according to Lemma 5 we need only show that

(i) $\lim_{t\to\infty} m_n(t)/t^{(n+1)/2} = \overline{m}_n \in [0, \infty)$ for each n,

(ii) $\overline{m}_2 > 0$, and

(iii) $\sup_t m_n(t)/t^{(n+1)/2} = O(n!\lambda_0^{-n})$ as $n \to \infty$ for some $\lambda_0 > 0$.

The proper scalings in three dimensions are

$$\varphi_n^t(x, s) = \varphi_n(x\sqrt{t}, st)/t^{(n/2)-2}, \qquad m_n^t(x, s) = m_n(x\sqrt{t}, st)/t^{(n/2)-1}.$$

With this modification (4.4) assumes essentially the same form, and the remainder of the proof of (i) goes through very much as before. There is an added factor of $\sqrt{t}$ in the α terms, but analogous estimates go through. As before $\overline{m}_1 = |A|$, now

$$0 < \overline{m}_2 = 2|A|^2 \int_0^1 du \int_0^u \int_0^u \frac{1}{[2\pi(v_1 + v_2)]^{3/2}} \, dv_1 \, dv_2$$

$$= \frac{8|A|^2}{3\pi^{3/2}} (\sqrt{2} - 1) < \infty \qquad \text{(cf. Theorem 1)},$$

and all applications of the basic trick involve similar integrals. Again $\overline{m}_n = |A|^n \overline{\overline{m}}_n$, which gives the independence of A in the statement of the theorem. Finally, (ii) is immediate from (3.3) for a suitable $\lambda_0 > 0$. Further details are left to the reader. □

PROOF OF THEOREM 5. Put $a_t = t, Y_t = T_t(A)$. To apply Lemma 6 we check that:

(i) $\lim_{t\to\infty} m_n(t)/t = \overline{m}_n \in [0, \infty)$ for each n,

(ii) $\overline{m}_2 > 0$, and

(iii) $\sup_t m_n(t)/t = O(n!\lambda_0^{-n})$ as $n \to \infty$ for some $\lambda_0 > 0$.

Use (3.3) to get (iii). Claim (i) follows easily from monotonicity: Since $\varphi_n(x, t)$ and $\varphi_n(t)$ are increasing in t, letting $\overline{\varphi}_n(x)$ and $\overline{\varphi}_n$ be the respective limits as $t \to \infty$, we have

$$\overline{m}_n = \lim_t m_n(t)/t = \lim_t t^{-1} \int_0^t \varphi_n(s) \, ds = \overline{\varphi}_n.$$

In fact it is easy to check that the $\overline{m}_n$ are determined by the inductive recipe: $\overline{\varphi}_1(x) = 1_A(x)$, and for $n \ge 2$,

$$\overline{\varphi}_n(x) = n 1_A(x) \int_0^\infty \int p_u(x, y) \overline{\varphi}_{n-1}(y) \, dy \, du$$

$$+ \sum_{k=1}^{n-1} \binom{n}{k} \int_0^\infty \int_0^\infty \iint p_u(x, y) p_v(x, z) \overline{\varphi}_k(y) \overline{\varphi}_{n-k}(z) \, dy \, dz \, du \, dv,$$

$\overline{m}_n = \overline{\varphi}_n = \int \overline{\varphi}_n(x)\,dx$. In particular,

$$\overline{m}_2 = \overline{\varphi}_2 = \int_0^\infty \int_A \int_A (1+s) p_s(x,y)\,dx\,dy\,ds > 0$$

(cf. Theorem 1), so (ii) holds. Further details are left to the reader. □

There remains the problem of dimension four. In this case the scaling argument of Theorems 2 and 3 does not work because

$$\int_0^1 \int_0^1 \frac{1}{(u+v)^2}\,du\,dv = \infty.$$

Nor do the properly normalized cumulant generating functions $\Psi_t(\lambda)$ enjoy any obvious monotonicity or subadditivity properties. This is an example where it may be very difficult to prove convergence of the $\Psi_t(\lambda)$ to a limit $\Psi(\lambda)$. In such cases one wants at least a technique that captures the *order of magnitude* of the large deviation tails. The idea is that by getting good upper and lower bounds on the $\Psi_t(\lambda)$ one should be able to obtain positive and finite bounds on the ostensible I function $I(\alpha)$, as in Theorem 4. In the notation of Lemma 6, for $d = 4$ we take $a_t = t/\log t$ and $Y_t = T_t(A)/|A|\log t$, so that

$$\Psi_t(\lambda) = \sum_{n=1}^{\infty} \frac{m_n(t)}{|A|^n t(\log t)^{n-1}} \frac{\lambda^n}{n!}.$$

Below we discuss only the situation above the mean ($\lambda > 0$); similar techniques give analogous results below the mean. Clearly, for $\lambda > 0$,

$$\Psi_t(\lambda) \ge \lambda + \frac{m_2(t)\lambda^2}{2|A|^2 t\log t}$$

$$\to \lambda + \lambda^2/4\pi^2 \quad \text{as} \quad t \to \infty,$$

by the mean and variance calculations of Theorem 1. Using the key estimate (3.3), on the other hand,

$$\Psi_t(\lambda) \le \lambda + J\sum_{n=2}^{\infty} \frac{t(K\log t)^{n-1}}{t(\log t)^{n-1}}\lambda^n$$

$$= \lambda + \frac{JK\lambda^2}{1-K\lambda}, \qquad 0 \le \lambda < K^{-1},$$

for suitable finite positive constants J and K. Thus the normalized cumulant generating functions $\Psi_t(\lambda)$ are sandwiched between strictly convex $\underline{\Psi}(\lambda)$ and $\overline{\Psi}(\lambda)$ with the same right derivative at 0. To prove Theorem 4 we can now apply the following general comparison result. We suspect that this last result may be useful for other complex large deviation problems where one is unable to prove existence of the limiting $\Psi(\lambda)$.

LEMMA 7. *Let Y_t, $t \geq 0$, be a sequence of random variables, and $a_t \to \infty$ a normalizing sequence, with*

$$\Psi_t(\lambda) = a_t^{-1} \log E[e^{\lambda Y_t}].$$

Let $\underline{\Psi}$ and $\overline{\Psi}$ be two functions such that for some $t_0 > 0, 0 < \lambda_0 < \infty$,

(i) *$\underline{\Psi}$ and $\overline{\Psi}$ are strictly convex on $[0, \lambda_0)$, and $\overline{\Psi}$ is differentiable on $[0, \lambda_0)$,*

(ii) *$\underline{\Psi}(0) = \overline{\Psi}(0) = 0, \underline{\Psi}'(0) = \overline{\Psi}'(0) = \mu$, and*

(iii) *$\underline{\Psi} \leq \Psi_t \leq \overline{\Psi}$ on $[0, \lambda_0)$ for all $t \geq t_0$.*

Then there exists an $\bar{\alpha} > \mu$ such that for all $\alpha \in (\mu, \bar{\alpha})$,

$$(5.3) \qquad \begin{aligned} -\infty &< \liminf_{t \to \infty} a_t^{-1} \log P(a_t^{-1} Y_t > \alpha) \\ &\leq \limsup_{t \to \infty} a_t^{-1} \log P(a_t^{-1} Y_t > \alpha) < 0. \end{aligned}$$

PROOF. Let $\bar{\alpha}_+ = \lim_{\lambda \uparrow \lambda_0} \overline{\Psi}(\lambda)/\lambda$, with $\bar{\alpha}_+ > \mu$. For $\alpha \in (\mu, \bar{\alpha}_+)$ let $\bar{I}(\alpha) = \alpha\lambda_\alpha - \overline{\Psi}(\lambda_\alpha)$, where λ_α is the unique solution of $\overline{\Psi}'(\lambda_\alpha) = \alpha$, and $\lambda_\alpha \in (0, \lambda_0)$. For $\alpha \in (\mu, \bar{\alpha}_+)$, by Chebyshev's inequality and (iii),

$$P(a_t^{-1} Y_t > \alpha) \leq \exp\{-a_t[\alpha\lambda - \overline{\Psi}(\lambda)]\}, \qquad t \geq t_0, 0 \leq \lambda < \lambda_0.$$

Set $\lambda = \lambda_\alpha$ to obtain

$$(5.4) \qquad P(a_t^{-1} Y_t > \alpha) \leq \exp\{-a_t \bar{I}(\alpha)\},$$

and the last inequality in (5.3) is now obvious.

The first inequality in (5.3) is also a consequence of Chebyshev's inequality, but requires more work. Let

$$\bar{\beta} = \lambda_0 \wedge \lim_{\alpha \uparrow \bar{\alpha}_+} \frac{\bar{I}(\alpha)}{\alpha - \mu}, \qquad \bar{\alpha} = \lim_{\lambda \uparrow \bar{\beta}} \frac{\underline{\Psi}(\lambda)}{\lambda}.$$

Then $0 < \bar{\beta} \leq \lambda_0$, $\mu < \bar{\alpha} \leq \bar{\alpha}_+$, and for each $\alpha \in (\mu, \bar{\alpha})$ there exist unique $\lambda^* \in (0, \bar{\beta})$, $\alpha^* \in (\alpha, \bar{\alpha}_+)$ such that

$$(5.5) \qquad \frac{\underline{\Psi}(\lambda^*)}{\lambda^*} = \alpha, \qquad \frac{\bar{I}(\alpha^*)}{\alpha^* - \mu} = \lambda^*.$$

These facts follow from convexity arguments.

Fix $\alpha \in (\mu, \bar{\alpha})$ for the remainder of the argument. For any $M \in (\alpha^*, \bar{\alpha}_+)$, integrate $e^{\lambda Y_t}$ separately over $\{a_t^{-1} Y_t \leq \alpha\}$, $\{\alpha < a_t^{-1} Y_t \leq M\}$, and (by parts) $\{M < a_t^{-1} Y_t\}$. The result is

$$(5.6) \qquad \begin{aligned} E[e^{\lambda Y_t}] &\leq e^{a_t \alpha \lambda} + e^{a_t M \lambda} P(a_t^{-1} Y_t > \alpha) \\ &\quad + e^{a_t M \lambda} P(a_t^{-1} Y_t > M) + a_t \lambda \int_M^\infty e^{a_t \lambda \rho} p(a_t^{-1} Y_t > \rho)\, d\rho, \end{aligned}$$

$0 \le \lambda < \lambda_0$. By convexity and (5.4) we have

$$P\left(a_t^{-1}Y_t > \rho\right) \le \exp\left\{-a_t \frac{\rho-\mu}{M-\mu}\bar{I}(M)\right\}, \qquad \rho \ge M.$$

With this estimate we find that

$$a_t\lambda \int_M^\infty e^{a_t\lambda\rho} P\left(a_t^{-1}Y_t > \rho\right) d\rho \le \frac{\lambda(M-\mu)}{\bar{I}(M)-\lambda(M-\mu)} e^{a_t[M\lambda - \bar{I}(M)]}$$

provided that $\bar{I}(M) - \lambda(M-\mu) > 0$. Rearranging (5.6) now yields

$$\begin{aligned}\lim_{t\to\infty} & a_t^{-1}\log P\left(a_t^{-1}Y_t > \alpha\right) \\ &\ge -\lambda M + \liminf_{t\to\infty} a_t^{-1}\log\Bigg(e^{a_t\underline{\Psi}(\lambda)} - e^{a_t\alpha\lambda} - e^{a_t[M\lambda-\bar{I}(M)]} \\ &\qquad\qquad - \frac{\lambda M}{\bar{I}(M)-\lambda(M-\mu)} e^{a_t[M\lambda - \bar{I}(M)]}\Bigg).\end{aligned}$$

In order to prove that for appropriate λ the first term in the parentheses dominates, i.e., that

$$\liminf_{t\to\infty} a_t^{-1}\log P\left(a_t^{-1}Y_t > \alpha\right) \ge -M\lambda + \underline{\Psi}(\lambda), \tag{5.7}$$

it suffices to show that

$$\underline{\Psi}(\lambda) > \lambda\alpha, \qquad \bar{I}(M) > \lambda(M-\mu), \qquad \underline{\Psi}(\lambda) > M\lambda - \bar{I}(M). \tag{5.8}$$

Since $\underline{\Psi}(\lambda) > \alpha\lambda$ for all $\lambda > \lambda^*$ (strict convexity of $\underline{\Psi}$), and since $\lambda^* < [\bar{I}(M)]/(M-\mu)$ for any $M > \alpha^*$ (strict convexity of $\bar{I}$), the first two requirements in (5.8) hold for all $\lambda \in (\lambda^*, \bar{I}(M)/(M-\mu))$. To deal with the third requirement in (5.8), let $\lambda \downarrow \lambda^*$ to obtain

$$\begin{aligned}\underline{\Psi}(\lambda) - M\lambda + \bar{I}(M) &\to \underline{\Psi}(\lambda^*) - M\lambda^* + \bar{I}(M) \\ &= \alpha\lambda^* - M\lambda^* + \bar{I}(M) \ge \alpha\lambda^* - M\lambda^* + \lambda^*(M-\mu) \\ &= \lambda^*(\alpha-\mu),\end{aligned}$$

which is clearly positive. Thus for all λ sufficiently close to λ^*, (5.8) is satisfied. Let $\lambda \downarrow \lambda^*$ and $M \downarrow \alpha^*$ in (5.7) to obtain

$$\begin{aligned}\liminf_{t\to\infty} a_t^{-1}\log P\left(a_t^{-1}Y_t > \alpha\right) &\ge -\lambda^*\alpha^* + \underline{\Psi}(\lambda^*) \\ &= -\lambda^*(\alpha^*-\alpha) > -\infty,\end{aligned}$$

which finishes the proof of (5.3). □

PROOF OF THEOREM 4. Use (5.1), (5.2), and Lemma 7. □

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