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ASYMPTOTIC BEHAVIOUR OF STABLE MEASURES NEAR THE ORIGIN

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We investigate the lower tail of $q_r = (\sum_{i=1}^{\infty}|\alpha_i\theta_i|^r)^{1/r}$ seminorms on R^∞, where $r \geq 1$ and θ_i are standard p-stable real random variables. We prove that for $p < r \leq 2$ we have $P\{q_r \leq t\} \geq \exp\{-ct^{-pr/(r-p)}\}$ in some neighbourhood of 0, where c is a nonnegative constant. If $r \leq p$, then for any positive, increasing function f, we can find q_r such that $P\{q_r \leq t\} \leq f(t)$ for $t \leq 1$. We also give a new characterization of Banach spaces of stable type p in terms of the behaviour of $\mu\{\|\cdot\| \leq t\}$ near 0, where μ is a symmetric and p-stable measure.

1. Introduction. In [2], Hoffmann-Jörgensen, Shepp, and Dudley have studied properties of Gaussian seminorms on R^∞. For example, they have shown that for norms $q = \sup|\alpha_n f_n|$ or $q = (\sum\alpha_n^2 f_n^2)^{1/2}$, where f_n are independent standard normal random variables, we have $P\{q \leq t\} \downarrow 0$ when $t \downarrow 0$ as rapidly as desired.

In this paper we consider p-stable seminorms, $0 < p < 2$. Contrary to the Gaussian case, distribution functions of norms as above, when f_n are standard p-stable, cannot tend to 0 as $t \downarrow 0$ in an arbitrary way.

In Section 4 we show that if $q_r = (\sum|\alpha_n|^r|f_n|^r)^{1/r}$, $p < r \leq 2$, then in some neighbourhood of 0, we have

$$P\{q_r \leq t\} \geq \exp\{-ct^{-rp/(r-p)}\},$$

where c is a constant determined by q_r. For $r \leq p$ the situation is different: For any increasing function f, $f(0) = 0$, we can find a seminorm q_r such that $P\{q_r \leq t\} \downarrow 0$, as $t \downarrow 0$ faster then f.

In Section 5 we show that for a separable Banach space E the following conditions are equivalent:

(i) E is of stable type p.

(ii) There exist positive numbers ρ and t_0 such that for every symmetric p-stable random vector X with spectral measure m, $m(E) = 1$, we have

$$P\{\|X\| \leq t\} \geq \exp(-t^{-\rho})$$

for $t \leq t_0$.

2. Preliminaries. $(E, \mathscr{B})$ is called a measurable vector space if E is a real vector space and $\mathscr{B}$ is a σ-field of subsets of E such that addition and multiplication by scalars are measurable operations.

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A function q: $E \to [0, \infty]$, $q(0) = 0$, is called a seminorm if it is subadditive and homogeneous. Throughout this paper we will only consider measurable seminorms.

A probability measure μ on $(E, \mathscr{B})$ is called p-stable, $0 < p \leq 2$, if for every independent E-valued random vectors X, Y with distributions μ and every $\alpha, \beta > 0$ we have

$$\mathscr{L}(\alpha X + \beta Y) = \mathscr{L}\big((\alpha^p + \beta^p)^{1/p} X + z\big), \tag{2.1}$$

where $z \in E$ and where $\mathscr{L}(X)$ denotes the distribution of X. If for every $\alpha, \beta > 0$, z can be taken to be 0, μ is called strictly p-stable.

Throughout this paper θ will denote a real stable random variable with the characteristic function $\exp(-|t|^p)$, $0 < p < 2$; $\{\theta_i\}$ will denote independent copies of θ.

A Banach space E is said to be of stable type p, $0 < p \leq 2$, if for each p', $0 < p' < p$, there exists a constant C such that for all $n \in \mathscr{N}$ and any $x_1, \dots, x_n \in E$

$$\left(E\left\|\sum_{i=1}^n \theta_i x_i\right\|^{p'}\right)^{1/p'} \leq C\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}.$$

A theorem of Maurey and Pisier [6] and Krivine [3] states that a Banach space E is of stable type p, $p < 2$, if and only if l_p is not finitely representable in E. We recall that l_p is finitely representable in E if for each $\varepsilon > 0$ and each $n \in \mathscr{N}$ one can find $x_1, \dots, x_n \in E$ such that for all $\beta_1, \dots, \beta_n \in R$ the following holds:

$$(1-\varepsilon)\left(\sum_{i=1}^n |\beta_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^n x_i \beta_i\right\| \leq (1+\varepsilon)\left(\sum_{i=1}^n |\beta_i|^p\right)^{1/p}. \tag{2.2}$$

We will also refer to Banach spaces of Rademacher type p, $1 \leq p \leq 2$. E is said to be of Rademacher type p, $1 \leq p \leq 2$, if there exists a constant B such that for all $n \in \mathscr{N}$ and any $x_1, \dots, x_n \in E$

$$\left(E\left\|\sum_{i=1}^n r_i x_i\right\|^{p}\right)^{1/p} \leq B\left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}, \tag{2.3}$$

where $\{r_i\}$ is a Rademacher sequence.

It is well known (see, e.g., [8]) that if E is a separable Banach space then for any p-stable, symmetric, E-valued random vector X the characteristic functional has the representation:

$$E[\exp(ix^*X)] = \exp - \int_E |x^*x|^p \, dm(x), \qquad x^* \in E^*,$$

where m is a finite measure concentrated on the unit sphere of E. m is called the spectral measure of X.

We will also use the following consequence of the Three Series Theorem. Let $0 < p < 2$. Then

(2.4)
(i) *For* $0 < r < p$ *the series* $\sum|\alpha_i\theta_i|^r$ *converges a.s. if and only if* $\sum|\alpha_i|^r < \infty$.
(ii) $\sum|\alpha_i\theta_i|^p$ *converges a.s. if and only if* $\sum|\alpha_i|^p \log(1 + 1/|\alpha_i|) < \infty$.
(iii) *For* $r > p$ *the series* $\sum|\alpha_i\theta_i|^r$ *converges a.s. if and only if* $\sum|\alpha_i\theta_i|^p < \infty$.

3. Lower tail of p-stable random vectors, $0 < p < 1$. We first consider the situation when $0 < p < 1$. Let q be a measurable seminorm on a measurable vector space $(E, \mathscr{B})$ and let X be an E-valued strictly p-stable random vector. Assume that $q(X)$ is finite a.s.

PROPOSITION 3.1. *If* $0 < p < 1$ *then for* $t \le s$ *we have*

$$P\{q(X) \le t\} \ge \exp\{-c(s)t^{-p/(1-p)}\}, \tag{3.1}$$

where $c(s) = -2s^{p/(1-p)}\log P\{q(X) \le s\} \ge 0$.

PROOF. Let Y be an independent copy of X. By (2.1) $\mathscr{L}(2^{1/p}X) = \mathscr{L}(X + Y)$. Therefore,

$$\begin{aligned}(P\{q(X) \le s\})^2 &= P\{q(X) \le s, q(Y) \le s\}\\ &\le P\{q(X + Y) \le 2s\} = P\{q(2^{1/p}X) \le 2s\}\\ &= P\{q(X) \le \alpha s\},\end{aligned} \tag{3.2}$$

where $\alpha = 2^{(p-1)/p} < 1$.

Iterating (3.2) we get

$$(P\{q(X) \le s\})^{2^n} \le P\{q(X) \le \alpha^n s\} \quad \text{for } n \ge 0. \tag{3.3}$$

Because $q(X) < \infty$ a.s., there exists a value s such that $P\{q(X) \le s\} > 0$ and since $\alpha^n \to 0$ then, by (3.3), $P\{q(X) \le s\} > 0$ for every $s > 0$. Taking logarithms of both sides of (3.3) we obtain

$$(\alpha^n s)^{p/(p-1)} s^{p/(1-p)} \log P\{q(X) \le s\} \le \log P\{q(X) \le \alpha^n s\}.$$

If $\alpha^{n+1}s < t \le \alpha^n s$ then $t^{p/(p-1)} \ge (\alpha^n s)^{p/(p-1)}$, so we have

$$\begin{aligned}t^{p/(p-1)} s^{p/(1-p)} \log P\{q(X) \le s\} &\le (\alpha^n s)^{p/(p-1)} s^{p/(1-p)} \log P\{q(X) \le s\}\\ &= (\alpha^{n+1}s)^{p/(p-1)} (\alpha s)^{p/(1-p)} \log P\{q(X) \le s\}\\ &\le \alpha^{p/(1-p)} \log P\{q(X) \le \alpha^{n+1}s\}\\ &\le (1/2)\log P\{q(X) \le t\}.\end{aligned}$$

Taking exponents of both sides we obtain (3.1).

As we will see later (Remark 4.7), the exponent $-p/(1-p)$ in the formula (3.1) cannot be improved even when $E = R$.

If $p \geq 1$ the following example taken from [7] shows that it is possible that $q(X) < \infty$ a.s. and $P\{q(X) \leq t\} = 0$ holds for some $t > 0$.

EXAMPLE 3.2. Let $\{a_k\}$, $\{n_k\}$ be sequences such that $(1/a_k)\sum_{i=1}^{n_k} |\theta_i| \to 1$ a.s. Such a choice is possible by virtue of a version of Weak Law of Large Numbers ([1], Chapter 7, page 236). Now, if we take $E = R^\infty$ with its cylindrical σ-field, $q(x) = \limsup_{k\to\infty}(1/a_k)\sum_{i=1}^{n_k} |x_i|$ and $X = (\theta_1, \theta_2, \ldots)$, then

$$P\{q(X) \leq t\} = \begin{cases} 0 & t < 1, \\ 1 & t \geq 1. \end{cases}$$

4. l_r seminorms on R^∞.

In this section E will stand for R^∞ with its cylindrical σ-field. We study here the behaviour of the lower tail of the seminorms $q_r(x) = (\sum_{i=1}^{\infty} |\alpha_i x_i|^r)^{1/r}$ for the R^∞-valued p-stable random vector $X = (\theta_1, \theta_2, \ldots)$.

We will use in the sequel the following well known fact (see [1], Chapter 6).

LEMMA 4.1. *Let $p < r \leq 2$ and η be a positive random variable with Laplace transform $\exp(-t^{p/r})$ and let ξ be a random variable with characteristic function $\exp(-t^r)$. If η and ξ are independent then*

$$\mathscr{L}(\theta) = \mathscr{L}(\xi\eta^{1/r}). \tag{4.1}$$

Now, we state and prove one important fact used in the sequel. First, we introduce some notation. Denote by

$$c(s, r, p) = -2(2E|\xi|^p)^{r/(r-p)} s^{p/(r-p)} \log P\{\eta \leq s\},$$

where $p < r \leq 2$, $s > 0$, and ξ, η are as above. Denote $\|\{\alpha_i\}\|_p = (\sum_{i=1}^{\infty} |\alpha_i|^p)^{1/p}$.

PROPOSITION 4.2. *If $p < r \leq 2$ and $\|\{\alpha_i\}\|_p < \infty$, then*

$$P\{q_r(X) \leq t\} \geq (1/2)\exp\Big\{-c(s, r, p)\big(t/\|\{\alpha_i\}\|_p\big)^{-rp/(r-p)}\Big\}, \tag{4.2}$$

for $t \leq 2^{1/p} s^{1/r} (E|\xi|^p)^{1/p} \|\{\alpha_i\}\|_p$.

PROOF. By part (iii) of (2.4) we have that $q_r(X)$ is finite a.s. Let $\{\xi_i\}$, $\{\eta_i\}$ be sequences of independent random variables defined on probability spaces $(\Omega_1, \mathscr{B}_1, P_1)$, $(\Omega_2, \mathscr{B}_2, P_2)$, respectively, such that $\mathscr{L}(\xi_i) = \mathscr{L}(\xi)$ and $\mathscr{L}(\eta_i) = \mathscr{L}(\eta)$. On the product space, $\{\xi_i\}$ and $\{\eta_i\}$ are independent, therefore, by (4.1),

$$\mathscr{L}(\{\theta_i\}) = \mathscr{L}\big(\{\xi_i \eta_i^{1/r}\}\big). \tag{4.3}$$

Since η is strictly (p/r)-stable, by (2.1) we have

$$\mathscr{L}\left(\sum_{i=1}^{n} \beta_i \eta_i\right) = \mathscr{L}\left(\left(\sum_{i=1}^{n} \beta_i^{p/r}\right)^{r/p} \eta_1\right), \qquad \beta_i \geq 0. \tag{4.4}$$

By (4.3), (4.4), and Fubini's theorem and since η_1 is positive and (p/r)-stable

$$
\begin{aligned}
P\{q_r(X) \le t\} &= P\left\{ \sum_{i=1}^{\infty} |\alpha_i|^r |\theta_i|^r \le t^r \right\} \\
&= P_1 \times P_2\left\{ \sum_{i=1}^{\infty} |\alpha_i|^r |\xi_i|^r \eta_i \le t^r \right\} \\
&= P_1 \times P_2\left\{ \left(\sum_{i=1}^{\infty} |\alpha_i|^p |\xi_i|^p \right)^{r/p} \eta_1 \le t^r \right\} \\
&= P_1 \times P_2\{Z\eta_1 \le t^r\},
\end{aligned}
$$

where $Z = (\sum_{i=1}^{\infty} |\alpha_i|^p |\xi_i|^p)^{r/p}$. Therefore, for $\alpha > 0$ we have

$$
\begin{aligned}
P\{q_r(X) \le t\} &\ge P_1 \times P_2\{Z \le 1/\alpha, \eta_1 \le \alpha t^r\} \\
&= P_1\{Z \le 1/\alpha\} P_2\{\eta_1 \le \alpha t^r\}.
\end{aligned}
\tag{4.5}
$$

By Chebyshev's inequality we have

$$
\begin{aligned}
P_1\{Z \le 1/\alpha\} &= 1 - P_1\{Z > 1/\alpha\} \ge 1 - EZ^{p/r}\alpha^{p/r} \\
&= 1 - \|\{\alpha_i\}\|_p^p E|\xi|^p \alpha^{p/r}.
\end{aligned}
$$

Applying (3.1) for $\alpha t^r \le s$, we obtain

$$
P_2\{\eta_1 \le \alpha t^r\} \ge \exp\left\{2s^{p/(r-p)} \log P\{\eta \le s\} \alpha^{p/(p-r)} t^{-rp/(r-p)}\right\}.
$$

Choosing $\alpha = (2^{1/p}\|\{\alpha_i\}\|_p (E|\xi|^p)^{1/p})^{-r}$, we get (4.2).

REMARK 4.3. If $r > 2$ then the inequality (4.2) holds whenever we put $r = 2$ in the right-hand side of this inequality. It is a consequence of the elementary fact that $q_r \le q_2$, $r > 2$. If $q_\infty = \sup |\alpha_i x_i|$ then $q_\infty(X)$ is finite a.s. if and only if $\|\{\alpha_i\}\|_p < \infty$ by the Borel–Cantelli lemma. Because $q_\infty \le q_2$, (4.2) holds for $r = 2$ in the right-hand side.

Next, we try to give an upper bound of $P\{q_r(X) \le t\}$. Our approach follows that in [2], which considers the Gaussian case.

Let λ_n denote the Lebesgue measure on R^n and

$$
B_n^r(t) = \left\{ x \in R^n : \left(\sum_{i=1}^{n} |x_i|^r \right)^{1/r} \le t \right\}, \qquad r, t > 0.
$$

As an easy application of the formula for the Dirichlet integral ([9], Section 7.7, page 178), we have that

$$
\lambda_n\{B_n^r(1)\} = (2/r)^n \left(\Gamma(r^{-1})\right)^n \left(\Gamma(nr^{-1} + 1)\right)^{-1}. \tag{4.6}
$$

LEMMA 4.4. *There exist positive constants A and B (determined by r) such that*

$$
(Bn^{-1/r})^n t^n \le \lambda_n\{B_n^r(t)\} \le (An^{-1/r})^n t^n, \tag{4.7}
$$

for all $t > 0$ and all $n \in \mathcal{N}$.

PROOF. Obviously, it is enough to prove (4.7) for $t = 1$. Let us estimate $\Gamma(nr^{-1} + 1)$:

$$\Gamma(nr^{-1} + 1) = \int_0^\infty x^{n/r} e^{-x}\, dx \geq \int_n^\infty x^{n/r} e^{-x}\, dx \geq n^{n/r} e^{-n}.$$

Because $\sup_{x \geq 0} x^{n/r} e^{-x/2} = (2nr^{-1})^{n/r} e^{-n/r}$, hence

$$\Gamma(nr^{-1} + 1) = \int_0^\infty x^{n/r} e^{-x/2} e^{-x/2}\, dx \leq 2\left((2r^{-1})^{1/r} e^{-1/r}\right)^n n^{n/r}.$$

If we put $A = 2r^{-1}\Gamma(r^{-1})e$ and $B = (2^{-1}er)^{1/r} r^{-1}\Gamma(r^{-1})$, then (4.7) follows immediately by (4.6).

Let g be the density of the distribution of θ and let $M = \sup_{x \in R} g(x) < \infty$. Assume that $\alpha_i > 0$ for all $i \in \mathcal{N}$. As a consequence of the above lemma and the following inequality

$$q_r = \left(\sum_{i=1}^{\infty} \alpha_i^r |\theta_i|^r\right)^{1/r} \geq \left(\sum_{i=1}^{n} \alpha_i^r |\theta_i|^r\right)^{1/r}, \qquad n \in \mathcal{N},$$

we get

LEMMA 4.5. *For all $n \in \mathcal{N}$ and $t > 0$*

$$P\{q_r(X) \leq t\} \leq \left(\prod_{i=1}^{n} 1/\alpha_i\right)(M \cdot An^{-1/r})^n t^n. \tag{4.8}$$

EXAMPLE 4.6. Let $r > p$ and $\alpha_i = (MAi^{(1/p+\varepsilon)})^{-1}$, $\varepsilon > 0$, where M and A are as in Lemma 4.5. Then

$$P\{q_r(X) \leq t\} \leq e \cdot \exp\left\{-(et)^{-rp/(r-p)+\delta}\right\} \quad \text{for } t \leq e^{-1},$$

where $\delta = r^2p^2\varepsilon\,(r - p + rp\varepsilon)^{-1}(r-p)^{-1}$.

Indeed, by part (iii) of (2.4) we have $q_r(X) < \infty$ a.s. Therefore, by (4.8),

$$P\{q_r(X) \leq t\} \leq \left(\prod_{i=1}^{n} i^{1/p+\varepsilon}\right)(n^{-1/r})^n t^n \leq (n^{1/p-1/r+\varepsilon} t)^n.$$

If we take $n = [(et)^{-pr/(r-p)+\delta}]$, where $[x]$ stands for the integer part of x, we get for $t \leq e^{-1}$

$$P\{q_r(X) \leq t\} \leq \exp\left\{-\left[(et)^{-pr/(r-p)+\delta}\right]\right\}$$
$$\leq e \cdot \exp\left\{-(et)^{-rp/(r-p)+\delta}\right\}.$$

This example shows that for every $\varepsilon > 0$ there exists a sequence $\{\alpha_i\}$ such that $q_r(X) < \infty$ a.s. and

$$P\{q_r(X) \leq t\} = o\left(\exp\{-t^{-pr/(r-p)+\varepsilon}\}\right) \quad \text{as } t \downarrow 0 \tag{4.9}$$

if $p < r$. Therefore, the exponent $-pr/(r-p)$ in the right-hand side of (4.2) cannot be improved.

REMARK 4.7. Let η be a positive random variable with the Laplace transform $\exp(-t^p)$, $p < 1$, then for every $\varepsilon > 0$

$$P\{\eta \le t\} = o\left(\exp\{-t^{-p/(1-p)+\varepsilon}\}\right) \quad \text{as } t \downarrow 0. \tag{4.10}$$

PROOF. If $q_1(X) < \infty$ a.s. by (4.5) there exists a positive constant K (determined by q_1) such that

$$KP\{\eta \le t\} \le P\{q_1(X) \le t\}.$$

If we take q_1 such that (4.9) is fulfilled (with $r = 1$) we get (4.10).

Note that (4.10) is stronger then the result in [1], page 448, obtained there via Tauberian theorems, for a more refined estimate see [4].

LEMMA 4.8. *Let $0 < b_1 < b_2$, $\varepsilon > 0$ and let $r \le p$. Then there exist an $a > 0$ and $n \in \mathcal{N}$ such that*

$$P\left\{b_1 < (1/a) \sum_{i=1}^{n} |\theta_i|^r \le b_2\right\} \ge 1 - \varepsilon.$$

PROOF. Suppose that we can find positive constants $\{a_n\}$ such that

$$(1/a_n) \sum_{i=1}^{n} |\theta_i|^r \to (b_1 + b_2)/2 \quad \text{in probability if } n \to \infty.$$

Indeed, for $r = p$ this follows from [1], page 236, for $r < p$ it is a consequence of the strong law of large numbers (with a.s. convergence). Therefore, we have

$$P\left\{b_1 < (1/a_n) \sum_{i=1}^{n} |\theta_i|^r \le b_2\right\} \to 1 \quad \text{if } n \to \infty.$$

EXAMPLE 4.9. Let $1 \le r \le p$ and let $f\colon R_+ \to R_+$ be increasing. Then there exists a sequence $\{\alpha_i\}$ such that q_r is finite a.s. and for $0 < t \le 1$

$$P\{q_r(X) \le t\} \le f(t).$$

Let us define numbers p_k by

$$p_0 = f(1/2), \qquad p_k = f(2^{-k-1})/f(2^{-k}) \quad \text{for } k \ge 1.$$

By Lemma 4.8 we find a_k and n_k such that

$$P\left\{(1/a_k) \sum_{i=1}^{n_k} |\theta_i|^r \le 2^{-kr}\right\} \le p_k$$

and

$$P\left\{(1/a_k) \sum_{i=1}^{n_k} |\theta_i|^r > 2^{(-k+1)r}\right\} \le 2^{-k}. \tag{4.11}$$

Let us define numbers $m_k = n_0 + \cdots + n_k$, $k \geq 0$ and put

$$N_0(x) = (1/a_0) \sum_{i=1}^{n_0} |x_i|^r, \qquad N_k(x) = (1/a_k) \sum_{i=m_{k-1}+1}^{m_k} |x_i|^r \quad \text{for } k \geq 1.$$

First, using arguments of Höffmann-Jorgensen et al. ([2], Theorem 3.5.), we will show that for $N = \sup_{k \geq 0} N_k$ we have

$$P\{N(X) \leq t^r\} \leq f(t) \quad \text{for } 0 < t \leq 1.$$

Let $F_k(t) = P\{N_k(X) \leq t\}$. Since $N_k(X)$ are independent, for $2^{-k-1} < t \leq 2^{-k}$ we obtain by (4.11):

$$\begin{aligned} P\{N \leq t^r\} &= \prod_{i=0}^{\infty} F_i(t^r) \leq \prod_{i=0}^{k} F_i(t^r) \leq \prod_{i=0}^{k} F_i(2^{-kr}) \\ &\leq \prod_{i=0}^{k} F_i(2^{-ir}) \leq \prod_{i=0}^{k} p_i = f(2^{-k-1}) \leq f(t). \end{aligned}$$

Next, if we take $q_r(x) = (\sum_{k=0}^{\infty} N_k(x))^{1/r}$, then $q_r^r \geq N$ and

$$P\{q_r(X) \leq t\} \leq P\{N(X) \leq t^r\} \leq f(t).$$

By (4.11) we have

$$P\{N_k(X) > 2^{(-k+1)r}\} \leq 2^{-k} \quad \text{for } k \geq 0,$$

So by the Borel–Cantelli lemma $q_r(X)$ is finite a.s.

Now, once again following methods developed in [2] we give a lower bound of $P\{q_r(X) \leq t\}$ for $r < p$.

First, we estimate the density g of the distribution of θ. Let η be a positive, $(p/2)$-strictly stable random variable with the Laplace transform $\exp(-t^{p/2})$ and let G be the distribution of $(2\eta)^{1/2}$. Then, by Lemma 4.1, we get the following estimate:

$$\begin{aligned} g(t) &= (2\pi)^{-1/2} \int_0^{\infty} x^{-1} \exp\{-2^{-1}(tx^{-1})^2\} G(dx) \\ &\geq (2\pi)^{-1/2} \int_1^{\infty} x^{-1} \exp\{-2^{-1}(tx^{-1})^2\} G(dx) \qquad (4.12) \\ &\geq C \exp\{-2^{-1}t^2\}, \qquad t > 0, \end{aligned}$$

where $C = (2\pi)^{-1/2} \int_1^{\infty} x^{-1} G(dx)$.

Next, let $\{\alpha_i\}$, $i \geq 1$ be a decreasing sequence of positive numbers such that $\sum \alpha_i^r < \infty$. Denote $m_r = E|\theta|^r$ and let $\psi_n = m_r \sum_{i=n}^{\infty} \alpha_i^r$. By part (i) of (2.4), $q_r(X)$ is finite a.s.

LEMMA 4.10. *There exists a positive constant D such that*

$$\begin{aligned} P\{q_r(X) \leq t\} \geq &\left(\prod_{i=1}^{n} \alpha_i^{-1}\right) D^n n^{-n/r} s^n \qquad (4.13) \\ &\cdot \exp\left\{-2^{-1}\left(s\alpha_n^{-1}\right)^2\right\}\left(1 - \psi_{n+1}\left(t^r - s^r\right)^{-1}\right) \end{aligned}$$

for all $n \in \mathcal{N}$ *and all* $0 < s < t$.

PROOF. Let $n \geq 1$ and let $0 < s < t$. Then

$$P\{q_r(X) \leq t\} \geq P\left\{\sum_{i=1}^{n} \alpha_i^r|\theta_i|^r \leq s^r\right\} P\left\{\sum_{i=n+1}^{\infty} \alpha_i^r|\theta_i|^r \leq (t^r - s^r)\right\}.$$

Since $r < p < 2$ and $\{\alpha_n\}$ is decreasing, we have for $(x_1, \ldots, x_n) \in B_n^r(s)$:

$$\sum_{i=1}^{n} \left(\alpha_i^{-1}x_i\right)^2 \leq \left\{\sum_{i=1}^{n} \left(\alpha_i^{-1}|x_i|\right)^r\right\}^{2/r} \leq \alpha_n^{-2}\left(\sum_{i=1}^{n} |x_i|^r\right)^{2/r} \leq \left(\alpha_n^{-1}s\right)^2.$$

By this inequality and by Lemma 4.4 and (4.12), we obtain

$$P\left\{\sum_{i=1}^{n} \alpha_i^r|\theta_i|^r \leq s^r\right\} = \int_{B_n^r(s)} \prod_{i=1}^{n} \left(\alpha_i^{-1}g\left(x_i\alpha_i^{-1}\right)\right)\lambda_n(dx)$$

$$\geq \left(\prod_{i=1}^{n} \alpha_i^{-1}\right)C^n \int_{B_n^r(s)} \exp\left\{-2^{-1}\sum_{i=1}^{n}\left(x_i\alpha_i^{-1}\right)^2\right\}\lambda_n(dx) \tag{4.14}$$

$$\geq \left(\prod_{i=1}^{n} \alpha_i^{-1}\right)C^n B^n \exp\left\{-2^{-1}\left(s\alpha_n^{-1}\right)^2\right\}s^n n^{-n/r},$$

where B is the constant appearing in (4.7).

By Chebyshev's inequality we get

$$P\left\{\sum_{i=n+1}^{\infty} \alpha_i^r|\theta_i|^r \leq (t^r - s^r)\right\} = 1 - P\left\{\sum_{i=1+n}^{\infty} \alpha_i^r|\theta_i|^r > (t^r - s^r)\right\} \tag{4.15}$$

$$\geq 1 - \psi_{n+1}(t^r - s^r)^{-1}.$$

Putting $D = CB$ and combining (4.14) and (4.15) we obtain (4.13).

Now we are able to construct the following example, which is a modification of Example 4.9 [2].

EXAMPLE 4.11. Let $f\colon R_+ \to R_+$ be an increasing function such that for all $n \in \mathcal{N}$

$$f(t) = O(t^n) \quad \text{if } t \to 0.$$

Suppose that $r > 0$. Then there exist a sequence $\{\alpha_i\}$, $i \geq 1$ and $t_0 > 0$ such that

$$P\{q_r(X) \leq t\} \geq f(t) \quad \text{for } t \leq t_0. \tag{4.16}$$

We first consider a situation when $r < p$. There exist constants A_n such that for all $n \in \mathcal{N}$

$$f(t) \leq A_n t^{n+r} \quad \text{for } 0 < t \leq 1.$$

Denote $E = \exp\{-2^{-1}((4/3)m_r)^{2/r}\}$ and for $n \geq 2$ let

$$\beta_n = (3^{-1/r}D)^n E(16m_r)^{-1}A_n^{-1}.$$

Next, define inductively a sequence $\{\alpha_n\}$. Put $\alpha_1 = 1$ and let $\alpha_n^r = \min\{\beta_n, \alpha_{n-1}^r 2^{-1}, \ldots, \alpha_1^r 2^{-n+1}\}$ for $n \geq 2$. Since $\alpha_n \leq 2^{(-n+1)/r}$, we get

$$\left(\prod_{i=1}^{n} \alpha_i^{-1}\right)n^{-n/r} \geq (2^{n-1}n^{-1})^{n/r} \geq 1. \tag{4.17}$$

Let $n \geq 2$ and let $2\psi_{n+1} \leq t^r \leq 2\psi_n$. Then we have

(4.18) $$t^r \leq 2\psi_n \leq 2m_r(\alpha_n^r + 2^{-1}\alpha_n^r + \cdots) = 4m_r\alpha_n^r \leq 4m_r\beta_n.$$

If we take $s = 3^{-1/r}t$ we obtain

(4.19) $$\exp\{-2^{-1}(s\alpha_n^{-1})^2\} \geq E \quad \text{and} \quad 1 - \psi_{n+1}(t^r - s^r)^{-1} \geq 4^{-1}.$$

Therefore, applying (4.13) and the inequalities (4.17), (4.18), and (4.19), for $2\psi_{n+1} \leq t^r \leq 2\psi_n$ and $0 < t \leq 1$, we get

$$\begin{aligned} P\{q_r(X) \leq t\} &\geq 3^{-n/r}4^{-1}Et^nD^n \\ &= A_nt^{n+r}t^{-r}4m_r\beta_n \geq A_nt^{n+r} \geq f(t). \end{aligned}$$

So (4.16) holds for $t \leq t_0 = \min\{1, \psi_2^{1/r}\}$.

For the case when $r \geq p$, we first find a sequence $\{\alpha_i\}$ and $t_0 > 0$ such that $P\{q_{p/2}(X) \leq t\} \geq f(t)$ for $t \leq t_0$. The conclusion now follows by the inequality $(\Sigma\alpha_i^{p/2}|\theta_i|^{p/2})^{2/p} \geq (\Sigma\alpha_i^r|\theta_i|^r)^{1/r}$ valid for $r \geq p$.

5. Banach spaces of stable type p. Throughout this section E will denote a separable Banach space, $\mathscr{B}$ will be its Borel σ-field.

Let X be a symmetric, p-stable, $0 < p < 2$, E-valued random vector with the spectral measure m. Let us denote $\|m\| = m(E)$ and $c'(s, r, p, B) = 2^{p/(r-p)}B^{rp/(r-p)}c(s, r, p)$, where $p < r$, $s, B > 0$, and $c(s, r, p)$ is as defined in Section 4.

THEOREM 5.1. *If E is of Rademacher type r, $p < r$, then*

(5.1) $$P\{\|X\| \leq t\} \geq (1/4)\exp\{-c'(s, r, p, B)(t/\|m\|^{1/p})^{-rp/(r-p)}\}$$

for $t \leq 2^{(1/p+1/r)}Bs^{1/r}(E|\xi|^p)^{1/p}\|m\|^{1/p}$, where B is the constant appearing in (2.3).

PROOF. Since every symmetric, p-stable measure on $(E, \mathscr{B})$ is the weak limit of some sequence of measures $\{\mu_n\}$, where $\mu_n = \mathscr{L}(\Sigma_{i=1}^{k_n} x_{in}\theta_i)$, $x_{in} \in E$, (see [5]), it is enough to prove (5.1) for X of the form $X = \Sigma_{i=1}^n x_i\theta_i$, $x_i \in E$.

Let $\{r_i\}$, $\{\theta_i\}$ be defined on probability spaces $(\Omega_1, \mathscr{B}_1, P_1)$, $(\Omega_2, \mathscr{B}_2, P_2)$, respectively. On the product space both sequences are independent and

$$\mathscr{L}(\{r_i\theta_i\}) = \mathscr{L}(\{\theta_i\}).$$

Then

$$\mathscr{L}(X) = \mathscr{L}\left(\sum_{i=1}^n x_ir_i\theta_i\right).$$

Next, for $t, \alpha > 0$, let us define events

$$A(t, \alpha) = \left\{\left\|\sum_{i=1}^n x_ir_i\theta_i\right\|^r \leq \alpha^r\sum_{i=1}^n \|x_i\|^r|\theta_i|^r\right\},$$

$$B(t, \alpha) = \left\{\sum_{i=1}^n \|x_i\|^r|\theta_i|^r \leq (t/\alpha)^r\right\}.$$

Thus, we have

$$P\{A(t,\alpha) \cap B(t,\alpha)\} \le P\{\|X\| \le t\}. \tag{5.2}$$

Using Fubini's theorem we get

$$\begin{aligned} P\{A(t,\alpha) \cap B(t,\alpha)\} &= E_1 E_2 \mathbb{1}_{A(t,\alpha)} \mathbb{1}_{B(t,\alpha)} \\ &= E_2\big(\mathbb{1}_{B(t,\alpha)} E_1 \mathbb{1}_{A(t,\alpha)}\big), \end{aligned} \tag{5.3}$$

where E_i denote the expectation with respect to P_i, $i = 1, 2$. By Chebyshev's inequality and (2.3)

$$\begin{aligned} E_1 \mathbb{1}_{A(t,\alpha)} &= 1 - P_1\left\{\left\|\sum_{i=1}^n x_i r_i \theta_i\right\|^r > \alpha^r \sum_{i=1}^n \|x_i\|^r |\theta_i|^r\right\} \\ &\ge 1 - \left(E_1\left\|\sum_{i=1}^n x_i r_i \theta_i\right\|^r\right)\left(\alpha^r \sum_{i=1}^n \|x_i\|^r |\theta_i|^r\right)^{-1} \\ &\ge 1 - (B/\alpha)^r \quad \text{a.s. } P_2. \end{aligned} \tag{5.4}$$

Thus, the inequalities (5.2), (5.3), and (5.4) yield:

$$P\{\|X\| \le t\} \ge \big(1 - (B/\alpha)^r\big) P\left\{\sum_{i=1}^n \|x_i\|^r |\theta_i|^r \le (t/\alpha)^r\right\}.$$

Taking $\alpha = 2^{1/r}B$ and writing $\Sigma\|x_i\|^p$ as $\|m\|$ we see that (5.1) is an immediate consequence of Proposition 4.2 (with $\alpha_i = \|x_i\|$).

Theorem 5.2. *Let $1 \le p < 2$. The following conditions are equivalent:*

(i) *There exist positive numbers ρ and t_0 such that for every symmetric, p-stable, E-valued random vector X with the spectral measure m, with $\|m\| = 1$, we have*

$$P\{\|X\| \le t\} \ge \exp - t^{-\rho} \text{ for } t \le t_0.$$

(ii) *E is of stable type p.*

Proof. (ii) $\Rightarrow$ (i). By assumption (and Theorem 1 of [6]), there exists a r, $p < r < 2$, such that E is of Rademacher type r. Therefore (i) follows from Theorem 5.1 and the elementary inequality: $A\exp(-\alpha t^{-\rho_1}) \ge \exp(-t^{-\rho_2})$ for $t \le t_0$ if $A, \alpha > 0$; $0 < \rho_1 < \rho_2$, and t_0 is small enough.

Now, we prove (i) $\Rightarrow$ (ii). Suppose that E is not of stable type p. Therefore, l_p is finitely representable in E. If $0 < \varepsilon < 1$, by (2.2) there exist $x_{in} \in E$ such that for every $\beta_i \in R$

$$(1-\varepsilon)\left(\sum_{i=1}^n |\beta_i|^p\right)^{1/p} \le \left\|\sum_{i=1}^n x_{in}\beta_i\right\| \le (1+\varepsilon)\left(\sum_{i=1}^n |\beta_i|^p\right)^{1/p}. \tag{5.5}$$

In particular,

$$\|x_{in}\| \le 1 + \varepsilon. \tag{5.6}$$

Let a sequence $\{\alpha_i\}$ be such that $\sum_{i=1}^{\infty}|\alpha_i|^p = 1$. Put $X_n = \sum_{i=1}^{n} x_{in}\alpha_i\theta_i$ and notice that X_n has the spectral measure $m_n = \sum_{i=1}^{n}|\alpha_i|^p\|x_{in}\|^p\delta_{\{\alpha_i x_{in}/\|\alpha_i x_{in}\|\}}$. Then if we take $Y_n = \|m_n\|^{-1/p}X_n$ by (5.5), we obtain

$$P\left\{(1-\varepsilon)\|m_n\|^{-1/p}\left(\sum_{i=1}^{n}|\alpha_i\theta_i|^p\right)^{1/p} \leq t\right\} \geq P\{\|Y_n\| \leq t\}.$$

Because the total variation of the spectral measure of Y_n equals 1, (i) implies

$$P\left\{(1-\varepsilon)\|m_n\|^{-1/p}\left(\sum_{i=1}^{n}|\alpha_i\theta_i|^p\right)^{1/p} \leq t_0\right\} \geq \exp - t_0^{-p} = \varepsilon' > 0.$$

By (5.6) $\|m_n\| \leq (1+\varepsilon)^p$ and therefore,

$$P\left\{\sum_{i=1}^{n}|\alpha_i\theta_i|^p \leq \left((1-\varepsilon)^{-1}(1+\varepsilon)t_0\right)^p\right\} \geq \varepsilon'.$$

Letting n to infinity we get

$$P\left\{\sum_{i=1}^{\infty}|\alpha_i\theta_i|^p < \infty\right\} > 0.$$

By The Kolmogorov 0–1 Law the last probability equals 1. Therefore, the convergence of $\sum|\alpha_i|^p$ would imply the a.s. convergence of $\sum|\alpha_i\theta_i|^p$. It contradicts part (ii) of (2.4).

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