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ON ITÔ STOCHASTIC INTEGRATION WITH RESPECT TO p-STABLE MOTION: INNER CLOCK, INTEGRABILITY OF SAMPLE PATHS, DOUBLE AND MULTIPLE INTEGRALS

By J. Rosiński[1] and W. A. Woyczyński

Case Western Reserve University

The paper studies in detail the sample paths of Itô-type stochastic integrals with respect to p-stable motion $M(t)$, $t \geq 0$. These results, in turn, permit an analysis of the concept of multiple p-stable integrals of the form

$$\int \cdots \int f(t_1, \ldots, t_n)\, dM(t_1) \cdots\, dM(t_n),$$

and, in particular, a full description of functions of two variables $f(t_1, t_2)$ for which the double stochastic integral $\int\int f(t_1, t_2)\, dM(t_1)\, dM(t_2)$ exists.

1. Introduction. In recent years several authors attempted to extend Itô's theory of multiple Wiener integrals to various classes of non-Gaussian processes [cf. e.g., Lin (1981), Surgailis (1981), and Engel (1982)] assuming, however, the existence of high moments of integrator processes. Stable processes pose a special problem in this context because of their poor integrability properties. The multiple integrals for them were studied by Surgailis (1981) and by the authors (1984) but the results obtained in these two papers were far from conclusive. Their completion requires an in-depth study of the sample path behavior of the single p-stable Itô-type stochastic integrals and this is the starting point of the present paper.

Section 2 contains a construction of the Itô-type stochastic integrals with respect to a p-stable motion M for processes with almost surely p-integrable sample paths. The existence of such integrals follows also from the work of Kallenberg (1975) on integration with respect to processes with stationary and independent increments. We feel, however, that the construction presented here is more natural and gives more insight into the inner structure of the integral. As an intermediate step in the construction, which corresponds to Itô's L^2-theory for Brownian integrals, we show that the stochastic integral $F \to \int^t F\, dM = X(t)$ is an isomorphic embedding of the class of adapted processes F for which $E\int |F|^p\, dt < \infty$ into the class of processes X with trajectories in $D[0, \infty)$ and such that $P(\sup_t |X(t)| > \lambda) = O(\lambda^{-p})$, $\lambda \to \infty$. The upper estimate needed to establish the above isomorphism is fundamental in the construction of the integral $\int F\, dM$ and was obtained by Giné and Marcus (1983). The final step in the construction adapts to L^p an approach to L^2-stochastic integrals that can be found [e.g., Ikeda and Watanabe (1981), page 52].

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[1]On leave from Wrocław University, Poland, now at the University of North Carolina at Chapel Hill.

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Section 3 introduces a notion of the inner clock for the p-stable integral. Such a random change of time is a familiar device in the theory of Brownian integrals and, more generally, in the study of local L^2-martingales with continuous sample paths. As far as processes with discontinuous paths are concerned, we have not encountered this method before. The inner clock permits the transfer of information about the sample path behavior of the p-stable motion to sample paths of stochastic integrals. In particular, the asymptotic behavior of stochastic integrals can be dealt with using this method (Corollary 3.1). However, our most important application of the change of time formula comes in Section 4 where we present a formal approach to stochastic integration and show that the p-integrability of sample paths of F is not only sufficient but also necessary for the existence of $\int F\,dM$.

The results of Sections 2–4 permit an analysis of the concept of multiple stochastic integral with respect to the p-stable motion. We deal here only with integrals over the n-dimensional tetrahedron $\{0 \le t_1 < t_2 < \cdots < t_n = T\}$ rather than over the cube $[0, T]^n$. The integration over the cube requires also consideration of integrals over diagonal sets which leads to problems that are of a different, partly combinatorial nature [cf. Engel (1982) and Rosiński and Woyczyński (1984) where also the abstract products of random measures and multiple integrals based thereon are studied]. In the case of double p-stable integrals, $1 \le p < 2$, we are able to give a full characterization of functions of two variables that are $dM \times dM$-integrable [see the paper by Cambanis, Rosiński, and Woyczyński (1983) for connections with the theory of p-stable quadratic forms]. Our approach here turns out to be more successful than a double Fourier–Haar expansion approach that was used by Szulga and Woyczyński (1983) and which, so far, yielded only sufficient conditions of $dM \times dM$-integrability.

The proof of Theorem 5.2 requires a study of measurability and integrability of sample paths of a symmetric p-stable process. We do it in Section 6 [see the paper of Cambanis and Miller (1980) for previous work in this direction and other references]. The section contains a description of p-stable measurable processes that have p-integrable sample paths. A characterization of such processes, which, as far as we know, was an open problem for quite a while, is equivalent to the characterization of deterministic M-integrable functions with values in L^p [cf. Rosiński (1984)] and to the characterization of θ_p-radonifying operators into L^p. This result should be compared with results of Kwapień [quoted without proof in Linde's book (1983)], Giné and Zinn (1983), and Cambanis, Rosiński, and Woyczyński (1983) concerning θ_p-radonifying operators into the sequence space l^p.

2. Construction of the Itô-type stochastic integrals for a p-stable motion. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\{\mathscr{F}_t\}_{t>0}$ be a right continuous, increasing family of P-complete sub-σ-fields of $\mathscr{F}$.

DEFINITION 2.1. An $\{\mathscr{F}_t\}$-adapted process $(M(t))_{t \ge 0}$ with all sample paths in $D[0, \infty)$ is said to be an $\{\mathscr{F}_t\}$-p-stable motion, $0 < p < 2$, if for every $0 \le s < t$, and $\lambda \in R$

$$E\{\exp[i\lambda(M(t) - M(s))] | \mathscr{F}_s\} = \exp[-(t-s)|\lambda|^p]. \tag{2.1}$$

The term "stable motion" sounds like a terrible misnomer, if one looks at the stable motion's trajectories, but it is handy, and was used before and certainly does not seem much worse in that respect than "Brownian motion." Observe that the above definition implies that the increments $M(t) - M(s)$, $s \le t$, are independent of $\mathscr{F}_s$ so that, in particular, they are also independent of $\sigma\{M(r): r \le s\}$. Hence, an $\{\mathscr{F}_t\}$-p-stable motion has always stationary and independent p-stable symmetric increments. The reason for considering the $\{\mathscr{F}_t\}$-p-stable motion, rather than the regular p-stable motion with respect to the natural σ-fields will be apparent later on and its introduction is very natural in the theory of stochastic integration. We will, however, suppress the prefix "$\{\mathscr{F}_t\}$-" whenever its appearance is not essential.

The space of all real measurable processes $F = \{F(t, \omega)\}_{t \ge 0}$ on $\Omega \times [0, \infty)$ adapted to $\{\mathscr{F}_t\}$, and such that for every $T > 0$

$$\|F\|_{p,T} =_{df} \left(E\int_0^T |F(s, \omega)|^p \, ds\right)^{1/p} < \infty \tag{2.2}$$

will be denoted by $L^p(L^p)$ and will be needed in the construction of the stochastic integral with respect to M. We identify F and F' in $L^p(L^p)$ if $\|F - F'\|_{p,T} = 0$ for every $T > 0$.

As usual, a *simple process* is a process of the form

$$F(t, \omega) = \varphi_0(\omega) I_{\{t=0\}}(t) + \sum_{i=0}^{\infty} \varphi_i(\omega) I_{(t_i, t_{i+1}]}(t), \tag{2.3}$$

where $0 = t_0 < t_1 < \cdots < t_n < \cdots \to \infty$ and φ_n is $\mathscr{F}_{t_n}$-measurable for every $n = 0, 1, 2, \ldots$, and for simple F the *stochastic integral* is defined as follows:

$$\int_0^t F(s, \omega) \, dM(s, \omega) =_{df} \sum_{i=0}^{n-1} \varphi_i(\omega)(M(t_{i+1}, \omega) - M(t_i, \omega)) + \varphi_n(\omega)(M(t, \omega) - M(t_n, \omega)) \tag{2.4}$$

if $t_n \le t \le t_{n+1}$, $n = 0, 1, 2, \ldots$. Clearly the above integral is a process with almost all sample paths in $D[0, \infty)$.

In the case when M is a Brownian motion the stochastic integral $F \to \int_0^t F\,dM$ is an isomorphic embedding from $L^2(L^2)$ into the space of square integrable martingales with almost all continuous sample paths. Our next result is a complete analogue of the above phenomenon.

Denote by $\Lambda^p(L^\infty)$ the set of all $\mathscr{F}_t$-adapted measurable processes $X(t)$, $t \ge 0$, with almost all sample paths in $D[0, \infty)$ such that for every $T > 0$ the weak-L^p norm

$$\left(\sup_{\lambda > 0} \lambda^p P\left\{\sup_{t \le T} |X(t)| > \lambda\right\}\right)^{1/p} \tag{2.5}$$

is finite. It is easy to see that $\Lambda^p(L^\infty)$ is a complete metric linear space for any p, $0 < p < 2$.

THEOREM 2.1. *There exist constants c_1, $c_2 > 0$ such that for any simple process $F \in L^p(L^p)$ and any $T > 0$*

$$c_1\|F\|_{p,T}^p \le \sup_{\lambda>0}\lambda^p P\left\{\sup_{t\le T}\left|\int_0^t F\,dM\right| > \lambda\right\} \le c_2\|F\|_{p,T}^p, \tag{2.6}$$

so that the mapping $F \to \int F\,dM$ extends to an isomorphic embedding of $L^p(L^p)$ into $\Lambda^p(L^\infty)$. In other words, for each $F \in L^p(L^p)$ there exists a stochastic process in $\Lambda^p(L^\infty)$, also denoted by $\int_0^t F\,dM$, $t \ge 0$, which satisfies the inequalities (2.6) *and which for simple F coincides with the stochastic integral* (2.4).

PROOF. The upper estimate of (2.6) is due to Giné and Marcus (1983), Theorem 3.5. We shall prove the lower estimate.

Let $F \in L^p(L^p)$ be a simple process given by (2.4) with $t_n \le T \le t_{n+1}$. Put $\Delta M_0 = M(t_1) - M(t_0), \dots, \Delta M_{n-1} = M(t_n) - M(t_{n-1})$, $\Delta M_n = M(T) - M(t_n)$. Then, for any $\lambda > 0$ and $K > 0$,

$$P\left\{\sup_{t\le T}\left|\int_0^t F\,dM\right| > \lambda\right\} \ge P\left\{\sup_{k\le n}\left|\sum_{j=0}^{k}\varphi_j\Delta M_j\right| > \lambda\right\} \ge P\left\{\sup_{k\le n}|\varphi_k\Delta M_k| > 2\lambda\right\}.$$

However, the right-hand side is equal to

$$\sum_{k=0}^{n} P\{|\varphi_0\Delta M_0| \le 2\lambda,\dots,|\varphi_{k-1}\Delta M_{k-1}| \le 2\lambda, |\varphi_k\Delta M_k| > 2\lambda\}$$

$$\ge \sum_{k=0}^{n} E\left\{I[|\varphi_0\Delta M_0| \le 2\lambda,\dots,|\varphi_{k-1}\Delta M_{k-1}| \le 2\lambda, |\varphi_k| \le K]\right.$$

$$\left.\times P\left[|\Delta M_k| > \frac{2\lambda}{|\varphi_k|}\,\middle|\,\mathscr{F}_{t_k}\right]\right\}.$$

Since ΔM_k is a p-stable symmetric r.v. there exists a constant $c_p > 0$ such that for $\alpha \ge 1$, $P(|\Delta M_k| > \alpha) \ge c_p\alpha^{-p}\Delta t_k$ for small enough Δt_k's, where $\Delta t_0 = t_1 - t_0, \dots, \Delta t_{n-1} = t_n - t_{n-1}$, $\Delta t_n = T - t_n$. Therefore if $2\lambda/K \ge 1$ we obtain

$$\sup_{\lambda>0}\lambda^p P\left\{\sup_{t\le T}\left|\int_0^t F\,dM\right| > \lambda\right\}$$

$$\ge c_p 2^{-p}\sum_{k=0}^{n} E\{I[|\varphi_1\Delta M_1| \le 2\lambda,\dots,|\varphi_{k-1}\Delta M_{k-1}| \le 2\lambda, |\varphi_k| \le K]|\varphi_k|^p\}\Delta t_k,$$

where the left-hand side is now independent of λ. Letting $\lambda \to \infty$, $K \to \infty$ on the right-hand side, but preserving the relationship $2\lambda/K \ge 1$, we obtain the required lower estimate with $c_1' = c_p 2^{-p}$. □

With the hindsight of the above theorem the proof of the following proposition is a straightforward adaptation of the proof of statement (iv) from Proposition

1.1 p. 50 of Ikeda and Watanabe (1981) (which dealt with the Brownian motion). We will omit it.

PROPOSITION 2.1. *Let $F \in L^p(L^p)$ and τ be an $(\mathscr{F}_t)$-stopping time. Then a.s.*

$$\int_0^{t\wedge\tau} F\,dM = \int_0^t I(\tau \geq s) F\,dM$$

for every $t \geq 0$.

The above proposition permits the construction of the p-stable stochastic integral for a process F from the class $L^p_{\text{a.s.}}$ (to be defined below) by gluing the integral together pathwise from pieces that are integrals of processes $F_n \in L^p(L^p)$ which form a certain particular approximating sequence for F. The class $L^p_{\text{a.s.}}$ is much wider than $L^p(L^p)$ and the results of Section 4 show that it is a maximal class for which $\int F\,dM$ can be sensibly defined. Also, as it turns out (Theorem 4.1), there is nothing special about the particular sequence (2.7) we use the construction that follows. Any other approximating sequence will do.

We denote by $L^p_{\text{a.s.}}$ the family of all real measurable $(\mathscr{F}_t)$-adapted processes F on $\Omega \times [0, \infty)$ such that for every $T > 0$, $\int_0^T |F(t, \omega)|^p\,dt < \infty$ a.s.

For every $F \in L^p_{\text{a.s.}}$ we define stopping times

$$\tau_n(\omega) = \inf\left\{t\colon \int_0^t |F(s, \omega)|^p\,ds \geq n\right\} \wedge n,$$

$n = 1, 2, \ldots$, so that $\tau_n \uparrow \infty$ a.s. Let

$$F_n(s, \omega) = I(\tau_n(\omega) \geq s) F(s, \omega). \tag{2.7}$$

Since

$$\int_0^\infty |F_n(s, \omega)|^p\,ds = \int_0^{\tau_n} |F_n(s, \omega)|^p\,ds \leq n,$$

we get that $F_n \in L^p(L^p)$. In view of Proposition 2.1 the following consistency condition is satisfied:

$$\int_0^{t\wedge\tau_m} F_n\,dM = \int_0^t F_m\,dM \quad \text{a.s.,} \qquad m < n.$$

Therefore the process

$$Y(t) =_{df} \int_0^t F_n\,dM \quad \text{for } t \leq \tau_n, \tag{2.8}$$

$n = 1, 2, \ldots$, is well defined and has a.a. sample paths in $D[0, \infty)$. We shall call the process $Y(t)$, $t \geq 0$, the stochastic integral of $F \in L^p_{\text{a.s.}}$ with respect to M, and denote it by $\int_0^t F\,dM$, $t \geq 0$.

REMARK 2.1. It should be noted that the above construction of the stochastic integral depends only on the tail behavior of the p-stable distribution and can be immediately extended to a wider class of integrators that includes p-stable processes as a particular case. Namely, let $X(t)$, $t \geq 0$, be an $\{\mathscr{F}_t\}$-adapted stochastic process with symmetric, independent increments and sample paths in $D[0, \infty)$, and let m be a Borel measure on R_+ which is finite on every finite interval. Assume that for each $T > 0$ there exist $a, b, \alpha_0 \geq 0$ such that for all $0 \leq s < t \leq T$ and $\alpha \geq \alpha_0$

$$(2.9) \qquad a\alpha^{-p}m((s,t]) \leq P\{|X(t) - X(s)| > \alpha\} \leq b\alpha^{-p}m((s,t]),$$

and denote by $L^p(L^p(m))$ the set of all measurable $\{\mathscr{F}_t\}$-adapted processes such that for every $T > 0$

$$\|F\|_{p,m,T} = \left[E\int_0^T |F(t)|^p \, dm(t)\right]^{1/p} < \infty.$$

Then, for any integrator X satisfying (2.9) there exist constants $c_1, c_2 > 0$ (depending on T) such that for every process $F \in L^p(L^p(m))$

$$c_1\|F\|_{p,m,T}^p \leq \sup_{\lambda > 0} \lambda^p P\left\{\sup_{t \leq T}\left|\int_0^t F\,dX\right| > \lambda\right\} \leq c_2\|F\|_{p,m,T}^p$$

[the upper estimate is again due to Giné and Marcus (1983)] and the extension of the integral $\int F\,dX$ to $F \in L^p_{\text{a.s.}}(m)$ follows verbatim the lines preceding this remark.

3. Inner clock of the *p*-stable stochastic integral. In this section we prove results about the time substitution, or, more precisely, random time change in the stochastic integrals that parallel the known results for Brownian integrals. The next proposition, concerning complex exponential martingales, is technical in nature but it is not without intrinsic interest. It is well known in case $p = 2$ [see e.g., Ikeda and Watanabe (1981), Theorem 5.3].

PROPOSITION 3.1. *Let $F \in L^p(L^p)$ be such that*

$$(3.1) \qquad E\exp\left\{\alpha\int_0^t |F|^p \, ds\right\} < \infty$$

for every $\alpha > 0$ and $t > 0$. Then for every $\lambda \in \mathbb{R}$

$$Z_F(t) =_{df} \exp\left\{i\lambda\int_0^t F\,dM + |\lambda|^p\int_0^t |F|^p \, ds\right\}$$

is a complex-valued $\{\mathscr{F}_t\}$-martingale.

PROOF. We will prove the proposition first for a bounded and simple process $F \in L^p(L^p)$ given by (2.3). Using the notation from the proof of Theorem 2.1

$$Z_F(t) = \exp\left\{i\lambda\sum_{k=0}^n \varphi_k \Delta M_k + |\lambda|^p \sum_{k=0}^n |\varphi_k|^p \Delta t_k\right\}.$$

Hence, for $s \leq t$, say $s \in [t_m, t_{m+1})$, $m \leq n$, we obtain that

$$E(Z_F(t)|\mathscr{F}_s) = Z_F(s)E\left(\prod_{j=m}^n V_j|\mathscr{F}_s\right),$$

where

$$V_m = \exp\{i\lambda\varphi_m(M(t_{m+1}) - M(s)) + |\lambda|^p|\varphi_m|^p(t_{m+1} - s)\}$$

and

$$V_k = \exp\{i\lambda\varphi_k\Delta M_k + |\lambda|^p|\varphi_k|^p\Delta t_k\}, \qquad k = m+1,\dots,n.$$

Since $E(V_m|\mathscr{F}_s) = 1$ and $E(V_k|\mathscr{F}_{t_k}) = 1$ for $k = m+1,\dots,n$, we get that $\{Z_F(t), \mathscr{F}_t\}$ is a martingale.

Let now $F \in L^p(L^p)$ be bounded, say $|F| < C$. Then there exists a sequence of simple processes $(F_n) \subset L^p(L^p)$ such that for every $T > 0$, $\|F_n - F\|_{p,T} \to 0$ as $n \to \infty$. Define $G_n(t) = F_n(t)$ if $|F_n(t)| < C$, $= -C$ if $F_n(t) \le -C$, and $= C$ if $F_n(t) \ge C$. Then $G_n \in L^p(L^p)$, $n = 1, 2, \dots$, are simple processes such that $|G_n| \le C$ for every n, and for every $T > 0$, $\|G_n - F\|_{p,T} \to 0$ as $n \to \infty$. Therefore $Z_{G_n}(t) \to Z_F(t)$ in P and since $|Z_{G_n}(t)| \le \exp\{t|\lambda C|^p\}$ for every n and since $(Z_{G_n}(t), \mathscr{F}_t)$ is a martingale (by the first part of the proof) we obtain that $(Z_F(t), \mathscr{F}_t)$ is also a martingale.

Finally, if F satisfies (3.1), then $Z_F(t)$ can be approximated in L^1 by $Z_{F^{(k)}}(t)$ as $k \to \infty$, where $F^{(k)} = FI(|F| \le k)$. □

THEOREM 3.1. *Let $F \in L^p_{\text{a.s.}}$ be such that $\tau(u) =_{df} \int_0^u |F|^p\,dt \to \infty$* a.s. *as $u \to \infty$. If*

$$\tau^{-1}(t) = \inf\{u: \tau(u) > t\} \quad \text{and} \quad \mathscr{A}_t = \mathscr{F}_{\tau^{-1}(t)},$$

then the time-changed stochastic integral

$$\tilde{M}(t) = \int_0^{\tau^{-1}(t)} F\,dM$$

is an $\{\mathscr{A}_t\}$-p-stable motion. Consequently, a.s. *for each $t > 0$*

$$\int_0^t F\,dM = \tilde{M}(\tau(t)),$$

i.e., the stochastic integral with respect to a p-stable motion is nothing but another p-stable motion with a randomly changed time scale.

PROOF. Observe that for every fixed t, $\tau^{-1}(t)$ is an $\mathscr{F}_{u+0} = \bigcap_{v>u}\mathscr{F}_v = \mathscr{F}_u$-stopping time, $u \ge 0$. Moreover, $t \to \tau^{-1}(t)$ is a right-continuous nondecreasing function, so that $\tilde{M}$ has almost all paths in $D[0,\infty)$.

To begin, assume additionally that F satisfies condition (3.1). The complex exponential martingale $(Z_F(u), \mathscr{F}_u)$ from Proposition 3.1 satisfies the equality

$$Z_F(\tau^{-1}(t)) = \exp\{i\lambda\tilde{M}(t) + |\lambda|^p t\}. \tag{3.2}$$

By the Optional Sampling Theorem [cf. e.g., Ikeda and Watanabe (1981), page 34] for each integer N and $s \le t$

$$E\left(Z_F(\tau^{-1}(t) \wedge N)|\mathscr{F}_{\tau^{-1}(s)\wedge N}\right) = Z_F(\tau^{-1}(s) \wedge N). \tag{3.3}$$

Since, for each t,

$$\begin{aligned} &E|Z_F(\tau^{-1}(t)\wedge N) - Z_F(\tau^{-1}(t))| \\ &\quad = \int_{\{\tau^{-1}(t)>N\}} |Z_F(N) - Z_F(\tau^{-1}(t))|\,dP \\ &\quad \le \int_{\{\tau^{-1}(t)>N\}} |Z_F(N)|\,dP + \int_{\{\tau^{-1}(t)>N\}} |Z_F(\tau^{-1}(t))|\,dP \\ &\quad \le 2\exp(t|\lambda|^p)P\{\tau^{-1}(t) > N\} \to 0 \end{aligned} \tag{3.4}$$

as $N \to \infty$, we have that

$$\begin{aligned} &E|E[Z_F(\tau^{-1}(t))|\mathscr{A}_s] - Z_F(\tau^{-1}(s))| \\ &\quad \le E|E[Z_F(\tau^{-1}(t))|\mathscr{A}_s] - E[Z_F(\tau^{-1}(t))|\mathscr{F}_{\tau^{-1}(s)\wedge N}]| \\ &\qquad + E|Z_F(\tau^{-1}(t)) - Z_F(\tau^{-1}(t)\wedge N)| \\ &\qquad + E|E[Z_F(\tau^{-1}(t)\wedge N)|\mathscr{F}_{\tau^{-1}(s)\wedge N}] - Z_F(\tau^{-1}(s)\wedge N)| \\ &\qquad + E|Z_F(\tau^{-1}(s)\wedge N) - Z_F(\tau^{-1}(s))| \to 0 \end{aligned}$$

as $N \to \infty$. Here, the convergence to 0 of the first summand is justified by the fact that $\mathscr{F}_{\tau^{-1}(s)\wedge N} \uparrow \mathscr{A}_s$ as $N \to \infty$, of the second and fourth by (3.4) and of the third by (3.3).

Therefore,

$$E[Z_F(\tau^{-1}(t))|\mathscr{A}_s] = Z(\tau^{-1}(s)) \quad \text{a.s.}$$

which, in conjunction with (3.2), gives that

$$E[\exp\{i\lambda(\tilde{M}(t) - \tilde{M}(s))\}|\mathscr{A}_s] = \exp[-(t-s)|\lambda|^p].$$

Hence $\tilde{M}$ is an $(\mathscr{A}_t)$-p-stable motion.

Now, the removal of the additional restriction that F satisfies (3.1) can be accomplished by the truncation argument, considering, instead of an arbitrary process F satisfying assumptions of the theorem, its restriction F_n defined by (2.7) which is the same as F up to time τ_n. □

COROLLARY 3.1. *Let $0 < p < 2$, the process $F \in L^p_{\text{a.s.}}$, $\tau(u) = \int_0^u |F|^p\,dt$, and let φ: $\mathbb{R}^+ \to \mathbb{R}^+$ be increasing. Then*

(i) $$\limsup_{t\to 0} \left|\int_0^t F\,dM\right| \Big/ \varphi(\tau(t)) = 0 \quad \text{or} \quad = \infty \quad \text{a.s.}$$

according as

$$\int_0^1 \varphi^{-p}(t)\,dt < \infty \quad \text{or} \quad = \infty.$$

(ii) *If* $\tau(u) \to \infty$ a.s. *as* $u \to \infty$ *then*

$$\limsup_{t\to\infty} \left| \int_0^t F\, dM \right| \Big/ \varphi(\tau(t)) = 0 \quad \text{or} \quad = \infty \quad \text{a.s.}$$

according as

$$\int_1^\infty \varphi^{-p}(t)\, dt < \infty \quad \text{or} \quad = \infty.$$

PROOF. The proof of the above corollary follows immediately from Theorem 3.2 and from the classical Khinchin's result [see e.g., Fristedt (1974), Theorem 11.2 and Corollary 11.3] describing asymptotic a.s. behavior at 0 and ∞ of the stable motion itself.

4. A necessary and sufficient condition for the existence of the p-stable stochastic integral. The inner clock formula proved in Section 3 makes a more formal theory of stochastic integration suggested below workable.

DEFINITION 4.1. An $\{\mathscr{F}_t\}$-adapted measurable process $F = \{F(t, \omega)\}_{t\geq 0}$ is said to be M-integrable (or $dM(t)$-integrable) if there exists a sequence (F_n) of simple $(\mathscr{F}_t)$-adapted processes such that for each $T > 0$

(i) $F_n \to F$ in measure $dP\,dt$ on $\Omega \times [0, T]$,
(ii) $\int_0^t F_n\, dM$ converge a.s. uniformly in $t \in [0, T]$ as $n \to \infty$,

and if the limiting stochastic process in (ii) does not depend on the choice of a sequence (F_n) satisfying conditions (i) and (ii). We will denote this limit process by $I_F(t)$, $t \geq 0$.

The next theorem describes the class of M-integrable processes and explains the relationship of the above definition to the integral $\int F\, dM$ constructed in Section 2.

THEOREM 4.1. *The process F is M-integrable if and only if $F \in L^p_{\text{a.s.}}$. Moreover*

$$I_F(t) = \int_0^t F\, dM$$

a.s. *for each* $t \geq 0$.

PROOF. Let F be M-integrable and F_n satisfy (i) and (ii) of Definition 4.1. Fix $T > 0$ and define $G_{nm} = F_n - F_m$ for $0 \leq t < T$ and $G_{nm} = 1$ for $t \geq T$. Clearly $\int_0^\infty |G_{nm}|^p\, ds = \infty$ and $G_{nm} \in L^p_{\text{a.s.}}$. By condition (ii) of Definition 4.1 $\int_0^t G_{nm}\, dM \to 0$ a.s. uniformly in $t \in [0, T]$. Put

$$\sigma_{nm}(u) = \int_0^u |G_{nm}|^p\, ds.$$

By Theorem 3.1

$$\tilde{M}_{nm}(t) = \int_0^{\sigma_{nm}^{-1}(t)} G_{nm}\, dM$$

is a new p-stable motion. Since for every $\varepsilon > 0$ we have

$$(4.1)\quad \begin{aligned} P\left\{\int_0^T |F_n - F_m|^p\,dt > \varepsilon\right\} &= P\{\sigma_{nm}(T) > \varepsilon\} \le P\{\sigma_{nm}^{-1}(\varepsilon) < T\} \\ &\le P\left\{|\tilde{M}_{nm}(\varepsilon)| \le \sup_{t<T}\left|\int_0^t G_{nm}\,dM\right|\right\} =_{df} p_{nm}, \end{aligned}$$

and since $\mathscr{L}(\tilde{M}_{nm}(\varepsilon)) = \mathscr{L}(M(\varepsilon))$ has no atom at 0, we get that $p_{nm} \to 0$ as $n, m \to \infty$ which implies that $\int_0^T |F|^p\,dt < \infty$ a.s. Since T is arbitrary this shows that $F \in L^p_{\text{a.s.}}$.

Now, to prove the converse, assume that $F \in L^p_{\text{a.s.}}$. The construction of the stochastic integral in Section 2 displayed a sequence of nonnecessarily simple processes (2.7) which satisfy (i) and (ii) of Definition 4.1. The upper estimate in Theorem 2.1 permits the a.s. uniform approximation in $[0, T]$ of $\int_0^t F_n\,dM$ by integrals of simple processes. Therefore, one can now obtain a sequence of simple processes G_n satisfying (i) and (ii) of Definition 4.1, and such that $\lim_{n\to\infty}\int_0^t G_n\,dM = \int_0^t F\,dM$ a.s. uniformly in $t \in [0, T]$. Now, let H_n be another sequence satisfying (i) and (ii). Exactly as in (4.1) we can show that for each T

$$\int_0^T |G_n - G_m|^p\,dt \to 0 \quad\text{and}\quad \int_0^T |H_n - H_m|^p\,dt \to 0$$

in probability. Hence by (i)

$$(4.2)\qquad \tau_n(T) = \int_0^T |H_n - G_n|^p\,dt \to 0$$

in probability as $n \to \infty$. Fix $T > 0$ and define $J_n = H_n - G_n$ for $0 \le t < T$ and $J_n = 1$ for $t \ge T$. Then

$$\tilde{M}_n(t) = \int_0^{\tau_n^{-1}(t)} J_n\,dM, \qquad t \ge 0,$$

is another p-stable motion. Let $\varepsilon > 0$, $\delta > 0$. We have

$$\begin{aligned} P\left\{\sup_{t\le T}\left|\int_0^t (H_n - G_n)\,dM\right| > \varepsilon\right\} &= P\left\{\sup_{t\le T}|\tilde{M}_n(\tau_n(t))| > \varepsilon\right\} \\ &= P\left\{\sup_{t\le T}|\tilde{M}_n(\tau_n(t))| > \varepsilon,\ \tau_n(T) \le \delta\right\} \\ &\quad + P\left\{\sup_{t\le T}|\tilde{M}_n(\tau_n(t))| > \varepsilon,\ \tau_n(T) > \delta\right\} \\ &\le P\left\{\sup_{t\le\delta}|\tilde{M}_n(t)| > \varepsilon\right\} + P\{\tau_n(T) > \delta\}. \end{aligned}$$

The last expression tends to zero as $n \to \infty$ and then $\delta \to 0$ ($\tilde{M}_n$ is a version of M which is right-continuous with start at 0), so that

$$I_F(t) = \lim_{n\to\infty}\int_0^t H_n\,dM = \lim_{n\to\infty}\int_0^t G_n\,dM = \int_0^t F\,dM$$

a.s. for each t. □

5. Multiple integration. Characterization of double p-stable stochastic integrals. In this section we will define an n-tuple integral of a nonrandom function of n-variables with respect to the p-stable motion.

DEFINITION 5.1. Let $f = f(t_1, \dots, t_n)$ be a real measurable function defined on the n-dimensional tetrahedron $\Delta^n = \{(t_1, \dots, t_n)\colon\ 0 \le t_1 < t_2 < \cdots < t_n \le T\}$. The n-tuple integral

$$\int_{\Delta^n} f\, dM \times \cdots \times dM$$

is defined as the iterated integral

$$\int_0^T \left(\int_0^{t_n} \cdots \left(\int_0^{t_2} f(t_1, t_2, \dots, t_n)\, dM(t_1) \right) \cdots\ dM(t_{n-1}) \right) dM(t_n)$$

provided each of the $\mathscr{F}_{t_k}$-adapted stochastic processes (inner integrals)

$$\int_0^{t_k} \cdots \int_0^{t_2} f(t_1, t_2, \dots, t_{k-1}, t_k, \dots, t_n)\, dM(t_1) \cdots\ dM(t_{k-1}),$$

$k = 2, \dots, n$ is $dM(t_k)$-integrable on $[0, T]$ in the sense of Definition 4.1. If this is the case, f is said to be $dM \times \cdots \times dM$-integrable on Δ^n.

Theorem 4.1 gives immediately the following criterion of $dM \times \cdots \times dM$-integrability of f:

THEOREM 5.1. *The function $f = f(t_1, \dots, t_n)$ is $dM \times \cdots \times dM$-integrable on Δ^n if and only if the following conditions are satisfied*:

$$\int_0^{t_2} |f(t_1, t_2, \dots, t_n)|^p\, dt_1 < \infty \quad \text{for each } t_2 < t_3 < \cdots < t_n \le T,$$

$$\int_0^{t_3} \left| \int_0^{t_2} f(t_1, t_2, \dots, t_n)\, dM(t_1) \right|^p dt_2 < \infty \quad \text{a.s.}$$

for each $t_3 < \cdots < t_n \le T$,

$$\int_0^T \left| \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, t_2, \dots, t_n)\, dM(t_1) \cdots\ dM(t_{n-1}) \right|^p dt_n < \infty \quad \text{a.s.}$$

The above theorem does not give, of course, explicit conditions for $dM \times \cdots \times dM$-integrability of f (except in case $n = 1$). Making these conditions explicit requires characterizing those fs for which the stochastic processes represented by inner integrals have sample paths in L^p. For $n > 2$ such processes are not p-stable, which makes the problem difficult to handle. In the case of double integrals ($n = 2$) such a characterization is equivalent to the problem of a.s. p-integrability on $[0, T]$ of sample paths of a general measurable p-stable symmetric stochastic process (see Proposition 6.1), and is given for $1 \le p < 2$ in Theorem 6.3 which yields the following:

THEOREM 5.2. *Let $f = f(t, s)$, $0 \le s < t \le T$ be a real measurable function. The double p-stable integral*

$$\int_{\Delta^2} f\,dM \times dM = \int_0^T \int_0^t f(t, s)\,dM(s)\,dM(t)$$

exists if and only if

$$(5.1)\quad \int_0^T \int_0^t |f(t, s)|^p \left[1 + \log_+ \frac{|f(t, s)|^p}{\int_0^t |f(t, u)|^p\,du \int_s^T |f(u, s)|^p\,du}\right] ds\,dt < \infty.$$

PROOF. Let $g(t, s) = f(t, s)I_{[0, t)}(s)$. Then by Theorem 5.1 the existence of $\int_{\Delta^2} f\,dM \times dM$ is equivalent to the p-integrability of a.a. sample paths of p-stable process $X(t) = \int_0^T g(t, s)\,dM(s\cdot)$. By Theorem 6.2 the condition (6.4) $[A_p(g) < \infty]$ is necessary and sufficient for such integrability and is clearly equivalent to (5.1). □

6. Measurability and integrability of sample paths of symmetric p-stable processes. Let $X(t)$, $t \in \mathbb{T}$, be a stochastic process with separable metric space $\mathbb{T}$ as a parameter set. Recall that X is said to be a measurable process if $X(t, \omega)$ is a jointly measurable mapping from $\mathbb{T} \times \Omega$ into $\mathbb{R}$. Cohn (1972) [see also the remark on page 206 of Hoffmann-Jørgensen (1973)] has shown that X has a measurable modification if and only if the mapping $\mathbb{T} \ni t \to X(t) \in L^0(\Omega)$ is Borel measurable and has separable range.

The following proposition complements the results of Bretagnolle, Dacunha-Castelle, and Krivine (1966), Kuelbs (1973), and Cambanis and Miller (1980) concerning the relationship between the measurability of a stable symmetric process and the joint measurability of the kernel in the process' integral representation.

PROPOSITION 6.1. *A symmetric and p-stable process $(X(t)\colon t \in \mathbb{T})$, $0 < p \le 2$, has a measurable modification if and only if it admits an integral representation*

$$\mathscr{L}(X(t)\colon t \in \mathbb{T}) = \mathscr{L}\left(\int_0^1 f(t, s)M(ds)\colon t \in \mathbb{T}\right),$$

where $f\colon \mathbb{T} \times [0,1] \to \mathbb{R}$ is jointly measurable and for each $t \in \mathbb{T}$, $f(t,\cdot) \in L^p[0,1]$.

PROOF. Let $Y(t) = \int_0^1 f(t, s)\,dM(s)$ be a version of $X(t)$, $t \in T$, where f is as above. Observe that $L^p[0,1] \ni g \to \int_0^1 g(s)\,dM(s) \in L^0(\Omega)$ is continuous and has a separable range. By Fubini's Theorem $\mathbb{T} \ni t \to f(t,\cdot) \in L^p[0,1]$ is Borel measurable. Thus the composition of these two functions: $\mathbb{T} \ni t \to \int_0^1 f(t, s)\,dM(s) \in L^0(\Omega)$ is Borel measurable and has a separable range. By the result of Cohn (1972) Y has a measurable modification and, since measurability depends only on two-dimensional distributions [cf. Hoffmann-Jørgensen (1973)], X also has a measurable modification.

Conversely, if a symmetric p-stable process X has a measurable modification then X has a separable range in $L^0(\Omega)$, and by Kuelbs' (1973) result there exists a collection of functions $\{f_t\}_{t\in\mathbb{T}} \subset L^p[0,1]$ and a p-stable motion M such that

$$Y(t) = \int_0^1 f_t(s)\, dM(s)$$

is a version of X[see also Cambanis and Miller (1980), Section 3]. Since $L^0(\Omega)$ and $L^q(\Omega)$ topologies are equivalent on $X(\mathbb{T})$ for $0 < q < p$, the function $\mathbb{T} \ni t \to X(t) \in L^q(\Omega)$ is also Borel measurable. Hence the function

$$t \to \left(\int |f_t(s) - f_{t'}(s)|^p\, ds\right)^{1/p} = c_{pq}\left(E|X(t) - X(t')|^q\right)^{1/q}$$

is measurable for fixed t'. Separability of $L^p[0,1]$ gives that $\mathbb{T} \ni t \to f_t \in L^p[0,1]$ is Borel measurable with separable range. By the continuity of the natural embedding of $L^p[0,1]$ into $L^0[0,1]$ the same holds for the function $\mathbb{T} \ni t \to f_t \in L^0[0,1]$. Therefore, by Cohn's (1972) theorem applied to the stochastic process $(f_t\colon t \in \mathbb{T})$, there exists a jointly measurable function $f(t,s)$, $t \in \mathbb{T}$, $s \in [0,1]$ such that for every $t \in \mathbb{T}$

$$|\{s \in [0,1]\colon f_t(s) = f(t,s)\}| = 1,$$

where $|\ |$ denotes the Lebesgue measure on $[0,1]$, so that $Y(t) = \int_0^1 f(t,s)\, dM(s)$ a.s. □

Let $(X(t)\colon t \in \mathbb{T})$ be a measurable symmetric p-stable process and let μ be a σ-finite Borel measure on $\mathbb{T}$. Without loss of generality, in view of Proposition 6.1, we shall assume that

$$X(t) = \int_0^1 f(t,s)\, dM(s), \qquad t \in \mathbb{T}. \tag{6.1}$$

Below, we study the integrability properties of sample paths of the process $X(t)$. The following theorem is well known [cf. Linde (1983) and Cambanis and Miller (1980) for other integrability results], and is easiest proved via techniques of the probability theory in infinite dimensional spaces.

Theorem 6.1. *Assume X is a p-stable process given by* (6.1). *If*

$$\int_{\mathbb{T}} |X(t)|^p \mu(dt) < \infty \quad \text{a.s.}$$

then

$$\int_0^1 \int_{\mathbb{T}} |f(t,s)|^p \mu(dt)\, ds < \infty.$$

However, the known results did not give a full characterization of symmetric p-stable processes with p-integrable sample paths. Let us observe that such a characterization problem is equivalent to the problem of description of θ_p-radonifying operators into L^p [cf. Linde (1983) and Cambanis, Rosiński, and

Woyczyński (1983)] and is of fundamental importance in the theory of double stochastic integrals with respect to a stable motion (see Section 5). Our next theorem gives a solution to the above problem.

THEOREM 6.2. *Let $1 \le p < 2$. For a measurable symmetric p-stable process*

$$X(t) = \int_0^1 f(t, s)\, dM(s), \qquad t \in \mathbb{T}, \tag{6.2}$$

we have that

$$\int_{\mathbb{T}} |X(t)|^p \mu(dt) < \infty \quad \text{a.s.} \tag{6.3}$$

if and only if

$$A_p(f) = \int_{\mathbb{T}} \int_0^1 |f(t, s)|^p \left[1 + \log_+ \frac{|f(t, s)|^p \int_{\mathbb{T}} \int_0^1 |f(u, v)|^p \mu(du)\, dv}{\int_0^1 |f(t, v)|^p\, dv \int_{\mathbb{T}} |f(u, s)|^p \mu(du)} \right] ds\, u(dt) < \infty. \tag{6.4}$$

PROOF. Note that by Theorem 6.1(ii) both (6.3) and (6.4) imply that

$$\|f\|_p^p = \int_0^1 \int_{\mathbb{T}} |f(t, s)|^p \mu(dt)\, ds < \infty.$$

The characteristic functional of X (or equivalently, of the cylindrical measure generated on $\mathbb{R}^{\mathbb{T}}$ by X) is given by the formula

$$E \exp\left[i \sum_{t \in \mathbb{T}} \alpha(t) X(t) \right] = \exp\left[- \int_{\mathbb{R}^{\mathbb{T}}} \left| \sum_{t \in \mathbb{T}} \alpha(t) \beta(t) \right|^p dm(\beta) \right], \tag{6.5}$$

$\alpha \in \mathbb{R}^{(\mathbb{T})}$ (= sequences with finite support), where the measure m on $\mathbb{R}^{\mathbb{T}}$ can be described as the distribution of the process $V(t) = V(t, \cdot\,)$ which is defined as follows:

$$V(t, s) = \|f\|_p \left(\int_{\mathbb{T}} |f(t, s)|^p\, d\mu(t) \right)^{-1/p} f(t, s), \tag{6.6}$$

where $t \in \mathbb{T}$, $s \in [0, 1]$, and $([0, 1], \nu)$ plays the role of underlying probability space with

$$\nu(ds) = \|f\|_p^{-p} \int_{\mathbb{T}} |f(t, s)|^p \mu(dt)\, ds. \tag{6.7}$$

Indeed,

$$\begin{aligned} \exp\left(- \int_{\mathbb{R}^{\mathbb{T}}} \left| \sum_{t \in \mathbb{T}} \alpha(t) \beta(t) \right|^p dm(\beta) \right) &= \exp\left(-E_\nu \left| \sum_{t \in \mathbb{T}} \alpha(t) V(t) \right|^p \right) \\ &= \exp\left(- \int_0^1 \left| \sum_{t \in \mathbb{T}} \alpha(t) f(t, s) \right|^p ds \right) \\ &= E \exp i \sum_{t \in \mathbb{T}} \alpha(t) X(t). \end{aligned}$$

Note that all the sample paths of $V(t)$ have a constant (independent of s) $L^p(\mathbb{T}, \mu)$-norm equal to $\|f\|_p$. Therefore, by Lemma 6.12 of Giné and Zinn (1983) or Remark 3.15 of Marcus and Pisier (1983) [the idea thereof really goes back to a paper by LePage, Woodroofe, and Zinn (1981)]; the process $X(t)$ has a.s. sample paths in $L^p(\mathbb{T}, \mu)$ if and only if the series

$$(6.8) \qquad \sum_j j^{-1/p} \varepsilon_j V_j(t), \qquad t \in \mathbb{T},$$

converges a.s. in the $L^p(\mathbb{T}, \mu)$-norm [or in $L^r(\Omega; L^p(\mathbb{T}, \mu))$, $0 < r < \infty$], where (ε_j) are Bernoulli r.v.s, V_js are independent copies of the process V, and the sequences (ε_j) and (V_j) are independent.

Proposition 5.2 of Cambanis, Rosiński, and Woyczyński (1983) states that for i.i.d. symmetric random variables $\xi, \xi_1, \xi_2, \dots$

$$(6.9) \quad c^{-1} E|\xi|^p \left(1 + \log_+ \frac{|\xi|^p}{E|\xi|^p}\right) \le E\left|\sum_{j=1}^{\infty} j^{-1/p} \xi_j\right|^p \le cE|\xi|^p \left(1 + \log_+ \frac{|\xi|^p}{E|\xi|^p}\right),$$

where $0 < p < 2$, and $c = c(p)$ is a numerical constant.

Since

$$E\left\|\sum_j j^{-1/p} \varepsilon_j V_j\right\|^p_{L^p(\mathbb{T}, \mu)} = \int_{\mathbb{T}} E\left|\sum_j j^{-1/p} \varepsilon_j V_j\right|^p \mu(dt),$$

and since

$$A_p(f) = \int_{\mathbb{T}} E_\nu |V(t)|^p \left(1 + \log_+ \frac{|V(t)|^p}{E|V(t)|^p}\right) \mu(dt),$$

[note that $E_\nu |V(t)|^p = \int_0^1 |f(t, s)|^p \, ds$], we get from (6.9) that

$$c^{-1} A_p(f) \le E\left\|\sum_j j^{-1/p} \varepsilon_j V_j\right\|^p_{L^p(\mathbb{T}, \mu)} \le cA_p(f),$$

which shows that the condition $A_p(f) < \infty$ is equivalent to the boundedness in $L^p(\Omega; L^p(\mathbb{T}, \mu))$ of the series (6.8), which in turn is equivalent to the convergence of (6.8) in the $L^p(\Omega; L^p(\mathbb{T}, \mu))$-norm by results of Hoffmann-Jørgensen (1974) and Kwapień (1974). □

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CENTER FOR STOCHASTIC PROCESSES
DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL
CHAPEL HILL, NORTH CAROLINA 27514

DEPARTMENT OF MATHEMATICS AND STATISTICS
CASE WESTERN RESERVE UNIVERSITY
UNIVERSITY CIRCLE
CLEVELAND, OHIO 44106