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ON THE INFLUENCE OF THE EXTREMES OF AN I.I.D. SEQUENCE ON THE MAXIMAL SPACINGS

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Let $X_1, X_2, \dots$ be an i.i.d. sequence of random variables with a continuous density f, positive on (A, B), and null otherwise. Under the assumption that $Y_n = \min\{X_1, \dots, X_n\}$ and $Z_n = \max\{X_1, \dots, X_n\}$ belong to the domain of attraction of extreme value distributions and that $f(x) \to 0$ as $x \to A$ or $x \to B$, we show that the weak limiting behavior of Y_n and Z_n characterizes completely the weak limiting behavior of the maximal spacing generated by $X_1, \dots, X_n$ and obtain the corresponding limiting distributions. We study as examples the cases of the normal, Cauchy, and gamma distributions.

1. Introduction. Let $X_1, X_2, \dots$ be an i.i.d. sequence of random variables, and let

$$X_{1,n} \le \cdots \le X_{n,n}$$

be the order statistics of $X_1, \dots, X_n$. For $n \ge 2$, define the corresponding spacings by

$$S_i^{(n)} = X_{i+1,n} - X_{i,n}, \qquad i = 1, \dots, n-1.$$

Denote the order statistics of $S_1^{(n)}, \dots, S_{n-1}^{(n)}$ by

$$M_{n-1}^{(n)} \le \cdots \le M_2^{(n)} \le M_1^{(n)}.$$

The aim of the following is to characterize the weak limiting behavior of the kth *maximal spacing* $M_k^{(n)}$ under various assumptions on the distribution function $F(x) = P(X_1 \le x)$.

Although a great deal is known about $M_k^{(n)}$ in the case of the uniform distribution on $(0, 1)$ (see Devroye, 1981, 1982; Deheuvels, 1982, 1983), very few results are available when $F(\cdot)$ is arbitrary.

Let $A = \inf\{x; F(x) > 0\}$, and $B = \sup\{x; F(x) < 1\}$. In the case where $F(\cdot)$ has a continuous density $f(\cdot) > 0$ on (A, B), strong limiting bounds for $M_k^{(n)}$ have been given by Deheuvels (1984) where it was proved, among other results, that, under general regularity assumptions on $F(\cdot)$, for any fixed $k \ge 1$, $M_k^{(n)} \to 0$ a.s. as $n \to \infty$, if and only if the extremes $X_{1,n}$ and $X_{n,n}$ of $X_1, \dots, X_n$ are strongly stable, i.e., iff there exist nonrandom sequences $\{\beta_n\}$ and $\{b_n\}$ such that

$$X_{1,n} - \beta_n \to 0 \quad \text{and} \quad X_{n,n} - b_n \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

This hints that, in the case where (A, B) is unbounded, the limiting behavior of $M_k^{(n)}$ as $n \to \infty$ should be closely related to the limiting behavior or the extremes $X_{1,n}$ and $X_{n,n}$ as $n \to \infty$.

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The main achievement of this paper is to show that it is the case, and that, under very general assumptions, the weak limiting behavior of $X_{1,n}$ and $X_{n,n}$ completely specifies the weak limiting behavior of $M_k^{(n)}$ as $n \to \infty$.

In Section 2 we present these results whose proofs are detailed in Section 3.

2. Theorems. We shall make, unless otherwise specified, the assumption:

(H1) $F(x) = P(X_1 \le x)$ has a continuous first derivative $f(x) > 0$ on (A, B), where

$$A = \inf\{x; F(x) > 0\} < B = \sup\{x; F(x) < 1\}.$$

We shall start with the case where

$$\text{(H2)} \quad -\infty < A < B = +\infty, \quad f(A+) = \lim_{x \downarrow A} f(x) \text{ exists and } f(A+) > 0.$$

It will be useful to introduce the function $G(u)$, $0 < u < 1$, defined by

$$G(u) = (1 - F)^{-1}(u) = \inf\{x; 1 - F(x) \le u\}.$$

Let us consider the upper extreme $X_{n,n}$ and assume that it belongs to the domain of attraction of an extreme value distribution. In this case (see Gnedenko, 1943; Galambos, 1978, pp. 49–62), the only possibilities are given by the Gumbel and Fréchet limiting types, for which we have:

(H3) *Gumbel limiting type.* We have

$$\lim_{n \to \infty} P\left(a_n^{-1}(X_{n,n} - b_n) \le x\right) = \lim_{n \to \infty} F^n(a_n x + b_n) = e^{-e^{-x}} = \Lambda(x), \quad -\infty < x < +\infty,$$

where

$$b_n = G\left(\frac{1}{n}\right) \quad \text{and} \quad a_n = G\left(\frac{1}{ne}\right) - G\left(\frac{1}{n}\right).$$

In this case, we shall make the complementary regularity assumption that $F(x) = P(X_1 \le x)$ has a continuous first derivative $f(x)$, ultimately nonincreasing in the upper tail.

(H4) *Fréchet limiting type.* We have, for $a > 0$,

$$\lim_{n \to \infty} P\left(b_n^{-1} X_{n,n} \le x\right) = \lim_{n \to \infty} F^n(b_n x) = e^{-x^{-a}} = \Phi_a(x), \qquad x > 0,$$

where

$$b_n = G\left(\frac{1}{n}\right).$$

Next, we consider the case of distributions with bounded support, assuming that:

$$\text{(H5)} \quad -\infty < A < B < +\infty, \quad f(A+) = \lim_{x \downarrow A} f(x) \text{ exists and } f(A+) > 0.$$

In this case, if $X_{n,n}$ belongs to the domain of attraction of an extreme value distribution, the only possibilities are given by the Gumbel limiting type (H3), and the Weibull limiting type:

(H6) *Weibull limiting type*. We have, for $a > 0$,

$$\lim_{n\to\infty} P\left(\frac{X_{n,n} - B}{B - b_n} \le x\right) = e^{-(-x)^a} = \Psi_a(x), \qquad x < 0,$$

where

$$b_n = G\left(\frac{1}{n}\right).$$

For each of these cases, the following theorems hold.

In Theorem 1, and other theorems and lemmas, ω_j are independent exponentially distributed random variables with parameter 1.

THEOREM 1 (Gumbel limiting type). *Under* (H1), (H2), *and* (H3), *or, under* (H1), (H3), *and* (H5), *we have*

$$\lim_{n\to\infty} P\left(a_n^{-1}M_1^{(n)} \le x\right) = \prod_{j=1}^{+\infty} (1 - e^{-jx}),$$

and, for any fixed $k > 1$,

$$\lim_{n\to\infty} P\left(a_n^{-1}M_k^{(n)} \le x\right) = H_k(x),$$

where $H_k(x)$ *is the distribution function of the* kth *maximum of* $\{\omega_j/j,\ j \ge 1\}$.

Furthermore, if $I = I_{n,k}$ *is the index such that* $S_I^{(n)} = M_k^{(n)}$, *then*

$$\lim_{n\to\infty} P(I_{n,k} = n - i) = P\left(\frac{\omega_i}{i} \quad \text{is the} \quad k\text{th maximum of} \left\{\frac{\omega_j}{j},\ j \ge 1\right\}\right),$$
$$i = 1, 2, \dots .$$

THEOREM 2 (Fréchet limiting type). *Let* $A = \inf\{x;\ F(x) > 0\} > -\infty$. *Then, under* (H4), *we have, for any* $k \ge 1$,

$$\lim_{n\to\infty} P\left(b_n^{-1}M_k^{(n)} \le x\right) = K_{k,a}(x),$$

where $K_{k,a}(x)$ *is the distribution function of the* kth *maximum of*

$$\left\{\zeta_j = \left(\sum_{i=1}^{j} \omega_i\right)^{-1/a} - \left(\sum_{i=1}^{j+1} \omega_i\right)^{-1/a},\ j \ge 1\right\}.$$

Furthermore, if $I = I_{n,k}$ *is the index such that* $S_I^{(n)} = M_k^{(n)}$, *then*

$$\lim_{n\to\infty} P(I_{n,k} = n - i) = P\left(\zeta_i \text{ is the } k\text{th maximum of } \{\zeta_j,\ j \ge 1\}\right), \qquad i = 1, 2, \dots .$$

THEOREM 3 (Weibull limiting type). *Under* (H1), (H5), *and* (H6), *if* $a > 1$, *then, for any fixed* $k > 1$, *we have*

$$\lim_{n\to\infty} P\left(\frac{M_k^{(n)}}{B - b_n} \le x\right) = L_{k,a}(x),$$

where $L_{k,a}(x)$ *is the distribution function of the* kth *maximum of*

$$\left\{\nu_j = \left(\sum_{i=1}^{j+1} \omega_i\right)^{1/a} - \left(\sum_{i=1}^{j} \omega_i\right)^{1/a}, j \geq 1\right\}.$$

Furthermore, if $I = I_{n,k}$ *is the index such that* $S_I^{(n)} = M_k^{(n)}$, *then*

$$\lim_{n\to\infty} P(I_{n,k} = n - i) = P\big(\nu_i \text{ is the } k\text{th maximum of } \{\nu_j,\ j \geq 1\}\big),$$

$$i = 1, 2, \ldots .$$

REMARK 1. (1°) In Theorem 3, it is obvious that as $j \to \infty$, $\nu_j \sim (1/a)j^{(1/a)-1}\omega_{j+1}$, which shows that the sequence $\{\nu_j, j \geq 1\}$ is bounded a.s. iff $a > 1$. If $X_{n,n}$ belongs to the domain of attraction of Ψ_a for $0 < a < 1$, then we must have

$$\limsup_{x \uparrow B} f(x) = \infty.$$

It is shown in Deheuvels (1984) that, in the case of a bounded support, the major influence on the limiting behavior of $M_k^{(n)}$ is exerted by the behavior of the density $f(\cdot)$ in the neighborhood of the point where it reaches its minimum. It follows that, in the case $0 < a < 1$, the limiting behavior of $X_{n,n}$ does not influence, in general, the limiting behavior of $M_k^{(n)}$.

(2°) If we assume that $a = 1$ in Theorem 3, then, the result subsists partially in the specific case of the uniform distribution on $(0, B)$. In this case, taking $B = 1$ without loss of generality, it is well known (see, e.g., Pyke (1965)) that $nM_k^{(n)}$ has the limiting distribution of the kth maximum of $\{\omega_j, 1 \leq j \leq n\}$. However when the distribution is not uniform, the behavior of $X_{1,n}$ and $X_{n,n}$ does not characterize completely the behavior of $M_k^{(n)}$.

(3°) It is easily seen that the conclusions of Theorems 1, 2, and 3 subsist if we assume that $-X_{1,n}$ is attracted by $\Psi_\alpha(\cdot)$ for some $0 < \alpha \leq 1$, and also if the distribution has a positive weight at $A > -\infty$.

(4°) By changing $X_1, X_2, \ldots$ into $-X_1, -X_2, \ldots$ we obtain from the preceding theorems the description of the influence of the minimum $X_{1,n}$ on $M_k^{(n)}$. We will not state the corresponding results, which can be deduced from Theorems 1, 2, and 3 in a straightforward manner.

In general, the preceding results can be combined to cover a wide range of situations, as is shown in the following theorem.

THEOREM 4. *Under* (H1), *let us assume that*:

(i) $X_{n,n}$ *belongs to the domain of attraction of an extreme value distribution* $M_S(\cdot) \in \{\Lambda(\cdot), \Phi_a(\cdot), \Psi_b(\cdot), a > 0, b > 1\}$.

(ii) $-X_{1,n}$ *belongs to the domain of attraction of an extreme value distribution* $M_I(\cdot) \in \{\Lambda(\cdot), \Phi_a(\cdot), \Psi_b(\cdot), a > 0, b > 1\}$.

Let $A < C < B$, *and consider for a fixed* $N \geq 1$, *the limiting behavior of the* N *maximal spacings* $M_{k,S}^{(n)}$ *generated by* $\{\max(C, X_i), 1 \leq i \leq n\}$ *and of the* N *maximal spacings* $M_{k,I}^{(n)}$ *generated by* $\{\min(C, X_i), 1 \leq i \leq n\}$, $1 \leq k \leq N$.

Assume that both cases are covered by Theorems 1–3 *and Remark* 1 (3°) *and that there exist two sequences* $\{c'_n, n \le 1\}$ *and* $\{c''_n, n \ge 1\}$ *and two independent sequences of random variables* $\{\nu'_i, i \ge 1\}$ *and* $\{\nu''_j, j \ge 1\}$ *such that the limiting joint distributions of* $\{c_n'^{-1}M_{k,S}^{(n)}, 1 \le k \le N\}$ *and of* $\{c_n''^{-1}M_{k,I}^{(n)}, 1 \le k \le N\}$ *coincide, respectively, with the joint distributions of the* N *largest values of* $\{\nu'_i, i \ge 1\}$ *and* $\{\nu''_j, j \ge 1\}$, *respectively.*

Then, for any fixed $k \ge 1$, *if* $\mu_k^{(n)}$ *stands for the* kth *largest value of* $\{c'_n\nu'_i, c''_n\nu''_j, i \ge 1, j \ge 1\}$, *we have*

$$\lim_{n\to\infty}\left\{\sup_x\left|P\left(M_k^{(n)} \le x\right) - P\left(\mu_k^{(n)} \le x\right)\right|\right\} = 0.$$

EXAMPLE 1°. *The normal distribution.* It is well known (see, e.g., Galambos (1978), p. 65), that for the $N(0,1)$ distribution,

$$\lim_{n\to\infty} P\left(a_n^{-1}(X_{n,n} - b_n) \le x\right) = \lim_{n\to\infty} P\left(a_n^{-1}(-X_{1,n} - b_n) \le x\right) = e^{-e^{-x}},$$

where

$$a_n = (2\log n)^{-1/2} \quad \text{and} \quad b_n = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}.$$

It follows that, in the case of the normal $N(0,1)$ distribution, the limiting distribution of $(2\log n)^{1/2}M_k^{(n)}$ as $n \to \infty$ is that of the kth maximum of $\{(1/i)\omega'_i, (1/j)\omega''_j, i \ge 1, j \ge 1\}$, where $\{\omega'_i, i \ge 1\}$ and $\{\omega''_j, j \ge 1\}$ are two independent i.i.d. sequences of exponentially distributed random variables with parameter one. In particular, for $k = 1$, we have

$$\lim_{n\to\infty} P\left(\sqrt{2\log n}\, M_1^{(n)} \le x\right) = \prod_{l=1}^{+\infty}\left(1 - e^{-lx}\right)^2, \qquad x > 0.$$

EXAMPLE 2°. *The Cauchy distribution.* We have

$$f(x) = \frac{1}{\pi(1 + x^2)},$$

and (see Galambos (1978), p. 68)

$$\lim_{n\to\infty} P\left(b_n^{-1}X_{n,n} \le x\right) = \lim_{n\to\infty} P\left(-b_n^{-1}X_{1,n} \le x\right) = e^{-1/x}, \qquad x > 0,$$

where

$$b_n = \tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right) \sim \frac{n}{\pi}, \qquad n \to \infty.$$

It follows that

$$\lim_{n\to\infty} P\left(\frac{\pi}{n}M_k^{(n)} \le x\right) = P(\theta_k \le x),$$

where θ_k is the kth maximum of

$$\left\{\left(\sum_{l=1}^{i}\omega'_l\right)^{-1} - \left(\sum_{l=1}^{i+1}\omega'_l\right)^{-1}, \left(\sum_{l=1}^{j}\omega''_l\right)^{-1} - \left(\sum_{l=1}^{j+1}\omega''_l\right)^{-1}, i \ge 1, j \ge 1\right\},$$

and where $\{\omega_i', i \geq 1\}$ and $\{\omega_j'', j \geq 1\}$ are two independent i.i.d. sequences of exponentially distributed random variables.

EXAMPLE 3°. *The gamma distribution.* Let

$$f(x) = \frac{x^{r-1}}{\Gamma(r)} e^{-x}, \qquad x > 0.$$

We have

$$\lim_{n \to \infty} P\left(M_1^{(n)} \leq x\right) = \prod_{k=1}^{\infty} (1 - e^{-kx}).$$

The result is independent of $r > 0$. For $r = 1$, it has been obtained by Devroye (1984).

Comments.

(1°) The preceding results show that, under (H1), if A or B is infinite and if the extreme values corresponding to an infinite tail of the distribution are attracted by the Gumbel $\Lambda(\cdot)$ limiting type, then, we must have (see Remark 4 in the sequel)

$$\lim_{n \to \infty} M_1^{(n)} \Big/ \{X_{n,n} - X_{1,n}\} = 0 \; P.$$

(2°) On the other hand, if at least one of the extreme values is attracted by a Fréchet Φ_a limiting type, then $M_1^{(n)}/\{X_{n,n} - X_{1,n}\}$ cannot converge to zero in probability.

It follows that the ratio $M_1^{(n)}/\{X_{n,n} - X_{1,n}\}$ has a limiting behavior which enables characterization of the limiting types of $X_{1,n}$ and $X_{n,n}$.

(3°) If $f(x) \downarrow L$ or $\uparrow L$ as $x \uparrow B$, then, if $X_{n,n}$ is attracted by an extreme value distribution, we must have $L = 0$ in all cases except when the limiting type of $X_{n,n}$ is Ψ_a for some $0 < a \leq 1$. It follows that the cases covered by Theorems 1–4 correspond to the cases where $f(x) \to 0$ as $x \to B$ or $x \to A$.

(4°) The preceding arguments directly yield the limiting distribution of the kth maximal j-spacing, where the j-spacings are defined by

$$S_{i,j}^{(n)} = X_{i+j,n} - X_{i,n}, \quad i = 1, \ldots, n - j.$$

(5°) The limiting distribution of Theorem 1 uses the infinite product

$$\prod_{k=1}^{+\infty} (1 - z^k).$$

This function has been introduced by Euler for applications in number theory.

2. Proofs of the theorems. Throughout this section, we shall assume that $U_1, U_2, \ldots$ is an i.i.d. sequence of uniformly distributed random variables on $(0, 1)$, and, without loss of generality, that $X_1 = G(U_1)$, $X_2 = G(U_2), \ldots$, where $G(u) = \inf\{x; 1 - F(x) \leq u\}$, $0 < u < 1$, and $F(x) = P(X_i \leq x)$, $i = 1, 2, \ldots$.

We shall denote the order statistics of $U_1, \dots, U_n$ by $0 = U_{0,n} < U_{1,n} < \cdots < U_{n,n} < 1 = U_{n+1,n}$, and the order statistics of $X_1, \dots, X_n$ by $X_{1,n} = G(U_{1,n}) \le \cdots \le X_{n,n} = G(U_{n,n})$.

We start with the proof of Theorem 1 which will be broken up in a series of lemmas.

LEMMA 1. *Let $\omega_1, \omega_2, \dots$ be an i.i.d. sequence of exponentially distributed random variables. Put $S_m = \omega_1 + \cdots + \omega_m$, $m \ge 1$, $S_0 = 0$. Then $\{U_{i,n}, 0 \le i \le n+1\}$ is identical in distribution with $\{S_i/S_{n+1}, 0 \le i \le n+1\}$.*

PROOF. See, e.g., Pyke (1965, p. 403).

LEMMA 2. *Let $\xi_{i,n} = \{U_{i,n}/U_{i+1,n}\}^i$, $1 \le i \le n$. Then the $\{\xi_{i,n}, 1 \le i \le n\}$ are independent and uniformly distributed on* $(0,1)$.

PROOF. See Malmquist (1950), David (1981, p. 21).

LEMMA 3. *Under* (H1), $X_{n,n} = \max\{X_1, \dots, X_n\}$ *belongs to the domain of attraction of the Gumbel distribution $\Lambda(\cdot)$, if and only if one of the following equivalent conditions holds.*

(i) *For any $x > 0$, $y > 0$, $y \ne 1$, we have*

$$\lim_{u \downarrow 0} \frac{G(ux) - G(u)}{G(uy) - G(u)} = \frac{\log x}{\log y}.$$

(ii) *There exists a function $g(\cdot)$, slowly varying at infinity, and a constant c, such that*

$$G(u) = c + g(1/u) + \int_1^{1/u} \frac{g(t)}{t}\, dt.$$

PROOF. See De Haan (1970, Theorems 1.4.1 and 2.4.1). The equivalence between (i) and (ii) requires $G(1/t)$ to be strictly increasing in the upper tail. This follows from (H1).

LEMMA 4. *For any fixed $N \ge 1$, for any $x_1 > 0, \dots, x_N > 0$, we have, under* (H1) *and* (H3),

$$\lim_{n \to \infty} P\left(\bigcap_{j=1}^{N} \left\{ a_n^{-1} S_{n-j}^{(n)} < x_j \right\} \right) = \prod_{j=1}^{N} (1 - e^{-jx}).$$

PROOF. We have

$$\frac{G(U_{j,n}) - G(U_{j+1,n})}{G(1/ne) - G(1/n)} = \frac{G\big(S_j(n/S_{n+1})/n\big) - G(1/n)}{G(1/ne) - G(1/n)} - \frac{G\big(S_{j+1}(n/S_{n+1})/n\big) - G(1/n)}{G(1/ne) - G(1/n)}.$$

Since $n/S_{n+1} \to 1$ in probability, it follows from Lemma 3(i) that the above expression tends to $-\log S_j + \log S_{j+1} = -j^{-1}\log \xi_{j,n}$ in distribution. An application of Lemma 2 completes the proof of Lemma 4.

LEMMA 5. *Under* (H1) *and* (H3), *for any* $\lambda > 0$, *the function* $G(\lambda u) - G(u)$ *is slowly varying as* $u \to 0$.

PROOF. See De Haan (1970, p. 34). It follows from Lemma 3(i).

LEMMA 6. *Let* (H1) *and* (H3) *be satisfied, jointly with* (H2) *or* (H5). *Let* $\gamma \in (0,1)$ *be fixed. Then, we have*

$$\sup_{0<u<1} \sup_{\gamma \le \lambda < 1} \left\{ \frac{1}{-\log \lambda} \frac{G(\lambda u) - G(u)}{G(u/e) - G(u)} \right\} = \Gamma_\lambda < \infty.$$

PROOF. We remark that, by (H3), there exists a $\delta \in (0,1)$, such that $-G'(u) = 1/f(G(u))$ is nonincreasing in $(0, \delta)$. It follows that, for any $u \in (0, \delta)$, we have

$$\frac{-uG'(u)}{G(u/e) - G(u)} \le \frac{-uG'(u)}{u(1/e - 1)G'(u)} = \frac{e}{e-1}.$$

Next, we have, for $0 < \lambda < 1$ and $0 < u < \delta$,

$$G(\lambda u) - G(u) = -u\int_\lambda^1 G'(us)\,ds \le -uG'(\lambda u)(1-\lambda) \le \frac{e}{e-1}\Theta(\lambda u)\frac{1-\lambda}{\lambda},$$

where $\Theta(u) = G(u/e) - G(u)$ is slowly varying as $u \to 0$ by Lemma 5.

Hence

$$\limsup_{u \downarrow 0} \sup_{\gamma \le \lambda < 1} \left\{ \frac{1}{-\log \lambda} \frac{G(\lambda u) - G(u)}{G(u/e) - G(u)} \right\}$$
$$\le \frac{e}{e-1} \limsup_{u \downarrow 0} \left\{ \sup_{\gamma \le \lambda < 1} \frac{\Theta(\lambda u)}{\Theta(u)} \right\} \left\{ \sup_{\gamma \le \lambda < 1} \frac{1-\lambda}{-\lambda \log \lambda} \right\} < \infty$$

by the uniform convergence theorem for slowly varying functions.

Now let $\Delta \in (0,1)$ be fixed. We have evidently, for any $u \in (\Delta, 1)$,

$$\frac{G(\lambda u) - G(u)}{G(u/e) - G(u)} \le \frac{\lambda - 1}{1/e - 1} \left\{ \frac{\sup_{\Delta\mu < v < 1} 1/f(G(v))}{\inf_{\Delta\mu < v < 1} 1/f(G(v))} \right\},$$

where $\mu = \min\{\lambda, 1/e\}$.

The proof of Lemma 6 follows directly from the above inequalities.

LEMMA 7. *For any* $\varepsilon > 0$, *there exists an* $M > 1$ *and a* $\gamma \in (0,1)$, *such that, with probability larger than* $1 - \varepsilon/2$, *we have, for any* j, $M + 1 \le j \le n - 1$,

$$0 < \gamma < \frac{U_{j,n}}{U_{j+1,n}} < 1 \quad \text{and} \quad \log\left(\frac{U_{j+1,n}}{U_{j,n}}\right) < \frac{2}{j}\log j.$$

PROOF. By Lemma 2, $\xi_{j,n}$ is uniformly distributed on $(0,1)$, and hence

$$P(-\log \xi_{j,n} \geq 2\log j) = P(\xi_{j,n} \leq j^{-2}) = j^{-2}.$$

It follows that

$$P\left(\bigcup_{j=M+1}^{n-1}\left\{-\log \xi_{j,n}^{1/j} \geq \frac{2}{j}\log j\right\}\right) = P\left(\bigcup_{j=M+1}^{n-1}\left\{\log\left(\frac{U_{j+1,n}}{U_{j,n}}\right) \geq \frac{2}{j}\log j\right\}\right) \leq \sum_{j=M+1}^{\infty} j^{-2}.$$

Let us now choose M such that $\sum_{j=M+1}^{\infty} j^{-2} < \varepsilon/2$. We can see that with probability larger than $1 - \varepsilon/2$, we have, for any $M + 1 \leq j \leq n - 1$,

$$0 < \gamma = \exp\left(-2 \sup_{j \geq M+1} \frac{\log j}{j}\right) \leq \exp\left(-\frac{2}{j}\log j\right) < \frac{U_{j,n}}{U_{j+1,n}} < 1.$$

The proof of Lemma 7 is now completed.

LEMMA 8. *For any $\varepsilon > 0$, there exists an $\alpha > 1$ and an n_0 such that, for $n \geq n_0$, we have*

$$P\left(\bigcap_{3 \leq j \leq n-1}\left\{j - \alpha\sqrt{j \log\log j} < nU_{j+1,n} < j + \alpha\sqrt{j \log\log j}\right\}\right) > 1 - \frac{\varepsilon}{2}.$$

PROOF. Using the notations of Lemma 1, we have $nU_{j+1,n} = S_{j+1}(n/S_{n+1})$. By the law of the iterated logarithm applied to the sequence $\{S_j\}$, we have

$$\lim_{\alpha \uparrow \infty} \sup_{N \geq 1} P\left(\bigcup_{j=3}^{N}\left\{|S_{j+1} - j| > \frac{\alpha}{2}\sqrt{j \log\log j}\right\}\right) = 0.$$

Since $n/S_{n+1} - 1 = O_p(n^{-1/2})$, and noting that $\log\log j$ is positive for $j \geq 3$, the lemma follows.

LEMMA 9. *Let $H(x)$, $x \geq 1$, be a continuous positive slowly varying function as $x \to \infty$. Then we have, for any $\nu > 0$,*

$$\lim_{T \to \infty} \sup_{T \leq t \leq x}\left\{\frac{\log t}{t^{\nu}} \frac{H(x/t)}{H(x)}\right\} = 0.$$

PROOF. By Karamata's Theorem (see, e.g., De Haan (1970); Seneta (1975)), we can assume that

$$H(x) = C(x)\exp\left(\int_1^x \frac{\varepsilon(t)}{t}\,dt\right), \quad \text{where, for } 1 \leq x < \infty,$$

$$0 < C_1 < C(x) < C_2 < \infty, \qquad \varepsilon(x) < \tfrac{1}{2}\nu,$$

and where

$$\lim_{x \to \infty} \varepsilon(x) = 0.$$

It follows that

$$\sup_{T\le t\le x}\ \sup_{x\ge 1}\left\{\frac{\log t}{t^{\nu}}\frac{H(x/t)}{H(x)}\right\}\le (C_2/C_1)\exp\left(\sup_{t\ge T}\left\{\log\log t-\frac{\nu}{2}\log t\right\}\right)\to 0$$

as $T\to\infty$,

which proves Lemma 9.

LEMMA 10. *Let* (H1) *and* (H3) *be satisfied, jointly with* (H2) *or* (H5). *Let* $x>0$ *be fixed. Then, for any* $\varepsilon>0$, *there exists an* $N>1$ *such that*

$$\limsup_{n\to\infty} P\left(\bigcup_{j=N+1}^{n-1}\left\{a_n^{-1}S_{n-j}^{(n)}>x\right\}\right)<\varepsilon.$$

PROOF. Let $\varepsilon>0$ be fixed. Define $M>1$ and $\gamma\in(0,1)$ as in Lemma 7, and Γ_γ as in Lemma 6. We then have, with probability larger than $1-\varepsilon/2$, for any $M+1\le j\le n-1$,

$$S_{n-j}^{(n)}=G(U_{j,n})-G(U_{j+1,n})=G(\lambda u)-G(u),$$

where

$$0<u=U_{j+1,n}<1\quad\text{and}\quad \gamma<\lambda=\frac{U_{j,n}}{U_{j+1,n}}<1.$$

It follows from Lemma 6 that

$$S_{n-j}^{(n)}\le\Gamma_\gamma\left\{\log\left(\frac{U_{j+1,n}}{U_{j,n}}\right)\right\}\left\{G\left(e^{-1}U_{j+1,n}\right)-G\left(U_{j,u}\right)\right\}.$$

By Lemma 7, we have also

$$\log\left(\frac{U_{j+1,n}}{U_{j,n}}\right)<\frac{2}{j}\log j.$$

It follows that

$$P\left(\bigcap_{j=M+1}^{n-1}\left\{S_{n-j}^{(n)}\le 2\Gamma_\gamma\frac{\log j}{j}\left(G\left(e^{-1}U_{j+1,n}\right)-G\left(U_{j+1,n}\right)\right)\right\}\right)\ge 1-\frac{\varepsilon}{2}.$$

Let now α and n_0 be defined as in Lemma 8. Using this lemma and the inequality above, defining $G(u)=A$ for $u\ge 1$, and event E_j by

$$\left\{a_n^{-1}S_{n-j}^{(n)}\le 2\Gamma_\gamma\frac{\log j}{j}\,\frac{G\left(\left(j-\alpha\sqrt{j\log\log j}\right)/ne\right)-G\left(\left(j+\alpha\sqrt{j\log\log j}\right)/n\right)}{G(1/ne)-G(1/n)}\right\},$$

one gets

$$P\left(\bigcap_{j=M+1}^{n-1}E_j\right)\ge 1-\varepsilon. \tag{A}$$

Let now $0 < \beta < 1$ be given. Then, there exists an M_0 such that $N \geq M_0$ implies, for any $j \geq N + 1$,

$$G\left(\frac{j - \alpha\sqrt{j \log\log j}}{ne}\right) - G\left(\frac{j + \alpha\sqrt{\log\log j}}{n}\right) \leq G\left(\frac{j(1-\beta)}{ne}\right) - G\left(\frac{j(1+\beta)}{n}\right).$$

By taking $H(x) = G((1-\beta)/(ex(1+\beta))) - G(1/x)$ in Lemma 9, it follows that

$$\text{(B)} \qquad \max_{N+1 \leq j \leq n} \left\{ \frac{\log j}{j} \frac{G((j(1-\beta))/ne) - G((j(1+\beta))/n)}{G((1/ne)(1-\beta)/(1+\beta)) - G(1/n)} \right\} \to 0$$

as $N \to \infty$.

Next, by Lemma 3(i), we remark that

$$\text{(C)} \qquad \lim_{n \to \infty} \frac{G((1/ne)(1-\beta)/(1+\beta)) - G(1/n)}{G(1/ne) - G(1/n)} = 1 + \log\left(\frac{1+\beta}{1-\beta}\right).$$

(A), jointly with (B) and (C), proves that, for any $x > 0$,

$$\liminf_{n \to \infty} P\left(\bigcap_{j=N+1}^{n-1} \left\{ a_n^{-1} S_{n-j}^{(n)} \leq x \right\} \right) \geq 1 - \varepsilon,$$

provided that $N \geq \max(M, M_0)$ is large enough. This is all we need for Lemma 10.

Proof of Theorem 1. The result follows as a direct consequence of Lemmas 4 and 10.

Remark 2. (i) In general, the conclusion of Lemma 6 is invalid when f is not assumed to be nonincreasing in the upper tail (as required by (H3)).

(ii) If f is assumed to be ultimately nonincreasing, then, a necessary and sufficient condition for $X_{n,n}$ to belong to the domain of attraction of Λ is (De Haan (1970), Theorem 2.7.3, p. 110) that

$$\text{(H7)} \qquad f(x) \sim \{1 - F(x)\}^2 \Big/ \int_x^B (1 - F(t))\, dt \quad \text{as } x \to B.$$

If f is arbitrary, then (H7) is still sufficient for $X_{n,n}$ to be attracted by Λ. If (H1) and (H7) are satisfied, then we have (see De Haan (1970, Corollary 2.5.1, p. 90)

$$f(G(u)) \sim u^2 \Big/ \int_{G(u)}^B (1 - F(t))\, dt \sim u \Big/ \left\{ G\left(\frac{u}{e}\right) - G(u) \right\} \sim u/g(1/u),$$

as $u \to 0$.

This last result can be deduced from Lemma 3(ii). It implies that $-uG'(u) = u/f(G(u)) \sim g(1/u)$ is slowly varying as $u \to 0$. A close look at the proof of Lemma 6 shows that its conclusion holds also under (H7). It follows that the result of Theorem 1 is also true under (H1), (H2), and (H7).

REMARK 3. If we do not assume f to exist, then the result of Theorem 1 may not hold. It suffices to assume that there exists $(c, d) \subset (A, B)$ with $P(X_i \in (c, d)) = 0$, and to take $a_n \to 0$ to obtain a counterexample.

REMARK 4. By Lemma 3(ii), $g(1/u) \sim G(u/e) - G(u) > 0$, as $u \to 0$. It follows that

$$g\left(\frac{1}{u}\right) = o\left(\int_1^{1/u} \frac{g(t)}{t}\, dt\right) \quad \text{as } u \to 0.$$

Hence, if (H2) is satisfied, then $a_n = G(1/ne) - G(1/n) = o(b_n) = o(G(1/n))$ as $n \to \infty$.

PROOF OF THEOREM 2. The domain of attraction of the Fréchet Φ_a distribution is characterized by:

LEMMA 11. *Let $X_1, X_2, \dots$ be an i.i.d. sequence with distribution function $F(x) = P(X_1 \le x)$. Put $G(x) = \inf\{x; 1 - F(x) \le u\}$. Then, $X_{n,n}$ belongs to the domain of attraction of the Fréchet $\Phi_a(\cdot)$ distribution if and only if one of the following equivalent conditions holds*:

(i) *For any $x > 0$, we have*

$$\lim_{t \uparrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-a}.$$

(ii) *For any $x > 0$, we have*

$$\lim_{u \downarrow 0} \frac{G(ux)}{G(u)} = x^{-1/a}$$

(iii) *There exists functions $c(\cdot)$ and $\varepsilon(\cdot)$ such that*

$$\lim_{u \downarrow 0} c(u) = c \in (0, \infty), \qquad \lim_{t \uparrow \infty} \varepsilon(t) = 0,$$

and

$$G(u) = u^{-1/a} c(u) \exp\left(\int_1^{1/u} \frac{\varepsilon(t)}{t}\, dt\right).$$

PROOF. See Gnedenko (1943) and De Haan (1970, Theorems 1.2.2 and 2.3.1, pp. 19 and 72).

By Lemmas 1 and 11, we get easily:

LEMMA 12. *Under* (H4), *for any fixed $N \ge 1$, the joint limiting distribution of $\{b_n^{-1} S_{n-j}^{(n)}, 1 \le j \le N\}$ is the distribution of*

$$\left\{\left(\sum_{i=1}^{j} \omega_i\right)^{-1/a} - \left(\sum_{i=1}^{j+1} \omega_i\right)^{-1/a}, 1 \le j \le N\right\}.$$

PROOF. We proceed as in the proof of Lemma 4.

LEMMA 13. *For any $x > 0$ and $\varepsilon > 0$, there exists an $N \geq 1$ such that*

$$\limsup_{n\to\infty} P\left(\bigcup_{j=N+1}^{n-1}\left\{b_n^{-1}S_{n-j}^{(n)} > x\right\}\right) < \varepsilon.$$

PROOF. Without loss of generality, assume that $A = 1$. By Lemma 8, for $n \geq n_0$, we have, with probability larger than $1 - \varepsilon/2$,

$$S_{n-j}^{(n)} \leq G\left(\frac{j - \alpha\sqrt{j}\log\log j}{n}\right) - G\left(\frac{j + \alpha\sqrt{j}\log\log j}{n}\right)$$
$$\leq G\left(\frac{j - \alpha\sqrt{j}\log\log j}{n}\right), \qquad M + 1 \leq j \leq n - 1.$$

By choosing M large enough, we get, for some $\beta > 0$,

$$S_{n-j}^{(n)} \leq G\left(\frac{j}{n}(1 - \beta)\right).$$

Hence, by Lemma 11, the result will be proved if we show that

$$\text{(D)} \qquad \lim_{M\to\infty} \sup_{M+1\leq j\leq n} \frac{G(j/n(1-\beta))}{G(1/n)} = 0.$$

Note that $G(u) = u^{-1/a}H(u)$, where $H(\cdot)$ is slowly varying at zero. It follows that (D) is equivalent to

$$\lim_{M\to\infty} \sup_{M+1\leq j\leq n} \left\{\frac{1}{j^{1/a}} \frac{H(j/n(1-\beta))}{H(1/n)}\right\}.$$

This, in turn, follows from Lemma 9.

The proof of Theorem 2 follows from Lemmas 12 and 13 in a straightforward way.

PROOF OF THEOREM 3. The domain of attraction of the Weibull Ψ_a distribution is characterized by:

LEMMA 15. *Let $X_1, X_2, \ldots$ be an i.i.d. sequence with distribution function $F(x) = P(X_1 \leq x)$. Put $G(x) = \inf\{x; 1 - F(x) \leq u\}$. Then $X_{n,n}$ belongs to the domain of attraction of the Weibull Ψ_a distribution if and only if $B = \sup\{x; F(x) < 1\} < \infty$, and if one of the following equivalent conditions holds*:

(i) *For any $x > 0$, we have*

$$\lim_{t\downarrow 0} \frac{1 - F(B - tx)}{1 - F(B - t)} = x^a.$$

(ii) *For any $x > 0$, we have*

$$\lim_{u \downarrow 0} \frac{B - G(ux)}{B - G(u)} = x^{1/a}.$$

(iii) *There exists functions $c(\cdot)$ and $\varepsilon(\cdot)$ such that*

$$\lim_{u \downarrow 0} c(u) = c \in (0, \infty), \qquad \lim_{t \uparrow \infty} \varepsilon(t) = 0,$$

and

$$G(u) = B - u^{1/a} c(u) \exp\left(\int_1^{1/u} \frac{\varepsilon(t)}{t} \, dt \right).$$

PROOF. See Gnedenko (1943) and De Haan (1970, Theorems 1.2.2 and 2.3.2, pp. 19 and 75).

The proof of Theorem 3 is essentially the same as the proof of Theorem 2, with Lemma 15 replacing Lemma 11. Details will be omitted.

PROOF OF THEOREM 4. Theorem 4 says that the spacings in the upper and lower tail of the distribution are asymptotically independent. This follows easily from the fact that, for the uniform distribution, these spacings are stochastically equivalent to $\{\omega_i/n, 2 \le i \le N\}$ and $\{\omega_{n-i}/n, 1 \le i < N + 1\}$ (see Lemma 3), where the $\{\omega_i, 1 \le i \le n + 1\}$ are i.i.d. exponentially distributed random variables. The result follows.

REMARK 5. Lemmas 4 and 13 can be considered as corollaries of the representation theorems of Hall (1978).

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