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# NORMAL AND STABLE CONVERGENCE OF INTEGRAL FUNCTIONS OF THE EMPIRICAL DISTRIBUTION FUNCTION

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We prove general invariance principles for integral functions of the empirical process. As corollaries we derive probabilistic proofs of the sufficiency criteria for a distribution to belong to the domain of attraction of the normal and stable laws with index $0 < \alpha < 2$. In the process we obtain equivalent statements of these criteria in terms of the tail behaviour of the underlying quantile function. We also give a representation of any stable random variable with index $0 < \alpha < 2$ in terms of a linear combination of two independent and identically distributed Poisson integrals. The role of a fixed number of extreme terms is exactly determined.

**0. Introduction.** Let $U_1, \ldots, U_n$ denote $n$ independent uniform $(0,1)$ random variables (rv) and let $G_n$ denote the right-continuous empirical distribution function based on these rv. Also let $F$ denote an arbitrary right-continuous distribution function with left-continuous inverse distribution or quantile function $Q$.

In this paper we will study invariance theorems for several integral functionals of $G_n$. For example, we will consider the integral functionals

$$I_n(s) = \int_0^s (G_n(u) - u)\, dQ(u),$$

$$J_n(s) = \int_s^1 (G_n(u) - u)\, dQ(u),$$

defined for $0 \le s \le 1$. (These functionals play a central role in our unified weak and strong approximation theory for empirical total time on test, mean residual life, empirical Lorenz and Goldie concentration processes which are of interest in reliability, and economic concentration theories. The reader is referred to M. Csörgő, S. Csörgő, Horváth, and Mason (CsCsHM, 1985).)

The functionals $I_n$ and $J_n$ are well defined, that is finite as long as

$$\int_{-\infty}^{\infty} |x|\, dF(x) = \int_0^1 |Q(u)|\, du < \infty. \tag{0.1}$$

In particular, (0.1) holds whenever $F$ satisfies the normal convergence criterion or

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is in the domain of attraction of a stable law with index $1 < \alpha < 2$. We refer to Rényi (1970) for proofs of these facts, and to the appendix for the statements of the normal convergence criterion and the necessary and sufficient conditions for a distribution to be in the domain of attraction of a stable law.

Our invariance theorems will be stated and proved using a weak approximation approach on an appropriate probability space constructed in the next section, where all the necessary notions are defined as well.

We will show in Section 2 that whenever $F$ satisfies the normal convergence criterion, then on our probability space the difference between normalized versions of $I_n$ and similar functionals of a sequence of Brownian bridges converges uniformly on $[0, 1]$ in probability to zero. This result may be viewed as a functional generalization of the sufficiency part of the classical Feller–Lévy–Hinčin result concerning when a distribution $F$ belongs to the domain of attraction of a normal law. Indeed, when setting $s = 1$, we obtain as a consequence a probabilistic proof of the sufficiency of the normal convergence criterion.

In Section 3 we prove that when $Q \leq 0$ (respectively $Q \geq 0$) and $F$ is in the domain of attraction of a stable law with index $1 < \alpha < 2$, then an appropriately time transformed and normalized version of $I_n$ (respectively that of $J_n$) converges uniformly on $[0, \infty)$ on the same constructed probability space as above to an integral function of a standard Poisson process $N^{(1)}$ (respectively to $N^{(2)}$), where $N^{(1)}$ and $N^{(2)}$ are independent processes. When $F$ is in the domain of attraction of a stable law with index $0 < \alpha \leq 1$, we obtain analogous results for integral functionals of $G_n$ that are similar to $I_n$ and $J_n$. These results lead to a probabilistic proof of the sufficiency of the classical criteria for a distribution to be in the domain of attraction of a stable law of index $0 < \alpha < 2$. We also identify the characteristic function of the obtained stable limit and give a general representation of all stable laws with index $0 < \alpha < 2$ as a linear combination of two integrals of independent Poisson processes. Finally, in Section 4 we investigate the effect of deleting a fixed number of upper and a fixed number of lower extreme values from the sum of variables in the domain of attraction of a stable law of index $0 < \alpha < 2$. We determine the limiting distribution of these truncated sums. The similar problem when a finite number of extremes determined by ordering the *moduli* of the terms in the sum is removed has been treated extensively in the literature.

In the process of proving these above mentioned results we derive alternative statements of the classic normal convergence and stable convergence criteria in terms of the tail behaviour of the quantile function $Q$ of $F$. These criteria have so far been stated in terms of the tail behaviour of $F$ only. The present quantile versions of the criteria should be of independent interest.

We will use the following convention concerning the integral sign: When $-\infty < a < b < \infty$ and $l$ is a left-continuous and $r$ is a right-continuous function then

$$\int_a^b r\,dl = \int_{[a,b)} r\,dl \quad \text{and} \quad \int_a^b l\,dr = \int_{(a,b]} l\,dr,$$

whenever these integrals make sense as Lebesgue–Stieltjes integrals. In this case the usual integration by parts formula

$$\int_a^b r\,dl + \int_a^b l\,dr = l(b)r(b) - l(a)r(a)$$

is valid. If $l$ or $r$ are not finite at least one of the endpoints or at least one of the endpoints themselves are not finite, then the corresponding integrals are meant as improper integrals.

**1. Preliminaries. Construction of the probability space upon which the theorems in this paper are valid.** Let $\{W^{(1)}(t);\ 0 \le t < \infty\}$ and $\{W^{(2)}(t);\ 0 \le t < \infty\}$ be two independent Wiener processes defined on the same probability space $(\Omega, \mathscr{A}, P)$. By means of $W^{(1)}$ and the Komlós, Major, and Tusnády (KMT) technique we construct a sequence of independent exponential rv with mean 1, say $Y_1^{(1)}, Y_2^{(1)}, \dots$, and in the same way by means of $W^{(2)}$ we construct a sequence of independent exponential rv with mean 1, say $Y_1^{(2)}, Y_2^{(2)}, \dots$. The thus constructed two sequences of i.i.d. rv $Y_1^{(1)}, Y_2^{(1)}, \dots$ and $Y_1^{(2)}, Y_2^{(2)}, \dots$ also live on $(\Omega, \mathscr{A}, P)$ and, since $W^{(1)}$ and $W^{(2)}$ are independent processes, they are also independent as sequences by construction.

For $i = 1, 2, \dots$, and $m = 1, 2, \dots$, we write

$$S_m^{(i)} = \sum_{j=1}^{m} Y_j^{(i)},$$

and set $S_0^{(1)} = S_0^{(2)} = 0$. We introduce two standard left-continuous Poisson processes $N^{(1)}$ and $N^{(2)}$ with time domain $[0, \infty)$, defined for $i = 1, 2$ and $0 \le s < \infty$ by

$$N^{(i)}(s) := \sum_{j=1}^{\infty} I\big(S_j^{(i)} < s\big),$$

and observe that $N^{(1)}$ and $N^{(2)}$ are independent processes on $(\Omega, \mathscr{A}, P)$ by construction.

For each integer $n \ge 2$, let

$$Y_j(n) = \begin{cases} Y_j^{(1)} & \text{for } j = 1, \dots, [n/2], \\ Y_{n+2-j}^{(2)} & \text{for } j = [n/2] + 1, \dots, n+1. \end{cases}$$

For $n \ge 2$ and $m = 1, \dots, n+1$ we write

$$S_m(n) = \sum_{j=1}^{m} Y_j(n),$$

and from now on, for the sake of notational simplicity, we will write $S_m = S_m(n)$ and $Y_j = Y_j(n)$ and set $S_0 = 0$.

For each integer $n \ge 2$ we define the stochastic process

$$W_n(s) = \begin{cases} W^{(1)}(s) & \text{for } 0 \le s \le [n/2], \\ W^{(1)}\left(\left[\frac{n}{2}\right]\right) + W^{(2)}\left(n + 1 - \left[\frac{n}{2}\right]\right) - W^{(2)}(n+1-s) & \\ & \text{for } [n/2] < s \le n+1. \end{cases}$$

Elementary calculations show that for each choice of $0 \le s \le t \le n+1$ we have $EW_n(s) = 0$, $EW_n(s)W_n(t) = s$. Thus $W_n$ is a standard Wiener process on $[0, n+1]$.

For each $n \ge 2$, we construct $n$ Uniform (0, 1) order statistics as follows:

$$U_{k,n} = S_k/S_{n+1} \quad \text{for } k = 1, \dots, n.$$

It is well known that $(U_{1,n}, U_{2,n}, \dots, U_{n,n})$ have the same joint distribution as the order statistics of $n$ independent Uniform (0, 1) rv. *Let* $(U_1, \dots, U_n)$ *denote a random permutation of* $(U_{1,n}, \dots, U_{n,n})$ *chosen with probability* $1/n!$. It is easily checked that $(U_1, \dots, U_n)$ have the same joint distribution as $n$ independent Uniform (0, 1) rv. Let $G_n$ denote *the empirical distribution function* based on *these* $U_1, \dots, U_n$ rv, defined by

$$G_n(s) := \sum_{i=1}^{n} I(U_i \le s)/n, \qquad 0 \le s \le 1,$$

and let $\alpha_n$ be the *corresponding empirical process* defined by

$$\alpha_n(s) := n^{1/2}(G_n(s) - s), \qquad 0 \le s \le 1.$$

Finally, we introduce a sequence of Brownian bridges $\{B_n\}$, defined for each $n \ge 2$ by

$$\{B_n(s); 0 \le s \le 1\} := n^{-1/2}\{W_n(ns) - sW_n(n); 0 \le s \le 1\},$$

and for each $n \ge 3$ a truncated version of $B_n$ given by

$$\overline{B}_n(s) = \begin{cases} B_n(s) & \text{for } 1/n \le s < 1 - 1/n, \\ 0 & \text{elsewhere.} \end{cases}$$

We observe, that all the stochastic processes and rv just constructed live on the initial probability space on which $W^{(1)}$ and $W^{(2)}$ are defined. *In the sequel, whenever we refer by name or symbol to any of these processes or rv, it will be understood that we are talking about the specially constructed versions just given.* In particular, we note that the here given empirical process $\alpha_n$ agrees in distribution with the classical uniform empirical process only for each $n \ge 2$. Nevertheless, when approximating our present $\alpha_n$ by the above introduced sequence of Brownian bridges $\{B_n\}$, the results of CsCsHM (1986) for example, concerning the classical uniform empirical process defined in terms of any sequence of independent Uniform (0, 1) rv hold true also on our present probability space for our present $\alpha_n$ and $B_n$. In this paper, we will be using only *one* of the many asymptotic properties of the above construction, stated as Theorem 1.1 below. For further details on the wide ranging ramifications and applications of this construction we refer to CsCsHM (1986).

Let $\mathscr{L}$ denote a class of left-continuous functions defined on (0, 1) such that each $l \in \mathscr{L}$ can be written as $l = l_1 - l_2$, where $l_1$ and $l_2$ are nondecreasing left-continuous functions defined on (0, 1). Also, let $L$ denote a strictly positive nonincreasing function defined on $(0, \frac{1}{2}]$ slowly varying at zero. We define also the following function on $(0, \frac{1}{2}]$: for $\delta \in (0, \frac{1}{2}]$ set

$$(1.1)\quad N(\delta) := \sup_{l \in \mathscr{L}} \sup_{0 < s \le \delta} s^{1/2}\{|l_1(s)| + |l_2(s)| + |l_1(1-s)| + |l_2(1-s)|\}/L(s).$$

We have, in terms of our $\alpha_n$ and $B_n$ as above,

THEOREM 1.1 (Corollary 3.2 of CsCsHM, 1986). *Whenever $\mathscr{L}$ and L are such that*

$$(1.2)\quad \lim_{\delta \downarrow 0} N(\delta) = 0$$

*then, as $n \to \infty$,*

$$(1.3)\quad \sup_{l \in \mathscr{L}} \left| \int_0^1 l(s)\, d\alpha_n(s) - \int_0^1 l(s)\, d\overline{B}_n(s) \right| \Big/ L(1/n) = o_P(1).$$

**2. Uniform convergence of integral functionals of the empirical process to an integral functional of a Brownian bridge.** Towards stating the main result of this section, let $F$ be a nondegenerate right-continuous distribution function with $Q$ as its left-continuous quantile function defined by

$$(2.1)\quad \begin{aligned} Q(s) &:= \inf\{x\colon F(x) \ge s\}, \qquad 0 < s < 1, \\ Q(0) &:= Q(0+), \qquad Q(1) := Q(1-). \end{aligned}$$

We define the following two functions on $(0, \frac{1}{2}]$: For any $0 < s \le \frac{1}{2}$ let (cf. the appendix)

$$S_4^2(s) = \int_s^{1-s}\int_s^{1-s} (u \wedge v - uv)\, dQ(u)\, dQ(v),$$

and choose any $0 < \gamma < \frac{1}{2}$ such that $0 < S_4^2(\gamma) < \infty$ (this is possible since $F$ is assumed to be nondegenerate) and set

$$(2.2)\quad L(s) = \begin{cases} S_4(\gamma) & \text{for } \gamma < s \le \frac{1}{2}, \\ S_4(s) & \text{for } 0 < s \le \gamma. \end{cases}$$

THEOREM 2.1. *Whenever $F$ is a nondegenerate distribution function that satisfies the normal convergence criterion* [*cf.* (A.1) *and* (A.2) *of the appendix*], *then, as $n \to \infty$,*

$$(2.3)\quad \Delta_n := \sup_{0 \le t \le 1} \left| \int_0^t \alpha_n(s)\, dQ(s) - \int_0^t \overline{B}_n(s)\, dQ(s) \right| \Big/ L(1/n) = o_P(1),$$

*where L is as in* (2.2).

PROOF. Observe that integration by parts gives for each $0 < t < 1$ [cf. the introduction]

$$\int_0^t \alpha_n(s)\, dQ(s) = -\int_0^t Q(s)\, d\alpha_n(s) + Q(t)\alpha_n(t),$$

which is equal to

$$\int_0^1 l_t(s)\, d\alpha_n(s),$$

where

$$l_t(s) = (Q(t) - Q(s))I(s \le t).$$

Similarly, we have for each $1/n \le t < 1 - 1/n$

$$\int_0^t \overline{B}_n(s)\, dQ(s) = \int_0^1 l_t(s)\, d\overline{B}_n(s).$$

Hence we have

$$\Delta_n = \sup_{0 \le t \le 1} \left| \int_0^1 l_t(s)\, d\alpha_n(s) - \int_0^1 l_t(s)\, d\overline{B}_n(s) \right| \Big/ L(1/n).$$

Let $\mathscr{L} = \{l_t\colon 0 < t < 1\}$. We observe that each $l_t(\cdot)$ is a nonincreasing left-continuous function defined on (0,1). Also, by applying some elementary bounds based on monotonicity of $Q$, we have for each fixed $0 < \delta < \frac{1}{2}$

$$\begin{aligned} N(\delta) &:= \sup_{0<t<1} \sup_{0<s\le\delta} s^{1/2}\{|l_t(s)| + |l_t(1-s)|\}/L(s) \\ &\le 4 \sup_{0<s\le\delta} s^{1/2}\{|Q(s)| + |Q(1-s)|\}/L(s). \end{aligned} \tag{2.4}$$

Proposition A.2 in the appendix implies that $L$ is slowly varying at zero, and $N(\delta)$ of (2.4) converges to zero as $\delta \downarrow 0$. Hence by Theorem 1.1 we have

$$\Delta_n = o_P(1), \qquad n \to \infty. \tag{2.5}$$

Thus the proof of the theorem is complete. □

COROLLARY 2.1 (Lemma 3.2 of CsCsHM, 1985). *Whenever $F$ is a distribution function that has a finite second moment*

$$D_n := \sup_{0 \le t \le 1} \left| \int_0^t \alpha_n(s)\, dQ(s) - \int_0^t B_n(s)\, dQ(s) \right| = o_P(1) \tag{2.6}$$

*as $n \to \infty$.*

PROOF. We will assume that $F$ is nondegenerate, otherwise $D_n$ of (2.6) is identically equal to zero. Since $F$ is a nondegenerate distribution function which has a finite second moment, $L^2(1/n)$ [cf. (2.2)] converges, as $n \to \infty$, to

$$\begin{aligned} 0 &< \int_0^1 \int_0^1 (u \wedge v - uv)\, dQ(u)\, dQ(v) \\ &= \int_{-\infty}^{\infty} x^2\, dF(x) - \left( \int_{-\infty}^{\infty} x\, dF(x) \right)^2 := \sigma^2 < \infty, \end{aligned}$$

the variance $\sigma^2$ of $F$, and since $F$ trivially satisfies the normal convergence criterion, we have immediately from Theorem 2.1 that

$$\sup_{0\le t\le 1}\left|\int_0^t \alpha_n(s)\,dQ(s) - \int_0^t \overline{B}_n(s)\,dQ(s)\right| = o_P(1). \tag{2.7}$$

The latter in turn implies that in order to complete the proof of (2.6) it is enough to show that

$$D_n^{(1)} := \sup_{0\le t\le 1/n}\left|\int_0^t B_n(s)\,dQ(s)\right| = o_P(1), \tag{2.8}$$

and

$$D_n^{(2)} := \sup_{1-1/n\le t\le 1}\left|\int_{1-1/n}^t B_n(s)\,dQ(s)\right| = o_P(1), \qquad n\to\infty. \tag{2.9}$$

The proofs of (2.8) and (2.9) follow by arguments based on the Birnbaum–Marshall inequality. For details we refer to CsCsHM (1985). This completes the proof of Corollary 2.1. □

Corollary 2.1 has wide ranging applications [cf. e.g., CsCsHM, 1985]. The reason for giving its above short proof is to underline the fact that, unlike in our quoted original proof, here only the Gaussian $D_n^{(1)}$ and $D_n^{(2)}$ tail rv have to be estimated, due to Theorem 2.1.

Our next corollary will illustrate the role of the, at zero, slowly varying function $L$ in Theorem 2.1. It is an easy sufficiency proof of the normal convergence criterion based on the latter theorem.

COROLLARY 2.2. *Let $X_1, X_2, \ldots$ denote a sequence of i.i.d. rv with nondegenerate distribution function $F$. Whenever $F$ satisfies the normal convergence criterion, there exist constants $A_n$ and $C_n$ such that*

$$A_n\left(\sum_{i=1}^n X_i - C_n\right) \to_{\mathscr{D}} N(0,1). \tag{2.10}$$

PROOF. Let $Q$ denote the quantile function of $F$. It is well known that

$$\begin{aligned} n^{1/2}\left(\sum_{i=1}^n X_i/n - \mu\right) &=_{\mathscr{D}} n^{1/2}\left(\sum_{i=1}^n Q(U_i)/n - \mu\right) \\ &= -\int_0^1 \alpha_n(s)\,dQ(s), \end{aligned} \tag{2.11}$$

where $\mu$ denotes the mean of $F$. Theorem 2.1 implies that, as $n\to\infty$,

$$\left|\int_0^1 \alpha_n(s)\,dQ(s) - \int_{1/n}^{1-1/n} B_n(s)\,dQ(s)\right| \Big/ L(1/n) = o_P(1). \tag{2.12}$$

Since

$$\int_{1/n}^{1-1/n} B_n(s)\,dQ(s)/L(1/n)$$

is a $N(0,1)$ rv, (2.11) and (2.12) imply (2.10) with $A_n = 1/(n^{1/2}L(1/n))$ and $C_n = n\mu$.

For an alternative probabilistic proof of the normal convergence criterion we refer to Root and Rubin (1973), where they consider the general nonidentically distributed case and base their proof on the Skorohod embedding.

**3. Stable laws and integral functionals of Poisson processes.** Let $Y_1^{(1)}, Y_2^{(1)}, \dots$ and $Y_1^{(2)}, Y_2^{(2)}, \dots$ be the two independent sequences of independent exponential rv constructed in Section 1. Let $\{N^{(i)}(s);\ 0 \le s < \infty\}$ $(i = 1, 2)$ be the two independent standard left-continuous Poisson processes of the same section. For each integer $n \ge 2$ let $U_{1,n}, \dots, U_{n,n}$ be the Uniform $(0,1)$ order statistics as constructed in Section 1. Let $G_n^{(1)}$ be the left-continuous empirical distribution function

$$G_n^{(1)}(u) := \sum_{i=1}^{n} I(U_{i,n} < u)/n, \qquad 0 \le u \le 1,$$

and for technical reasons we introduce also

$$G_n^{(2)}(u) := \sum_{i=1}^{n} I(1 - U_{n+1-i,n} < u)/n, \qquad 0 \le u \le 1.$$

For $i = 1, 2$ let

$$\Gamma_n^{(i)}(s) := \begin{cases} nG_n^{(i)}(s/n) & \text{for } 0 \le s \le n, \\ 0 & \text{for } s > n. \end{cases}$$

Let $\mathscr{K}_1$ denote the class of all quantile functions $Q$ defined on $[0,1)$ as in (2.1), such that

(K.1) $$Q \ge 0,$$

(K.2) $Q(1-u) = u^{-1/\alpha}L(u)$ for some $0 < \alpha < 2$ and function $L$ which is slowly varying near zero, i.e., for any $t > 0$,

(3.1) $$\lim_{u \downarrow 0} L(tu)/L(u) = 1.$$

Also, let $\mathscr{K}_2$ denote the class of all quantile functions $Q$, defined on $[0,1)$ as in (2.1), such that (K.1) is satisfied, i.e., $Q \ge 0$, but instead of (K.2) we have

(K.3) $Q(1-u) = o(u^{-1/\alpha}L(u))$ for some $0 < \alpha < 2$ and function $L$ which is slowly varying near zero.

For any $Q \in \mathscr{K}_1 \cup \mathscr{K}_2$ we write

(3.2) $$H(u) := -Q(1-u),$$

and set

$$d\mu_{\alpha,n}(s) := n^{-1/\alpha}\, dH(s/n)/L(1/n).$$

Note that $H$ is right-continuous.

When $1 < \alpha < 2$, we write for $i = 1, 2$

$$\Gamma_{n,\alpha}^{(i)}(t) := \begin{cases} \int_0^t \left(\Gamma_n^{(i)}(s) - s\right) d\mu_{\alpha,n}(s) & \text{if } 0 \le t \le n, \\ \Gamma_{n,\alpha}^{(i)}(n) & \text{if } t > n, \end{cases}$$

$$M_{n,\alpha}^{(i)}(t) := \int_0^t \left(N^{(i)}(s) - s\right) d\mu_{\alpha,n}(s) \qquad \text{for } t \ge 0,$$

$$M_{\alpha}^{(i)}(t) := \alpha^{-1} \int_0^t \left(N^{(i)}(s) - s\right) s^{-1-1/\alpha}\, ds \quad \text{for } t \ge 0.$$

Let also

$$U_{1,n}^{(i)} := Y_1^{(i)} / S_{n+1} \quad \text{for } i = 1, 2 \quad \text{and} \quad n \ge 2,$$

where $S_{n+1} = S_{n+1}(n)$ is as in Section 1.

When $\alpha = 1$, we write for $i = 1, 2$

$$\Gamma_{n,1}^{(i)}(t) := \begin{cases} 0 & \text{if } 0 \le t \le nU_{1,n}^{(i)}, \\ \int_{nU_{1,n}^{(i)}}^t \left(\Gamma_n^{(i)}(s) - s\right) d\mu_{1,n}(s) & \text{if } nU_{1,n}^{(i)} < t \le n, \\ \Gamma_{n,1}^{(i)}(n) & \text{if } t > n, \end{cases}$$

$$M_{n,1}^{(i)}(t) := \begin{cases} \int_{nU_{1,n}^{(i)}}^t \left(N^{(i)}(s) - s\right) d\mu_{1,n}(s) & \text{if } t > nU_{1,n}^{(i)}, \\ 0 & \text{if } 0 \le t \le nU_{1,n}^{(i)}, \end{cases}$$

and

$$M_1^{(i)}(t) := \begin{cases} \int_{Y_1^{(i)}}^t \left(N^{(i)}(s) - s\right) s^{-2}\, ds & \text{if } t > Y_1^{(i)}, \\ 0 & \text{if } 0 \le t \le Y_1^{(i)}. \end{cases}$$

Finally, when $0 < \alpha < 1$, we write for $i = 1, 2$

$$\Gamma_{n,\alpha}^{(i)}(t) := \begin{cases} \int_0^t \Gamma_n^{(i)}(s)\, d\mu_{\alpha,n}(s) & \text{if } 0 \le t \le n, \\ \Gamma_{n,\alpha}^{(i)}(n) & \text{if } t > n, \end{cases}$$

$$M_{n,\alpha}^{(i)}(t) := \int_0^t N^{(i)}(s)\, d\mu_{\alpha,n}(s) \quad \text{for } t \ge 0,$$

and

$$M_{\alpha}^{(i)}(t) := \alpha^{-1} \int_0^t N^{(i)}(s) s^{-1-1/\alpha}\, ds \quad \text{for } t \ge 0.$$

We have

THEOREM 3.1. *Whenever $Q \in \mathscr{K}_1$, then with $0 < \alpha < 2$ and $L$ as in condition* (K.2)

$$\sup_{0 \le t < \infty} \left| \Gamma_{n,\alpha}^{(i)}(t) - M_{\alpha}^{(i)}(t) \right| = o_P(1) \quad \text{as } n \to \infty \quad \text{for } i = 1, 2. \tag{3.3}$$

*Whenever* $Q \in \mathscr{K}_2$, *then with* $0 < \alpha < 2$ *and* $L$ *as in condition* (K.3)

$$\sup_{0 \le t < \infty} \left|\Gamma_{n,\alpha}^{(i)}(t)\right| = o_P(1) \quad \text{as } n \to \infty \quad \text{for } i = 1, 2.$$

PROOF. First we consider the first statement in Theorem 3.1. For any $0 < T < \infty$ we have

$$\begin{aligned}
&\sup_{0 \le t < \infty} \left|\Gamma_{n,\alpha}^{(i)}(t) - M_\alpha^{(i)}(t)\right| \\
(3.4) \qquad &\le \sup_{0 \le t \le T} \left|M_{n,\alpha}^{(i)}(t) - M_\alpha^{(i)}(t)\right| + \sup_{0 \le t \le T} \left|\Gamma_{n,\alpha}^{(i)}(t) - M_{n,\alpha}^{(i)}(t)\right| \\
&\quad + \sup_{T < t < \infty} \left|\Gamma_{n,\alpha}^{(i)}(t) - \Gamma_{n,\alpha}^{(i)}(T)\right| + \sup_{T < t < \infty} \left|M_\alpha^{(i)}(t) - M_\alpha^{(i)}(T)\right|.
\end{aligned}$$

Lemmas 3.3, 3.4, and 3.5 given below imply that the right side of the inequality of (3.4) is equal to

$$o_P(1) + O_P(1)\left(T^{\beta - 1/\alpha} + o(1)\right)$$

for some $0 < \beta < 1/\alpha$, where the $O_P(1)$ terms does not depend on $T$. Since $T$ can be chosen arbitrarily large and $\beta - 1/\alpha < 0$, the said lemmas imply (3.3).

In order to complete the proof of the first statement of Theorem 3.1, we will now establish the aforementioned lemmas together with Lemmas 3.1 and 3.2, preliminary to them.

LEMMA 3.1. *Assume* $Q \in \mathscr{K}_1$. *For every* $0 < T_1 < T_2 < \infty$, $0 < \alpha < 2$, *and* $-\infty < \beta < \infty$, *where* $\beta \ne 1/\alpha$

$$(3.5) \qquad \int_s^t u^\beta \, d\mu_{\alpha,n}(u) \to \frac{1}{\alpha}\left(s^{\beta - 1/\alpha} - t^{\beta - 1/\alpha}\right)/(1/\alpha - \beta)$$

*uniformly in* $s$ *and* $t$ *in* $[T_1, T_2]$ *as* $n \to \infty$, *and* $T_1$ *can be chosen to be zero if* $\beta - 1/\alpha > 0$.

PROOF. Integrating by parts, the left-hand side expression of (3.5) is equal to

$$(3.6) \qquad n^{-1/\alpha} H(u/n) u^\beta / L(1/n)\Big|_s^t - \beta \int_s^t u^{\beta - 1} n^{-1/\alpha} H(u/n)\, du / L(1/n).$$

We will use the fact that for every $0 < T_1 < T_2 < \infty$ and $-\infty < \gamma < \infty$ [cf. item 4 on pages 21 to 22 of de Haan (1970)]

$$(3.7) \qquad \sup_{T_1 \le u \le T_2} \left|u^\gamma L(u/n)/L(1/n) - u^\gamma\right| = o(1), \qquad n \to \infty,$$

where $T_1$ can be chosen to be zero if $\gamma > 0$.

Applying (3.7) we get

$$\begin{aligned}
&\sup_{T_1 \le u \le T_2} \left|u^{\beta - 1/\alpha} + n^{-1/\alpha} u^\beta H(u/n)/L(1/n)\right| \\
(3.8) \qquad &= \sup_{T_1 \le u \le T_2} \left|u^{\beta - 1/\alpha} - u^{\beta - 1/\alpha} L(u/n)/L(1/n)\right| = o(1),
\end{aligned}$$

and for every $0 < \varepsilon < |\beta - 1/\alpha|/2$ and $T_1 \le s \le t \le T_2$

$$\begin{aligned}\left|\int_s^t u^{\beta-1}n^{-1/\alpha}H(u/n)\,du/L(1/n) + \int_s^t u^{\beta-1-1/\alpha}\,du\right| \\ \le \left(\int_s^t u^{\beta-1-(1/\alpha)-\varepsilon}\,du\right)\sup_{T_1\le u\le T_2}|u^{\varepsilon} - u^{\varepsilon}L(u/n)/L(1/n)| \\ = \left|T_2^{\beta-(1/\alpha)-\varepsilon} - T_1^{\beta-(1/\alpha)-\varepsilon}\right|o(1).\end{aligned} \tag{3.9}$$

Now (3.8) and (3.9) imply (3.5).

LEMMA 3.2. *Assume* $Q \in \mathscr{K}_1$. *For each* $0 < \alpha < 2$, $0 < \beta < 1/\alpha$, *and* $0 < T < \infty$

$$\lim_{n\to\infty}\int_T^n s^{\beta}\,d\mu_{\alpha,n}(s) = \alpha^{-1}T^{\beta-1/\alpha}/(1/\alpha - \beta). \tag{3.10}$$

PROOF. On applying integration by parts, the left-hand side of expression (3.10) is equal to the sum of

$$\lim_{n\to\infty}\left\{T^{\beta-1/\alpha}L(T/n)/L(1/n) + H(1)n^{\beta-1/\alpha}/L(1/n)\right\} \tag{3.11}$$

and

$$-\lim_{n\to\infty}\beta n^{\beta-1/\alpha}\frac{\int_{T/n}^1 H(u)u^{\beta-1}\,du}{T^{\beta-1/\alpha}L(T/n)}\frac{L(T/n)}{L(1/n)}T^{\beta-1/\alpha}. \tag{3.12}$$

Hence by (3.1) and the assumption that $\beta - 1/\alpha < 0$, the expression of (3.11) is equal to $T^{\beta-1/\alpha}$, and by the same two conditions

$$\int_{T/n}^1 L(u)u^{\beta-1-1/\alpha}\,du/\left((T/n)^{\beta-1/\alpha}L(T/n)\right) \to 1/(1/\alpha - \beta) \tag{3.13}$$

[cf. Remark 1.2.1 on pages 18–19 of de Haan (1970)]. Consequently, on account of (3.1) and (3.13), expression (3.12) equals $\beta T^{\beta-1/\alpha}/(1/\alpha - \beta)$. This also completes the proof of Lemma 3.2.

LEMMA 3.3. *Assume* $Q \in \mathscr{K}_1$. *For each* $0 < \alpha < 2$ *and* $0 < T < \infty$

$$\sup_{0\le t\le T}\left|M_{n,\alpha}^{(i)}(t) - M_{\alpha}^{(i)}(t)\right| = o_P(1), \qquad n\to\infty, \tag{3.14}$$

*for* $i = 1, 2$.

PROOF. We will only supply a proof for the case when $1 < \alpha < 2$. The proofs for the other two cases of $\alpha = 1$ and $0 < \alpha < 1$ are almost the same. First we observe that the left side of (3.14) is less than or equal to the sum of

$$\sup_{0\le t\le T}\left|\int_0^t \alpha^{-1}N^{(i)}(s)s^{-1-1/\alpha}\,ds - \int_0^t N^{(i)}(s)\,d\mu_{\alpha,n}(s)\right| \tag{3.15}$$

and

$$\sup_{0\le t\le T}\left|\int_0^t \alpha^{-1}s^{-1/\alpha}\,ds-\int_0^t s\,d\mu_{\alpha,n}(s)\right|. \tag{3.16}$$

Now by setting $\beta = 1$ in Lemma 3.1, we see that expression (3.16) converges to zero. As to (3.15), we note that almost surely the Poisson processes $N^{(i)}$ $(i = 1,2)$ have at most a finite number of jumps in $[0, T]$ at distinct points. Let $0 < b_1 < \cdots < b_m < T$ be the jump points of $N^{(i)}$ in $[0, T)$, and set $b_{m+1} = T$ and $b_0 = 0$. Then expression (3.15) is equal to

$$\sup_{0\leqq t\leqq T}\left|\sum_{b_j\leqq t} N^{(i)}(b_j)\left\{\alpha^{-1}\int_{b_j}^{b_{j+1}\wedge t} s^{-1-1/\alpha}\,ds-\int_{b_j}^{b_{j+1}\wedge t} d\mu_{\alpha,n}(s)\right\}\right|,$$

which by Lemma 3.1 also converges to zero almost surely. Hence we have also (3.14).

LEMMA 3.4. *Assume $Q \in \mathscr{K}_1$. For each $0 < \alpha < 2$ and $0 < T < \infty$*

$$\sup_{0\leqq t\leqq T}\left|\Gamma_{n,\alpha}^{(i)}(t)-M_{n,\alpha}^{(i)}(t)\right| = o_P(1), \qquad n\to\infty, \tag{3.17}$$

*for $i = 1,2$.*

PROOF. Let, for $i = 1,2$,

$$m_i := \text{first integer } k \geqq 0 \text{ such that } S_k^{(i)} \leqq T \text{ and } S_{k+1}^{(i)} > T,$$

where $S_k^{(i)}$ $(i = 1,2)$, $k = 0,1,\dots$, are as in Section 1.

It is easily seen that $m_i < \infty$ almost surely. We note also that almost surely

$$\begin{gathered} S_{m_i}^{(i)} < T, \quad S_{m_i+1}^{(i)} > T, \quad \text{and} \\ 0 = S_0^{(i)} < S_1^{(i)} < \cdots < S_{m_i+1}^{(i)}, \qquad i = 1,2. \end{gathered} \tag{3.18}$$

Let, for $i = 1,2$,

$$m_i(n) := \text{first integer } k \geqq 0 \text{ such that } nS_k^{(i)}/S_{n+1} \leqq T \text{ and } nS_{k+1}^{(i)}/S_{n+1} > T.$$

Again we note that almost surely

$$\begin{gathered} \frac{nS_{m_i(n)}^{(i)}}{S_{n+1}} < T, \quad \frac{nS_{m_i(n)+1}^{(i)}}{S_{n+1}} > T, \quad \text{and} \\ 0 = \frac{S_0^{(i)}}{S_{n+1}} < \frac{nS_1^{(i)}}{S_{n+1}} < \cdots < \frac{nS_{m_i(n)+1}^{(i)}}{S_{n+1}}, \qquad i = 1,2. \end{gathered} \tag{3.19}$$

Let

$$A_n^{(i)} := \{m_i = m_i(n)\} \qquad (i = 1,2),$$

$$B_n^{(1)} := \{nS_j/S_{n+1} = nS_j^{(1)}/S_{n+1}\colon 0 \le j \le m_1 + 1\}$$

and

$$B_n^{(2)} := \{n(S_{n+1} - S_{n+1-j})/S_{n+1} = nS_j^{(2)}/S_{n+1}\colon 0 \le j \le m_2 + 1\}.$$

Then we have for $i = 1,2$ that

$$P\{B_n^{(i)}\} \to 1 \quad \text{as } n \to \infty. \tag{3.20}$$

For $i = 1,2$, $n = 1,2,\dots$, $j = 1,\dots,m_i + 1$ we set

$$a_{j,n}^{(i)} := \min\left(S_j^{(i)}, nS_j^{(i)}/S_{n+1}\right),$$
$$b_{j,n}^{(i)} := \max\left(S_j^{(i)}, nS_j^{(i)}/S_{n+1}\right),$$

and

$$C_n^{(i)} := \left\{a_{j,n}^{(i)} < b_{j,n}^{(i)} < a_{j+1,n}^{(i)} \text{ for } j = 1,\dots,m_i\right\}.$$

For $i = 1,2$ and $j = 1,\dots,m_i + 1$ we let

$$I_{j,n}^{(i)} := \left(a_{j,n}^{(i)}, b_{j,n}^{(i)}\right]$$

and for $i = 1,2$ and $j = 1,\dots,m_i$ we let

$$J_{j,n}^{(i)} := \left(b_{j,n}^{(i)}, a_{j+1,n}^{(i)}\right].$$

For any choice of $1 < \lambda < \infty$, we set

$$I_j^{(i)}(\lambda) := \left(S_j^{(i)}/\lambda, \lambda S_j^{(i)}\right] \qquad (i = 1,2),\ j = 1,\dots,m_i + 1,$$

and let

$$D_n^{(i)} := \left\{I_{j,n}^{(i)} \subset I_j^{(i)}(\lambda) \text{ for } j = 1,\dots,m_i + 1\right\} \qquad (i = 1,2).$$

Finally, let

$$E_n^{(i)} := A_n^{(i)} \cap B_n^{(i)} \cap C_n^{(i)} \cap D_n^{(i)}, \qquad i = 1,2.$$

Since

$$S_{n+1}/n \to_{\text{a.s.}} 1 \quad \text{as } n \to \infty, \tag{3.21}$$

an elementary argument shows that

$$P\{E_n^{(i)}\} \to 1 \qquad (i = 1,2) \quad \text{as } n \to \infty. \tag{3.22}$$

We observe that when $\omega \in E_n^{(i)}$ and $s \in I_{j,n}^{(i)}$ $(i = 1,2)$, then both

$$j - 1 \le N^{(i)}(s) \le j \tag{3.23}$$

and

$$j - 1 \le \Gamma_n^{(i)}(s) \le j, \tag{3.24}$$

and for $s \in J_{j,n}^{(i)}$

$$\Gamma_n^{(i)}(s) = N^{(i)}(s) = j. \tag{3.25}$$

Now notice that for any $\omega \in E_n^{(i)}$ $(i = 1,2)$ and $t \in (0,T)$

$$\left|\int_0^t \left(N^{(i)}(s) - \Gamma_n^{(i)}(s)\right) d\mu_{\alpha,n}(s)\right| \le \sum_{j=1}^{m_i+1} \int_{I_{j,n}^{(i)}} \left|N^{(i)}(s) - \Gamma_n^{(i)}(s)\right| d\mu_{\alpha,n}(s)$$
$$+ \sum_{j=1}^{m_i+1} \int_{J_{j,n}^{(i)}} \left|N^{(i)}(s) - \Gamma_n^{(i)}(s)\right| d\mu_{\alpha,n}(s),$$

which, on account of (3.23), (3.24), (3.25), and $\omega \in D_n^{(i)}$, is less than or equal to

$$\sum_{j=1}^{m_\iota+1} \int_{I_j^{(i)}(\lambda)} d\mu_{\alpha,n}(s) \qquad (i = 1,2).$$

On applying Lemma 3.1 with $\beta = 0$, we see that the last expression converges, as $n \to \infty$, to

$$\alpha^{-1} \sum_{j=1}^{m_\iota+1} \left(S_j^{(i)}\right)^{-1/\alpha} (\lambda^{1/\alpha} - \lambda^{-1/\alpha}).$$

Since $\lambda$ can be chosen arbitrarily close to 1, on account of (3.22), the proof of (3.17) is now complete.

For the proof of our next lemma we will need the following fact, contained in the proof of Theorem 1 of Mason (1983).

FACT. *For each* $\frac{1}{2} < \beta \leq 1$ *and* $i = 1,2$

(3.26) $$\sup_{0 \leq s < \infty} |N^{(i)}(s) - s| s^{-\beta} < \infty \quad \text{a.s.},$$

*and*

(3.27) $$\sup_{0 \leq s < \infty} |\Gamma_n^{(i)}(s) - s| s^{-\beta} \to_{\mathscr{D}} \sup_{0 \leq s < \infty} |N^{(i)}(s) - s| s^{-\beta}.$$

LEMMA 3.5. *Assume* $Q \in \mathscr{K}_1$. *For each* $0 < \alpha < 2$, *there exists a* $0 < \beta < 1/\alpha$ *such that for every* $0 < T < \infty$

(3.28) $$\sup_{T < t < \infty} |M_\alpha^{(i)}(t) - M_\alpha^{(i)}(T)| = O_P(1) T^{\beta - 1/\alpha}$$

*and*

(3.29) $$\sup_{T < t < \infty} |\Gamma_{n,\alpha}^{(i)}(t) - \Gamma_{n,\alpha}^{(i)}(T)| = O_P(1)\{T^{\beta-1/\alpha} + o(1)\}$$

*for* $i = 1,2$ *where the* $O_P(1)$ *terms in* (3.28) *and* (3.29) *are independent of* $T$.

PROOF. First choose $\frac{1}{2} < \beta < 1/\alpha \leq 1$. In this case the left sides of expressions (3.28) and (3.29) are less than or equal to

(3.30) $$\sup_{T < t < \infty} \int_T^t \alpha^{-1} |N^{(i)}(s) - s| s^{-1-1/\alpha}\, ds$$

and

(3.31) $$\sup_{T < t \leq n} \int_T^t |\Gamma_n^{(i)}(s) - s|\, d\mu_{\alpha,n}(s),$$

respectively ($i = 1,2$). Expression (3.30) is less than or equal to

(3.32) $$\sup_{0 \leq s < \infty} s^{-\beta} |N^{(i)}(s) - s| \left(\alpha^{-1} T^{\beta - 1/\alpha} / (1/\alpha - \beta)\right),$$

and expression (3.31) is less than or equal to

(3.33) $$\sup_{0 \leq s < \infty} s^{-\beta} |\Gamma_n^{(i)}(s) - s| \int_T^n t^\beta\, d\mu_{\alpha,n}(t).$$

Consequently (3.26), (3.27), and Lemma 3.2 complete the proof for this case of $1 \leq \alpha < 2$.

Now choose $1 = \beta < 1/\alpha < \infty$. In this case the left sides of expressions (3.28) and (3.29) are less than or equal to

$$\sup_{0 \leq s < \infty} s^{-1} N^{(i)}(s)\left(\alpha^{-1} T^{1-1/\alpha}/(1/\alpha - 1)\right) \tag{3.34}$$

and

$$\sup_{0 \leq s < n} s^{-1} \Gamma_n^{(i)}(s) \int_T^n s \, d\mu_{\alpha, n}(s), \tag{3.35}$$

respectively ($i = 1, 2$), and (3.26), (3.27), and Lemma 3.2 also complete the proof for this case of $0 < \alpha < 1$.

In the light of Lemmas 3.3, 3.4, and 3.5 the proof of the first statement of Theorem 3.1 in (3.3) is now also complete.

In order to prove the second statement of Theorem 3.1 we require the following analogue of Lemma 3.2.

LEMMA 3.6. *Assume $Q \in \mathscr{K}_2$. For each $0 < \alpha < 2$, $0 < \beta < 1/\alpha$, and $0 < T < \infty$*

$$\lim_{n \to \infty} \int_T^n s^\beta \, d\mu_{\alpha, n}(s) = 0. \tag{3.36}$$

PROOF. Integration by parts gives that the left side of expression (3.36) is equal to the sum of

$$\lim_{n \to \infty} \left\{ H(1) n^{\beta - 1/\alpha}/L(1/n) + T^\beta n^{-1/\alpha} Q(1 - T/n)/L(1/n) \right\} \tag{3.37}$$

and

$$- \lim_{n \to \infty} n^{\beta - 1/\alpha} \beta \int_{T/n}^1 H(u) u^{\beta - 1} \, du / L(1/n). \tag{3.38}$$

By (K.3) and the assumption that $\beta - 1/\alpha < 0$, expression (3.37) equals zero. For each $0 < \lambda < 1$, expression (3.38) equals

$$- \lim_{n \to \infty} n^{\beta - 1/\alpha} \beta \int_{T/n}^\lambda H(u) u^{\beta - 1} \, du / L(1/n)$$
$$- \lim_{n \to \infty} n^{\beta - 1/\alpha} \beta \int_\lambda^1 H(u) u^{\beta - 1} \, du / L(1/n),$$

where the second term is obviously zero and the first one is less than or equal to

$$\sup_{0 \leq s \leq \lambda} \frac{Q(1 - s)}{L(s) s^{-1/\alpha}} \lim_{n \to \infty} n^{\beta - 1/\alpha} \beta \int_{T/n}^\lambda L(u) u^{\beta - 1 - 1/\alpha} \, du / L(1/n)$$
$$= \sup_{0 \leq s \leq \lambda} \frac{Q(1 - s)}{L(s) s^{-1/\alpha}} \frac{\beta T^{\beta - 1/\alpha}}{(1/\alpha) - \beta},$$

where the last equality is obtained by the same argument as given in the proof of

Lemma 3.2. Since $0 < \lambda < 1$ can be chosen arbitrarily close to zero, we have (3.36) by (K.3).

Now we turn to the proof of the second statement of Theorem 3.1. It is enough to show that

$$(3.39) \qquad \int_0^n |\Gamma_n^{(i)}(s) - s|\, d\mu_{\alpha,n}(s) = o_P(1) \quad \text{for } i = 1,2 \quad \text{if } 1 < \alpha < 2,$$

$$(3.40) \qquad \int_{nU_{1,n}}^n |\Gamma_n^{(i)}(s) - s|\, d\mu_{1,n}(s) = o_P(1) \quad \text{for } i = 1,2 \quad \text{if } \alpha = 1,$$

$$(3.41) \qquad \int_0^n \Gamma_n^{(i)}(s)\, d\mu_{\alpha,n}(s) = o_P(1) \quad \text{for } i = 1,2 \quad \text{if } 0 < \alpha < 1.$$

To prove (3.39), chose any $0 < T < n$. Then the integral in question is equal to

$$B_n^{(i)}(T) + C_n^{(i)}(T) := \int_0^T |\Gamma_n^{(i)}(s) - s|\, d\mu_{\alpha,n}(s) + \int_T^n |\Gamma_n^{(i)}(s) - s|\, d\mu_{\alpha,n}(s).$$

Choose now any $\frac{1}{2} < \beta < 1/\alpha$. We see that

$$\begin{aligned} C_n^{(i)}(T) &\le \sup_{0 \le s < \infty} s^{-\beta} |\Gamma_n^{(i)}(s) - s| \int_T^n t^\beta \, d\mu_{\alpha,n}(t) \\ &= O_P(1) o(1) = o_P(1) \end{aligned}$$

by (3.27) and Lemma 3.6. Hence, to complete the proof of (3.39) it is enough to show that for any $0 < \varepsilon < \infty$,

$$(3.42) \qquad \lim_{T \downarrow 0} \limsup_{n \to \infty} P\{B_n^{(i)}(T) > \varepsilon\} = 0.$$

Since

$$(3.43) \qquad \lim_{T \downarrow 0} \limsup_{n \to \infty} P\{nU_{1,n} > T\} = 1,$$

to establish (3.42) it is sufficient to prove that

$$(3.44) \qquad \lim_{n \to \infty} \int_0^T s\, d\mu_{\alpha,n}(s) = 0 \quad \text{for each } 0 < T < \infty.$$

Here the left side equals the limit of

$$-n \int_0^{T/n} u\, dQ(1-u) / (n^{1/\alpha} L(1/n)),$$

which by integration by parts equals

$$\frac{-TQ\left(1 - \frac{T}{n}\right)}{n^{1/\alpha} L(1/n)} + \frac{n \int_0^{T/n} Q(1-u)\, du}{n^{1/\alpha} L(1/n)}.$$

By (K.3) the first term converges to zero as $n \to \infty$. The second term is not greater than

$$(3.45) \qquad \begin{aligned} &\sup_{0 \le u \le T/n} Q(1-u) (u^{1/\alpha} L(u))^{-1} \int_0^{T/n} nL(v) v^{-1/\alpha}\, dv (n^{1/\alpha} L(1/n))^{-1} \\ &= o(1) \int_0^{T/n} nL(v) v^{-1/\alpha}\, dv (n^{1/\alpha} L(1/n))^{-1}, \end{aligned}$$

where the last equality is by (K.3). Applying now Theorem 1.2.1 of de Haan (1970), page 15, we have

$$\lim_{n\to\infty}\int_0^{T/n} nL(v)v^{-1/\alpha}\,dv/\big(n^{1/\alpha}L(1/n)\big) = T^{1-1/\alpha}/(1-1/\alpha).$$

Thus (3.44) holds by (3.45), and (3.39) is proved.

Now consider (3.40) and (3.41). By (3.43) it is enough to show that for any $0 < T < \infty$,

$$\text{(3.46)}\qquad \int_T^n |\Gamma_n^{(i)}(s) - s|\,d\mu_{1,n}(s) = o_P(1), \qquad i = 1,2, \quad \text{when } \alpha = 1,$$

$$\text{(3.47)}\qquad \int_T^n \Gamma_n^{(i)}(s)\,d\mu_{\alpha,n}(s) = o_P(1), \qquad i = 1,2, \quad \text{when } 0 < \alpha < 1.$$

Choose any $\frac{1}{2} < \beta < 1$. Then the integral in (3.46) is less than or equal to

$$\sup_{0\le s<\infty} s^{-\beta}|\Gamma_n^{(i)}(s) - s|\int_T^n t^\beta\,d\mu_{1,n}(t),$$

and when $0 < \alpha < 1$, the integral in (3.47) is not greater than

$$\sup_{0\le s<\infty} s^{-1}\Gamma_n^{(i)}(s)\int_T^n t\,d\mu_{\alpha,n}(t).$$

Applying again Lemma 3.6 and (3.27), we see that both of these bounds are $o_P(1)$ for any $0 < T < \infty$. This completes the proof of the second statement of Theorem 3.1. □

The rest of this section is devoted to establishing results for stable laws paralleling that of Corollary 2.2 for the normal convergence criterion. These results will illustrate Theorem 3.1 as an analogue of Theorem 2.1.

Let $X_1, X_2, \dots$ be a sequence of i.i.d. rv with common distribution $F$ and quantile function $Q$. Let

$$G(y) := P\{|X_1| \le y\} \quad \text{for } y > 0,$$

and

$$K(u) = G^{-1}(u) := \inf\{y\colon G(y) \ge u\} \quad \text{for } u \in [0,1).$$

Put

$$Q_1(u) := (-Q(1-u)) \vee 0,$$
$$Q_2(u) := Q(u) \vee 0,$$
$$H_1(u) = -Q_1(u+) \quad \text{and} \quad H_2(u) = -Q_2(1-u).$$

Note that $Q(u) = -Q_1(1-u) + Q_2(u)$.

Assume that for some $0 < \alpha < 2$ and function $L$, slowly varying near zero, we have

$$\text{(3.48)}\qquad K(1-u) = u^{-1/\alpha}L(u).$$

Assume also that for each $i = 1,2$

$$\text{(3.49)}\qquad \lim_{u\downarrow 0} Q_i(1-u)/K(1-u) = w_i \text{ exists.}$$

We observe that whenever conditions (3.48) and (3.49) hold true and $w_i > 0$, then for this $i$

$$Q_i(1-u) = u^{-1/\alpha} L_i(u) \tag{3.50}$$

for some function $L_i$, slowly varying near zero such that

$$\lim_{u \downarrow 0} L_i(u)/L(u) = w_i. \tag{3.51}$$

It is shown in Proposition A.3 of the appendix that *the above conditions on $Q$ are equivalent to the classical necessary and sufficient conditions on $F$ for it to be in the domain of attraction of a stable law with $\alpha \in (0,2)$.*

Whenever $1 < \alpha < 2$, let

$$S_{n,\alpha} := n^{-1/\alpha} \sum_{j=1}^{n} (X_j - \mu)/L(1/n),$$

where

$$\mu := \int_0^1 Q(u)\,du = \int_{-\infty}^{\infty} x\,dF(x).$$

We notice that

$$\begin{aligned} S_{n,\alpha} &=_{\mathscr{D}} n^{-1/\alpha} \sum_{j=1}^{n} \big(Q(U_{j,n}) - \mu\big)/L(1/n) \\ &= n^{-1/\alpha} \sum_{j=1}^{n} \big(Q_2(U_{j,n}) - \mu_2\big)/L(1/n) \\ &\quad - n^{-1/\alpha} \sum_{j=1}^{n} \big(Q_1(1-U_{j,n}) - \mu_1\big)/L(1/n), \end{aligned}$$

where

$$\mu_i = \int_0^1 Q_i(u)\,du \quad \text{for } i = 1,2.$$

Thus by integration by parts we have

$$\begin{aligned} S_{n,\alpha} &=_{\mathscr{D}} n^{-1/\alpha} \int_0^n \big(\Gamma_n^{(2)}(s) - s\big)\,dH_2(s/n)/L(1/n) \\ &\quad - n^{-1/\alpha} \int_0^n \big(\Gamma_n^{(1)}(s) - s\big)\,dH_1(s/n)/L(1/n). \end{aligned} \tag{3.52}$$

When $\alpha = 1$, let

$$S_{n,1} := n^{-1} \sum_{j=1}^{n} \big(X_j - \mu(n)\big)/L(1/n),$$

where

$$\mu(n) := -\int_{1/n}^{1} Q_1(1-u)\,du + \int_{1/n}^{1} Q_2(1-u)\,du.$$

Applying integration by parts again, we see that

$$
\begin{aligned}
S_{n,1} =_{\mathscr{D}} & n^{-1}\int_{n(1-U_{n,n})}^{n} \left(\Gamma_n^{(2)}(s) - s\right) dH_2(s/n)/L(1/n) \\
& -n^{-1}\int_{nU_{1,n}}^{n} \left(\Gamma_n^{(1)}(s) - s\right) dH_1(s/n)/L(1/n) \\
& +n^{-1}\left\{\int_1^{nU_{1,n}} s\, dH_1(s/n) + H_1(1/n)\right\}\Big/L(1/n) \\
& -n^{-1}\left\{\int_1^{n(1-U_{n,n})} s\, dH_2(s/n) + H_2(1/n)\right\}\Big/L(1/n).
\end{aligned} \tag{3.53}
$$

Finally, whenever $0 < \alpha < 1$, we let

$$S_{n,\alpha} := n^{-1/\alpha}\sum_{j=1}^{n} X_j/L(1/n)$$

and integrating by parts as above we get

$$
\begin{aligned}
S_{n,\alpha} =_{\mathscr{D}} & n^{-1/\alpha}\left\{\int_0^n \Gamma_n^{(2)}(s)\, dH_2(s/n) - nH_2(1)\right\}\Big/L(1/n) \\
& -n^{-1/\alpha}\left\{\int_0^n \Gamma_n^{(1)}(s)\, dH_1(s/n) - nH_1(1)\right\}\Big/L(1/n).
\end{aligned} \tag{3.54}
$$

Whenever $0 < \alpha < 1$, we write

$$\Delta_{\alpha,i} := \alpha^{-1}\int_0^\infty N^{(i)}(s)s^{-1-1/\alpha}\, ds \qquad (i = 1,2),$$

When $\alpha = 1$, we write

$$\Delta_{1,i} := \int_{Y_1^{(i)}}^\infty \left(N^{(i)}(s) - s\right)s^{-2}\, ds - \left(\log Y_1^{(i)} - 1\right) \qquad (i = 1,2),$$

and whenever $1 < \alpha < 2$, we write

$$\Delta_{\alpha,i} := \alpha^{-1}\int_0^\infty \left(N^{(i)}(s) - s\right)s^{-1-1/\alpha}\, ds \qquad (i = 1,2).$$

The following corollary gives a probabilistic proof of the sufficiency part of the classical criteria for a distribution to be in the domain of attraction of a stable law of index $0 < \alpha < 2$.

COROLLARY 3.1. *Assume that Q is such that* (3.48) *and* (3.49) *hold. Then for* $0 < \alpha < 2$ *we have*

$$S_{n,\alpha} \to_{\mathscr{D}} - w_1\Delta_{\alpha,1} + w_2\Delta_{\alpha,2}. \tag{3.55}$$

PROOF. First we consider the case when both $w_1$ and $w_2$ are positive. We first note that whenever $1 < \alpha < 2$, (3.55) follows directly from the in distribution representation of $S_{n,\alpha}$ given in (3.52) in combination with condition (3.49) and

(3.3) of Theorem 3.1 for the case when $1 < \alpha < 2$. Similarly, since for $0 < \alpha < 1$

$$n^{1-1/\alpha}H_i(1)/L(1/n) = o(1), \qquad n \to \infty,$$

for $i = 1, 2$, (3.55) in this case follows from (3.54), (3.51), and Theorem 3.1 for the said case.

We consider now the case when $\alpha = 1$. Applying (3.3) of Theorem 3.1 in the latter case in combination with Lemma 3.5, along with (3.49), we get

$$\left(n^{-1}/L(1/n)\right)\int_{nU_{1,n}}^{n}\left(\Gamma_n^{(1)}(s) - s\right)dH_1(s/n) \to_P w_1\int_{Y_1^{(1)}}^{\infty}\left(N^{(1)}(s) - s\right)s^{-2}\,ds,$$

and

$$\left(n^{-1}/L(1/n)\right)\int_{n(1-U_{n,n})}^{n}\left(\Gamma_n^{(2)}(s) - s\right)dH_2(s/n) \to_P w_2\int_{Y_1^{(1)}}^{\infty}\left(N^{(2)}(s) - s\right)s^{-2}\,ds.$$

Also, for $i = 1, 2$

$$n^{-1}H_i(1/n)/L_i(1/n) = -1.$$

Hence, in order to complete the proof of (3.55), it suffices to show that when $\alpha = 1$

(3.56) $$n^{-1}\int_1^{nU_{1,n}} s\,dH_1(s/n)/L_1(1/n) \to_P \log Y_1^{(1)},$$

and

(3.57) $$n^{-1}\int_1^{n(1-U_{n,n})} s\,dH_2(s/n)/L_2(1/n) \to_P \log Y_1^{(2)}.$$

We will only provide a proof for (3.56). The proof for (3.57) is, *mutatis mutandis* in notation, exactly the same. We notice that by integration by parts the left side of expression (3.56) is equal to

$$\left(U_{1,n}H_1(U_{1,n}) - n^{-1}H_1(1/n)\right)/L_1(1/n) - n^{-1}\int_1^{nU_{1,n}} H_1(s/n)\,ds/L_1(1/n),$$

which, in turn, by definition of $H_1$ and (3.50) is equal to

$$\Delta_n := 1 - L_1(U_{1,n})/L_1(1/n) + \int_1^{nU_{1,n}} L_1(s/n)s^{-1}\,ds/L_1(1/n).$$

Choose any $\lambda > 1$ and set

$$A_n(\lambda) := \{1/n\lambda < U_{1,n} < \lambda/n\},$$

and

$$\delta_n(\lambda) := \sup_{\lambda^{-1} \le u \le \lambda} \left|1 - L_1(u/n)/L_1(1/n)\right|.$$

We note that (3.7) implies that $\delta_n(\lambda) = o(1)$ for any $\lambda > 1$. Also, we observe that

(3.58) $$nU_{1,n} \to_P Y_1^{(1)} \quad \text{as } n \to \infty.$$

Thus for any $\omega \in A_n(\lambda)$

$$\begin{aligned}\left|\Delta_n - \log Y_1^{(1)}\right| &\le \delta_n(\lambda)\left\{1 + \left|\log(nU_{1,n})\right|\right\} + \left|\log(nU_{1,n}) - \log Y_1^{(1)}\right| \\ &= o_P(1).\end{aligned}$$

Since (3.58) implies that

$$\lim_{\lambda\to\infty}\ \limsup_{n\to\infty} P\{A_n(\lambda)\} = 1, \tag{3.59}$$

the proof of (3.56), and hence also that of Corollary 3.1 is now complete in the case when $w_1 > 0$ and $w_2 > 0$.

Now we consider the case when $w_1 = 0$ or $w_2 = 0$. Suppose that $w_1 = 0$. This entails that $w_2 = 1$ (cf. Proposition A.3 in the appendix).

First assume that $\alpha = 1$. Since $w_2 = 1$, we have as before

$$n^{-1}\int_{n(1-U_{1,n})}^{n}\big(\Gamma_n^{(2)}(s) - s\big)\,dH_2(s/n)/L(1/n)$$
$$-n^{-1}\left\{\int_1^{n(1-U_{1,n})} s\,dH_2(s/n) + H_2(1/n)\right\}\Big/L(1/n) \to_P \Delta_{1,2}$$

as $n \to \infty$. Since when $w_1 = 0$, $Q_1$ satisfies (K.3), we have by the second statement of Theorem 3.1 that

$$n^{-1}\int_{nU_{1,n}}^{n}\big(\Gamma_n^{(1)}(s) - s\big)\,dH_1(s/n)/L(1/n) \to_P 0.$$

Hence to complete the proof for this case it is enough to show that

$$n^{-1}\left\{\int_1^{nU_{1,n}} s\,dH_1(s/n) + H_1(1/n)\right\}\Big/L(1/n) \to_P 0.$$

First note that by (K.3)

$$n^{-1}H_1(1/n)/L(1/n) \to 0 \quad \text{as } n \to \infty.$$

By (3.59) it is therefore sufficient to show that for every $1 < \lambda < \infty$,

$$n^{-1}\int_{1/\lambda}^{\lambda} s\,dH_1(s/n)/L(1/n) \to 0 \quad \text{as } n \to \infty. \tag{3.60}$$

Integrating by parts, the left side here is

$$n^{-1}\{\lambda H_1(\lambda/n) - H_1(1/(\lambda n))/\lambda\}/L(1/n) - n^{-1}\int_{1/\lambda}^{\lambda} H_1(s/n)\,ds/L(1/n).$$

Now (K.3) implies that the first term converges to zero as $n \to \infty$, while the second term is less than or equal to

$$\sup_{1/\lambda\le s\le\lambda}\left\{\frac{s}{n}Q_1\left(1-\frac{s}{n}\right)\Big/L\left(\frac{s}{n}\right)\right\}\int_{1/\lambda}^{\lambda} L\left(\frac{t}{n}\right)t^{-1}\,dt\Big/L\left(\frac{1}{n}\right).$$

The latter bound, by (K.3) and the fact that

$$\sup_{1/\lambda\le t\le\lambda} L(t/n)/L(1/n) \to 1 \quad \text{as } n \to \infty,$$

equals $o(1)2\log\lambda = o(1)$. This completes the proof of (3.60), and hence that of the case $w_1 = 0$, $w_2 = 1$ when $\alpha = 1$.

The method of the proof for the case when $0 < \alpha < 2$ with $\alpha \ne 1$ should be obvious from the aforegoing proof, so we omit the details. Also the case when $w_1 = 1$ and $w_2 = 0$ is the same. This completes the proof of Corollary 3.1. □

Finally, we identify the limiting stable distribution of the $S_{n,\alpha}$ in Corollary 3.1. From now on $i$ will denote the imaginary unit.

The canonical form for a nonnormal stable characteristic function [see Feller (1966) and the words of caution given by Hall (1981) concerning other standard references] is

(3.61)

$$\phi_{\alpha,\beta,\gamma,\theta}(t) = \exp\{i\theta t - \gamma|t|^\alpha[1 - i\beta\,\mathrm{sgn}(t)\omega(t,\alpha)]\}, \qquad -\infty < t < \infty,$$

where $\alpha \in (0,2)$ is the characteristic exponent, $\beta \in [-1,1]$ is the skewness parameter, $\gamma > 0$ is the scale parameter, $\theta \in (-\infty,\infty)$ is the location parameter, and

$$\omega(t,\alpha) = \begin{cases} \tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1, \\ -\frac{2}{\pi}\log|t| & \text{if } \alpha = 1. \end{cases}$$

In particular, if $\beta = 0$ then the corresponding distribution is symmetric about $\theta$, and distributions with $|\beta| = 1$ are commonly called completely asymmetric stable distributions. In case $0 < \alpha < 1$ the stable laws with $|\beta| = 1$ are one-sided, namely their support is $[\theta,\infty)$ in case $\beta = 1$ and $(-\infty,\theta]$ in case $\beta = -1$.

The limiting characteristic function of $S_{n,\alpha}$ in Corollary 3.1 is the product $\psi_\alpha(-w_1 t)\psi_\alpha(w_2 t)$, where

$$\psi_\alpha(s) = E\exp(is\Delta_{\alpha,1}) = E\exp(is\Delta_{\alpha,2}), \qquad -\infty < s < \infty.$$

Note first that Proposition A.3 (in the appendix) identifies the constants $w_1$ and $w_2$ in terms of the tail behaviour of the underlying distribution function. We have $w_1 = q^{1/\alpha} = (1-p)^{1/\alpha}$ and $w_2 = p^{1/\alpha}$, where $p$ is given in (A.30). On the other hand, borrowing ideas from Ferguson and Klass (1972), the characteristic function $\psi_\alpha$ of the completely asymmetric stable variables $\Delta_{\alpha,j}$, $j = 1,2$, can be obtained by routine calculations not detailed here. The final result is that for any $-\infty < t < \infty$,

$$\begin{aligned}\lim_{n\to\infty} E\exp(itS_{n,\alpha}) &= \psi_\alpha\big(-(1-p)^{1/\alpha}t\big)\psi_\alpha\big(p^{1/\alpha}t\big) \\ &= \phi_{\alpha,1-2p,\gamma(\alpha),\theta(\alpha,p)}(t),\end{aligned}$$

where

$$\gamma(\alpha) = \begin{cases} \Gamma(1-\alpha)\cos\frac{\pi\alpha}{2} & \text{if } 0 < \alpha < 1, \\ \frac{\pi}{2} & \text{if } \alpha = 1, \\ \alpha(\alpha-1)^{-1}\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2} & \text{if } 1 < \alpha < 2, \end{cases} \tag{3.62}$$

and

$$\theta(\alpha,p) = \begin{cases} 0 & \text{if } 0 < \alpha < 2,\ \alpha \neq 1, \\ A(2p-1) & \text{if } \alpha = 1, \end{cases}$$

where

$$A = \int_0^\infty \left( \frac{\sin x}{x^2} - \frac{1}{x(1+x)} \right) dx. \tag{3.63}$$

Given now any possible configuration of the parameters $0 < \alpha < 2$, $-1 \le \beta \le 1$, $\gamma > 0$, and $-\infty < \theta < \infty$, a routine calculation gives the following representation of the corresponding stable rv in terms of the independent and identically distributed Poisson integrals $\Delta_{\alpha,1}$ and $\Delta_{\alpha,2}$ of Corollary 3.1. Let $\gamma(\alpha)$ be given as in (3.62) and set

$$C_\alpha(\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha \neq 1, \\ \beta\left\{A + \frac{2}{\pi}\left(\log\frac{2}{\pi} + \gamma\log\gamma\right)\right\} & \text{if } \alpha = 1, \end{cases}$$

where the constant $A$ is given in (3.63).

THEOREM 3.2. *The characteristic function of the rv*

$$\theta + C_\alpha(\beta,\gamma) + \left(\frac{\gamma}{\gamma(\alpha)}\right)^{1/\alpha}\left\{\left(\frac{1-\beta}{2}\right)^{1/\alpha}\Delta_{\alpha,1} + \left(1 - \frac{1-\beta}{2}\right)^{1/\alpha}\Delta_{\alpha,2}\right\}$$

*is* $\phi_{\alpha,\beta,\gamma,\theta}$ *as given in* (3.61).

We note that this representation can also be obtained from the Itô–Lévy integral representation of a stable process $X(t)$ [see Itô (1969), pages 1.12.4–1.12.5] after setting $t = 1$, changing variables, integrating by parts, and identifying the parameters.

Since, as shown in Proposition A.3 of the appendix, conditions (3.48) and (3.49) are equivalent to the classical criteria for a distribution function to be in the domain of attraction of a stable law with index $0 < \alpha < 2$, Corollary 3.1 gives a probabilistic proof of the sufficiency of these criteria for the existence of sequences of constants $\{A_n\}$ and $\{C_n\}$ such that $A_n(\sum_{i=1}^n X_n - C_n)$ converges in distribution to a stable law of index $0 < \alpha < 2$. In fact these constants are given explicitly in the statement of Corollary 3.1. Previously, probabilistic proofs of the sufficiency of stable convergence criteria were given by Simons and Stout (1978) and LePage, Woodroofe, and Zinn (1981). The Simons and Stout (1978) proof is obtained by means of a special weak invariance principle, whereas the LePage et al. (1981) proof is based on order statistics methodology. In neither of these papers is the characteristic function of the limiting distribution identified, nor are the normalizing constants $A_n$ and $C_n$ given as explicitly as in Corollary 3.1.

Our approach is more closely related to that of LePage et al. (1981). Their method of proof and corresponding representation of the limiting distribution is based on the asymptotic independence of the upper extreme values of $|X_1|, \dots, |X_n|$ and the sign of the $X$s that correspond to these extremes, whereas our approach is in a sense based on the asymptotic independence of the upper and lower extreme values of $X_1, \dots, X_n$. Only for the case when $w_1 = 0$ or 1 and $0 < \alpha < 2$ is the representation we obtain for the limiting stable rv algebraically equivalent, modulo a constant factor, to theirs (our $w_1 = q^{1/\alpha}$, where our $q$ corresponds to their $q$; cf. Proposition A.3).

**4. The role of the extreme terms in nonnormal stable convergence.** Let $X_1, \ldots, X_n$ be independent and identically distributed rv and consider the truncated partial sum

$$\tilde{Z}_{n,k} = \sum_{i=1}^{n} X_i - \sum_{i=1}^{k} \tilde{X}_{n+1-i,n},$$

where $\tilde{X}_{n+1-k,n}, \ldots, \tilde{X}_{n,n}$ denote the rv that appear inside the modulus of the upper $k$ order statistics of $|X_1|, \ldots, |X_n|$. Several authors such as Darling (1952), Arov and Bobrov (1960), Hall (1978), and Teugels (1981) considered the problem of the asymptotic distribution of $\tilde{Z}_{n,k}$ when $k$ is kept fixed and $n \to \infty$, and the common distribution function of the $X$ is in the domain of attraction of a stable law of index $0 < \alpha < 2$. They have shown that for suitable constants $\tilde{a}_n$ and $\tilde{b}_n$ the rv $(\tilde{Z}_{n,k} - \tilde{b}_n)/\tilde{a}_n$ converge in distribution to a *nondegenerate* rv. Most of these works are based on characteristic function techniques. Using the asymptotic independence results of LePage et al. (1981) referred to above we could easily reprove all these results by our probabilistic techniques. However, from the point of view of statistical motivation it is perhaps more natural to consider the problem when a certain number of the largest and the smallest order statistics are removed from the partial sum, i.e., sums of the form

$$\sum_{i=k+1}^{n-m} X_{i,n},$$

where $X_{1,n} \leq \cdots \leq X_{n,n}$ are the order statistics corresponding to $X_1, \ldots, X_n$. As far as we know this problem has not yet been considered. The following result is a generalization of Corollary 3.1.

THEOREM 4.1. *Assume that $Q$ is such that* (3.48) *and* (3.49) *hold. Then for $0 < \alpha < 2$ and any fixed nonnegative integers $k$ and $m$ we have*

$$\frac{1}{n^{1/\alpha}L(1/n)} \sum_{i=k+1}^{n-m} (X_{i,n} - \mu_{n,\alpha}) \to_{\mathscr{D}} - w_1\left\{\Delta_{\alpha,1} - \sum_{j=1}^{k} \left(S_j^{(1)}\right)^{-1/\alpha}\right\} + w_2\left\{\Delta_{\alpha,2} - \sum_{j=1}^{m} \left(S_j^{(2)}\right)^{-1/\alpha}\right\}$$

*where*

$$\mu_{n,\alpha} = \begin{cases} 0 & \text{if } 0 < \alpha < 1, \\ \int_{1/n}^{1} Q_2(1-u)\,du - \int_{1/n}^{1} Q_1(1-u)\,du & \text{if } \alpha = 1, \\ \int_0^1 Q(u)\,du & \text{if } 1 < \alpha < 2. \end{cases}$$

**Proof.** We need the fact that for each fixed $1 \le j \le n$

$$Q(U_{j,n})/(L(1/n)n^{1/\alpha}) \to_P -w_1\left(S_j^{(1)}\right)^{-1/\alpha} \tag{4.1}$$

and

$$Q(U_{n+1-j,n})/(L(1/n)n^{1/\alpha}) \to_P -w_2\left(S_j^{(2)}\right)^{-1/\alpha} \tag{4.2}$$

as $n \to \infty$. We prove (4.1). The proof of (4.2) is the same. Condition (3.49) implies that

$$\lim_{s \downarrow 0} s^{1/\alpha} Q(s)/L(s) = -w_1.$$

Also, it is easy to see that

$$nU_{j,n} \to_P S_j^{(1)}. \tag{4.3}$$

The last two facts imply that for every $\varepsilon > 0$, as $n \to \infty$,

$$P\left\{(-w_1 - \varepsilon)\frac{U_{j,n}^{-1/\alpha}L(U_{j,n})}{n^{1/\alpha}L(1/n)} \le \frac{Q(U_{j,n})}{n^{1/\alpha}L(1/n)} \le (-w_1 + \varepsilon)\frac{U_{j,n}^{-1/\alpha}L(U_{j,n})}{n^{1/\alpha}L(1/n)}\right\} \to 1. \tag{4.4}$$

Now (4.3) implies that

$$U_{j,n}^{-1/\alpha}/n^{1/\alpha} \to_P \left(S_j^{(1)}\right)^{-1/\alpha} \tag{4.5}$$

and that

$$\lim_{\lambda \to \infty} \limsup_{n \to \infty} P\{1/\lambda \le nU_{j,n} \le \lambda\} = 1. \tag{4.6}$$

Thus, since $L$ is slowly varying near zero, we conclude from (4.6) that

$$L(U_{j,n})/L(1/n) \to_P 1. \tag{4.7}$$

Hence from (4.4), (4.5), (4.6), and (4.7) we have (4.1).

To complete the proof of the theorem we note that

$$\frac{n^{-1/\alpha}}{L(1/n)}\left\{\sum_{i=k+1}^{n-m} X_{i,n} - \mu_{n,\alpha}\right\} =_{\mathscr{D}} S_{n,\alpha}^* - \frac{n^{-1/\alpha}}{L(1/n)}\left\{\sum_{j=1}^{k} Q(U_{i,n}) + \sum_{j=1}^{m} Q(U_{n+1-j,n})\right\}, \tag{4.8}$$

where $S_{n,\alpha}^*$ denotes the right side of (3.52), (3.53), or (3.54) depending on the value of $\alpha$. Putting now together (4.1), (4.2), and the proof of Corollary 3.1, we see that the right side of the latter distributional equality converges in probability to

$$-w_1\left\{\Delta_{\alpha,1} - \sum_{j=1}^{k}\left(S_j^{(1)}\right)^{-1/\alpha}\right\} + w_2\left\{\Delta_{\alpha,2} - \sum_{j=1}^{m}\left(S_j^{(2)}\right)^{-1/\alpha}\right\}. \qquad \square$$

The joint limiting behaviour of the sums

$$\sum_{i=k_n}^{l_n-1} X_{i,n}, \qquad \sum_{i=l_n}^{n+1-l_n} X_{i,n}, \qquad \sum_{i=n-l_n}^{n+1-k_n} X_{i,n},$$

where $k_n \to \infty$, $l_n/k_n \to \infty$, $l_n/n \to 0$, and $k_n/n \to 0$ as $n \to \infty$, is investigated by S. Csörgő, Horváth, and Mason (1986), and for related papers on central limit theorems for sums of extreme values see S. Csörgő and Mason (1985, 1986).

## APPENDIX

**Quantile equivalence for the normal and stable convergence criteria.** In this section, let $X_1, X_2, \dots$ be i.i.d. nondegenerate rv with common distribution function $F$. One of the classical central limit theorems of probability theory says [cf. e.g., Feller (1966), page 545 or Gnedenko and Kolmogorov (1954), page 172]: *There exist sequences of constants* $\{A_n\}$ *and* $\{C_n\}$ $(n = 1, 2, \dots)$ *such that*

$$A_n\left(\sum_{i=1}^{n} X_i - C_n\right) \to_{\mathscr{D}} N(0,1) \tag{A.1}$$

*if and only if*

$$x^2 P\{|X_1| \ge x\}/E\{|X_1|^2 I(|X_1| < x)\} \to 0 \quad \text{as } x \to \infty. \tag{A.2}$$

Let $Q$ be the quantile function of $F$, let $G$ denote the distribution function of $|X_1|$, and let $K = G^{-1}$ be its quantile function. For $0 < s \le \frac{1}{2}$ we write

$$S_1^2(s) := \int_0^{G(K(1-s)-)} K^2(u)\,du,$$

$$S_2^2(s) := \int_0^{1-s} K^2(u)\,du,$$

$$S_3^2(s) := \int_s^{1-s} Q^2(u)\,du,$$

and

$$S_4^2(s) := \int_s^{1-s}\int_s^{1-s} (u \wedge v - uv)\,dQ(u)\,dQ(v).$$

In the following, we will use the general change of variables formula

$$\int_{a\pm}^{b\pm} g(x)\,dF(x) = \int_{F(a\pm)}^{F(b\pm)} g(Q(u))\,du, \tag{A.3}$$

where $-\infty \le a < b \le \infty$, $g$ is any $F$ integrable function defined on the domain in question, and $F$ is any distribution function with quantile function $Q$.

PROPOSITION A.1. *Whenever condition* (A.2) *holds, we have*

$$\text{each } S_i^2 \text{ for } i = 1,2,3 \text{ is slowly varying at zero,} \tag{A.4}$$

$$\lim_{s\downarrow 0} sK^2(1-s)/S_1^2(s) = \lim_{s\downarrow 0} sK^2(1-s)/S_2^2(s) = 0, \tag{A.5}$$

*and*

$$\lim_{s\downarrow 0} s\big(Q^2(s) + Q^2(1-s)\big)/S_3^2(s) = 0. \tag{A.6}$$

PROOF. We will only provide a proof for the difficult cases, when

(A.7) $$\lim_{s \downarrow 0} Q(1 - s) = \infty$$

and

(A.8) $$\lim_{s \downarrow 0} Q(s) = -\infty.$$

The details of the proof for the other cases can be worked out along the same lines.

First we observe that

(A.9) $$x^2 P\{|X_1| \geq x\} = x^2(1 - G(x-)) = x^2(F(-x) + 1 - F(x-))$$

and

$$E\{|X_1|^2 I(|X_1| < x)\} = \int_0^{x-} y^2 \, dG(y),$$

which by (A.3) equals

(A.10) $$\int_0^{G(x-)} K^2(u) \, du.$$

Let $x = K(1 - s)$ in (A.9) and (A.10). We notice that $G(K(1 - s)) \geq 1 - s$, whereas $G(K(1 - s) - ) \leq 1 - s$. Hence

$$K^2(1 - s)\{1 - G(K(1 - s) -)\} \geq sK^2(1 - s),$$

and

$$S_1^2(s) \leq \int_0^{G(K(1-s)-)} K^2(u) \, du + \int_{G(K(1-s)-)}^{1-s} K^2(u) \, du = S_2^2(s).$$

Thus we have

(A.11) $$K^2(1 - s)\{1 - G(K(1 - s) -)\}/S_1^2(s) \geq sK^2(1 - s)/S_1^2(s) \geq sK^2(1 - s)/S_2^2(s).$$

Since by (A.7) and (A.8), $K(1 - s) \to \infty$ as $s \downarrow 0$, (A.2) implies that the left side of (A.11) goes to zero as $s \downarrow 0$. Thus we have shown (A.5).

We will now show that $S_2^2$ is slowly varying at zero. Choose any $1 < \lambda < \infty$, and notice that for small enough $s$

$$S_2^2(\lambda s) = S_2^2(s) - \int_{1-\lambda s}^{1-s} K^2(u) \, du,$$

but

$$\int_{1-\lambda s}^{1-s} K^2(u) \, du / S_2^2(s) \leq (\lambda - 1) sK^2(1 - s)/S_2^2(s),$$

which by (A.5) goes to zero as $s \downarrow 0$. Hence we have that

$$S_2^2(\lambda s)/S_2^2(s) \to 1 \quad \text{as } s \downarrow 0,$$

i.e., $S_2^2$ is slowly varying at zero.

In order to verify that $S_1^2$ is also slowly varying at zero, it is enough to demonstrate that

(A.12) $$\left(S_2^2(s) - S_1^2(s)\right)/S_1^2(s) \to 0 \quad \text{as } s \downarrow 0.$$

We observe that

$$\begin{aligned}\left(S_2^2(s) - S_1^2(s)\right)/S_1^2(s) &= \int_{G(K(1-s)-)}^{1-s} K^2(u)\,du/S_1^2(s)\\ &\le \{1 - s - G(K(1-s)-)\}K^2(1-s)/S_1^2(s)\\ &\le \{1 - G(K(1-s)-)\}K^2(1-s)/S_1^2(s),\end{aligned}$$

which by (A.2) converges to zero as $s \downarrow 0$. Hence (A.12) is true.

We show now that $S_3^2$ is also slowly varying at zero. We notice that since $G(x) \le F(x)$ for all $-\infty < x < \infty$, $K(1-s) \ge Q(1-s)$ for every $0 < s < 1$, so that by (A.7) we have

(A.13) $$K^2(1-s) \ge Q^2(1-s) \quad \text{for all sufficiently small } s > 0.$$

We also note that by (A.8), for all sufficiently small $s > 0$

$$1 - G(-Q(s)-) = F(Q(s)) + 1 - F(-Q(s)-),$$

which by definition of $Q(s)$ is greater than or equal to $s$. We also note that, on account of all quantile functions in this paper being defined to be left-continuous inverses of right-continuous distribution functions, we have

$$K(1-s) = \sup\{x\colon 1 - G(x-) \ge s\}.$$

Thus we have $K(1-s) \ge -Q(s)$ for all small enough $s > 0$, which implies

(A.14) $$K^2(1-s) \ge Q^2(s) \quad \text{for all sufficiently small } s > 0.$$

We will now show that

(A.15) $$\left(S_3^2(s) - S_1^2(s)\right)/S_1^2(s) \to 0 \quad \text{as } s \downarrow 0.$$

First observe that

$$S_1^2(s) = \int_0^{K(1-s)-} y^2\,dG(y) = \int_{-K(1-s)+}^{K(1-s)-} x^2\,dF(x),$$

which by right continuity of $F$ and (A.3) equals

$$\int_{F(-K(1-s))}^{F(K(1-s)-)} Q^2(u)\,du.$$

Hence expression (A.15) equals

$$\int_{1-s}^{F(K(1-s)-)} Q^2(u)\,du/S_1^2(s) + \int_{s}^{F(-K(1-s))} Q^2(u)\,du/S_1^2(s) =: \Delta_1(s) + \Delta_2(s).$$

Next observe that whenever $F(K(1-s)-) \ge 1 - s$

$$|\Delta_1(s)| \le \int_{1-s}^{F(K(1-s))} Q^2(u)\,du/S_1^2(s).$$

By (A.7) and the fact that $K(1-s) \to \infty$ as $s \downarrow 0$, this last expression is, for sufficiently small $s > 0$, less than or equal to

$$sQ^2(F(K(1-s)))/S_1^2(s). \tag{A.16}$$

Also, since by (A.7) again, we have for all sufficiently large positive $x$ that $0 < Q(F(x)) \le x$, expression (A.16) is, for all sufficiently small $s > 0$, less than or equal to

$$sK^2(1-s)/S_1^2(s). \tag{A.17}$$

Now suppose that $F(K(1-s)-) < 1-s$. Then

$$|\Delta_1(s)| \le \int_{F(K(1-s)-)}^{1-s} Q^2(u)\,du/S_1^2(s),$$

which by (A.7) is, for small enough $s > 0$, less than or equal to

$$\{1 - F(K(1-s)-)\}Q^2(1-s)/S_1^2(s).$$

The latter expression by (A.13) is, for all sufficiently small $s > 0$, less than or equal to

$$\{1 - F(K(1-s)-)\}K^2(1-s)/S_1^2(s),$$

which in turn is less than or equal to [cf. (A.9)]

$$\{1 - G(K(1-s)-)\}K^2(1-s)/S_1^2(s). \tag{A.18}$$

Thus, in either of the cases $F(K(1-s)-) \ge 1-s$ and $F(K(1-s)-) < 1-s$, $|\Delta_1(s)|$ is, for all sufficiently small $s > 0$, bounded by expressions [cf. (A.17) and (A.18)] that go to zero as $s \downarrow 0$ by (A.2) [cf. also (A.11)].

As to estimating $|\Delta_2(s)|$, we notice that whenever $s > 0$ is sufficiently small so that $K^2(1-u) \ge Q^2(u)$ [cf. (A.14)] and $Q(u) < 0$ for all $0 < u \le s$, then we can only have $F(-K(1-s)) \le s$. In case of equality here, $\Delta_2(s) = 0$. Hence it suffices to consider only the case of $F(-K(1-s)) < s$. Then we have

$$\begin{aligned}|\Delta_2(s)| &= \int_{F(-K(1-s))}^{s} Q^2(u)\,du/S_1^2(s)\\ &\le \int_{F(Q(s)-)}^{s} Q^2(u)\,du/S_1^2(s) + \int_{F(-K(1-s)-)}^{F(Q(s)-)} Q^2(u)\,du/S_1^2(s),\end{aligned}$$

which by (A.3) and by the fact that $Q$ is constant on $(F(Q(s)-), s]$ is less than or equal to

$$Q^2(s)s/S_1^2(s) + \int_{-K(1-s)-}^{Q(s)-} x^2\,dF(x)/S_1^2(s).$$

The second integral is properly defined since $F(-K(1-s)) < s$ implies that $Q(s) > -K(1-s)$. By (A.14), the last sum is less than or equal to

$$\frac{K^2(1-s)s}{S_1^2(s)} + \frac{K^2(1-s)F(Q(s)-)}{S_1^2(s)} < 2K^2(1-s)s/S_1^2(s) \to 0 \quad \text{as } s \downarrow 0,$$

by the already proven (A.5). Hence $\Delta_2(s) \to 0$ as $s \downarrow 0$, and (A.15) is proven. Consequently $S_3^2(s)$ is also slowly varying at zero.

Now (A.6) also follows immediately by (A.13), (A.14), (A.15), and (A.5). This completes the proof of Proposition A.1. □

PROPOSITION A.2. *Whenever condition* (A.2) *holds*

(A.19) $$S_4^2 \text{ is slowly varying at zero},$$

*and*

(A.20) $$\lim_{s \downarrow 0} s\big(Q^2(s) + Q^2(1-s)\big)/S_4^2(s) = 0.$$

PROOF. We will first show that (A.2) implies (A.20). First observe that if in addition to (A.2) we assume that

(A.21) $$\int_0^1 Q^2(u)\,du < \infty,$$

then

$$s\big(Q^2(s) + Q^2(1-s)\big) \to 0 \quad \text{as } s \downarrow 0,$$

so that in this case (A.20) holds automatically and subsequently there is nothing to prove. Hence we will assume from now on that

(A.22) $$\lim_{s \downarrow 0} \int_s^{1-s} Q^2(u)\,du = \lim_{s \downarrow 0} S_3^2(s) = \infty.$$

Write

$$\mu(s) = \int_s^{1-s} Q(u)\,du \quad \text{for } 0 < s < \tfrac{1}{2}.$$

It is shown in Rényi (1970) that (A.2) implies that

(A.23) $$\int_0^1 |Q(u)|\,du < \infty.$$

Thus, on account of (A.22) and (A.23) we have

(A.24) $$\lim_{s \downarrow 0} \mu(s)/S_3(s) = 0,$$

and

(A.25) $$s(Q(s) + Q(1-s)) \to 0 \quad \text{as } s \downarrow 0.$$

For any $0 < s < \frac{1}{2}$ we write

$$K_s(t) := \begin{cases} Q(1-s) & \text{for } 1-s \le t < 1, \\ Q(t) & \text{for } s < t < 1-s, \\ Q(s) & \text{for } 0 < t \le s. \end{cases}$$

Then $S_4^2(s)$ can be rewritten as

$$S_4^2(s) = \int_0^1 K_s^2(t)\,dt - \left(\int_0^1 K_s(t)\right)^2,$$

which, in turn, equals

$$s\left(Q^2(s) + Q^2(1-s)\right) + S_3^2(s) - \left(s\left(Q(s) + Q(1-s)\right) + \mu(s)\right)^2.$$

Now by (A.6) of Proposition A.1, (A.24), and (A.25) we have

$$\lim_{s \downarrow 0} S_4^2(s)/S_3^2(s) = 1. \tag{A.26}$$

Thus, on account of (A.6) and (A.4), we have that both (A.19) and (A.20) hold true when (A.22) is satisfied. Hence it remains to be shown that (A.19) is true when (A.21) holds. In this case

$$0 < \lim_{s \downarrow 0} S_4^2(s) = \operatorname{Var} X_1 < \infty,$$

thus for any $0 < \lambda < \infty$, we have

$$\left(S_4^2(s) - S_4^2(\lambda s)\right)/S_4^2(s) \to 0 \quad \text{as } s \downarrow 0, \tag{A.27}$$

which implies (A.19). This completes the proof of Proposition A.2. □

REMARK A.1. We could have proved more precise statements of Propositions A.1 and A.2, which would describe the chain of equivalences between slowly varying at zero of the functions in question and their tail condition. In particular, we have: (A.2) *implies* $S_3^2$ *is slowly varying at zero and* (A.6), *which, as seen from the proof of Proposition* A.2, *in turn imply* (A.19) *and* (A.20) *which by Theorem* 2.1 *imply* (A.1) (cf. Corollary 2.2) *for appropriate* $\{A_n\}$ *and* $\{C_n\}$, *which in turn implies* (A.2). *Thus we see that the converses of Propositions* A.1 *and* A.2 *are also true.*

Turning now to stable laws in the light of our quantile function approach, we will show that conditions (3.48) and (3.49), which are assumptions on the tail behaviour of the quantile function $Q$ of $F$, can be reformulated in terms of an equivalent set of conditions describing the tail behaviour of $F$. This is the usual way that the assumptions of Corollary 3.1 are presented [cf., e.g., Feller (1966)]. Namely, we have: *There exist sequences of constants* $\{A_n\}$ *and* $\{C_n\}$ $(n = 1, 2, \dots)$ *such that*

$$A_n\left(\sum_{i=1}^{n} X_i - C_n\right) \to_{\mathscr{D}} D(\alpha), \tag{A.28}$$

*where* $D(\alpha)$ *is a stable rv of index* $\alpha \in (0,2)$ *if and only if there exist a function* $l$ *slowly varying at infinity, an* $0 < \alpha < 2$, *and a* $0 \le p \le 1$ *such that*

$$1 - G(y) = l(y)y^{-\alpha} \quad \text{for } y \ge 0, \tag{A.29}$$

$$\lim_{y \to \infty} (1 - F(y))/(1 - G(y)) = p, \tag{A.30}$$

$$\lim_{y \to \infty} F(-y)/(1 - G(y)) = 1 - p := q. \tag{A.31}$$

As to our equivalent quantile tail conditions, we have

PROPOSITION A.3. *There exist a function $l$ slowly varying at infinity, an $0 < \alpha < 2$ and a $0 \le p \le 1$ such that conditions* (A.29), (A.30), *and* (A.31) *hold if and only if for some function $L$ slowly varying near zero we have* [cf. (3.48) *and* (3.49)]

$$K(1-u) = u^{-1/\alpha}L(u), \tag{A.32}$$

$$\lim_{u \downarrow 0} Q_1(1-u)/K(1-u) = q^{1/\alpha} := w_1, \tag{A.33}$$

*and*

$$\lim_{u \downarrow 0} Q_2(1-u)/K(1-u) = p^{1/\alpha} := w_2 \tag{A.34}$$

*with the same $\alpha$ and $p$ as appear in* (A.29) *and* (A.30).

The proof is an easy application of Lemma 1.10 in Seneta (1976), so the details are omitted.

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