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ON THE PREVALENCE OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH UNIQUE STRONG SOLUTIONS

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It is shown that in the sense of Baire category, almost all stochastic differential equations with uniformly bounded measurable coefficients and uniformly nondegenerate diffusions have unique strong solutions.

1. Introduction. The basic existence and uniqueness theorem of Itô for stochastic differential equations of the Markov type:

$$d\xi(t) = \sigma(\xi(t), t)\, dW(t) + b(\xi(t), t)\, dt, \tag{1.1}$$

where $\xi(t)$ is in R^d, states that when $\sigma(x, t)$ and $b(x, t)$ are Borel measurable in (x, t) and locally Lipschitz continuous with respect to x for each t, then (1.1) has a unique strong solution [4]. In the case where continuity of $\sigma(x, t)$ and $b(x, t)$ with respect to x is discarded and replaced with the requirement that $\sigma(x, t)$ is the unit operator, results about the existence of strong solutions have been obtained by Zvonkin [13] and Veretennikov [12] who have studied the uni- and multidimensional cases, respectively. By using methods drawn from the theory of partial differential equations, Zvonkin obtains a result of pleasing generality, to the effect that the one-dimensional version of (1.1) with $\sigma(x, t)$ set to unity, has a pathwise unique strong solution for each bounded and Borel measurable $b(x, t)$. For the multidimensional case of (1.1) with $\sigma(x, t)$ set to a unit matrix of appropriate dimension, Veretennikov obtains conditions on $b(x, t)$ that ensure the existence of a pathwise unique solution of (1.1); the method used in this case is essentially probabilistic, relying as it does on the verification of pathwise uniqueness, after which a theorem of Yamada and Watanabe (see e.g., [4], Theorem 1.1, Chapter IV) along with the Girsanov theorem furnishes the existence of a unique strong solution.

This note examines the prevalence of stochastic differential equations with measurable coefficients and nondegenerate diffusions, which enjoy the property of having a unique strong solution. Prevalence questions of this general nature were first studied by Orlicz [8] in the case of ordinary differential equations, and subsequently by Alexiewicz and Orlicz [1] for partial differential equations and by Lasota and Yorke [7] for ordinary differential equations assuming values in an infinite dimensional Banach space.

In this note it will be shown that the set of Markov stochastic differential equations of the form (1.1) whose coefficients are Borel measurable in (x, t) and uniformly bounded, and whose diffusions are uniformly nondegenerate, may be regarded as a Baire space (by defining a natural complete metric on it), which

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contains a dense residual subset of the second category of Baire, such that the equation corresponding to each point in this subset has the following property: given an initial data point (x, s) in $R^d \times [0, \infty)$ and an arbitrary triple (E, C, μ) carrying a Wiener process $W(t)$, there is exactly one process $\xi(\cdot)$ defined on $[s, \infty) \times \Omega$, satisfying $\xi(s) = x$, and (1.1) for $t \geq s$, such that $\xi(t)$ is measurable with respect to the completion [in (E, C, μ)] of the σ-algebra $\sigma(W(u), u \leq t)$ for each $t \geq s$. Since a set of the second category in a Baire space contains "almost all" of the points in the space (it may be thought of as the topological analogue of the measure theoretical concept of a set whose complement is of measure zero), this result is saying that in a certain sense almost all stochastic differential equations of Markov type, whose diffusions are uniformly nondegenerate, have a unique strong solution.

2. Genericity of equations allowing unique strong solutions. Stochastic differential equations of the form (1.1) will be considered subject to the requirement that the diffusion coefficients are nondegenerate in some uniform sense to be made precise in the next paragraph. The results to be obtained depend on the use of inequalities due to Krylov [5] and [6], which give estimates on the distribution of a stochastic integral, and the final genericity claim is established by adapting to the present situation an oscillation function used by Lasota and Yorke [7].

For a given natural number d, and real M and $m > 0$, let Y_1 be the set of all Lebesgue measurable functions $\sigma(x, t)$ defined on $R^d \times [0, \infty)$, assuming values in the set of real d by d matrices, such that $\langle z, \sigma(x, t)z \rangle \geq m\|z\|^2$ a.e. on $R^d \times [0, \infty)$ for each z in R^d, and also satisfying $\operatorname{ess\,sup}_{(x,t)}\|\sigma(x, t)\| \leq M$.

Likewise let Y_2 be the set of all Lebesgue measurable and R^d-valued functions $b(x, t)$ defined on $R^d \times [0, \infty)$, and satisfying $\operatorname{ess\,sup}_{(x,t)}\|b(x, t)\| \leq M$.

Define a metric ρ_1 on Y_1 as follows:

$$\rho_1(\sigma_1, \sigma_2) \triangleq \sum_{N=1}^{\infty} 2^{-N} \min\left\{1, \left[\int_{S_N} \|\sigma_1(x, u) - \sigma_2(x, u)\|^{2d+2} \, dx \, du\right]^{1/2d+2}\right\},$$

where $S_N \triangleq \{x \mid \|x\| \leq N\} \times [0, N]$, and on Y_2 define metric ρ_2 in likewise manner with b replacing σ. In these definitions, $\|\cdot\|$ denotes, depending on the context, either the operator norm on the set of real d by d matrices or the euclidean (L_2) norm on the set of real d-vectors. Finally, let Y be the cartesian product of Y_1 and Y_2 and let ρ_y be the product metric on Y. Clearly (Y, ρ_y) is a complete metric space.

Now define the set Y_1^L to be the subset of Y_1 consisting of all functions $\sigma(x, t)$ which are locally Lipschitz continuous on $R^d \times [0, \infty)$. Define Y_2^L in likewise manner,with b replacing σ and Y_2 replacing Y_1, and let Y^L be the cartesian product of Y_1^L and Y_2^L.

PROPOSITION 1. *Y^L is a dense subset of (Y, ρ_y).*

PROOF. Let $\theta(x, t)$ be a nonnegative infinitely differentiable function defined on R^{d+1} with an integral of unity (on R^{d+1}), and assuming the value zero outside the unit ball of R^{d+1}, and define $\sigma^{(\varepsilon)}(x, t) \triangleq \varepsilon^{-(d+1)}\theta(x/\varepsilon, t/\varepsilon) * \sigma(x, t)$, $\varepsilon > 0$, where $*$ denotes convolution on R^{d+1} with respect to (x, t). $\sigma(x, t)$ is a fixed element of Y_1 which is taken to be zero for all x in R^d and $t < 0$ in the above convolution. Clearly $\sigma^{(\varepsilon)}$ belongs to Y^L and for each natural number N, $\lim_{\varepsilon \to 0}\sigma^{(\varepsilon)}(x, t) = \sigma(x, t)$ a.e. on S_N. The claim follows from the dominated convergence theorem. □

Henceforth let (E, C, μ) be a fixed complete probability space with a fixed Wiener process $\{W(t), t \geq 0\}$, defined on it. For $t \geq 0$, let

$$F_t^W \triangleq \sigma\{W(s), 0 \leq s \leq t\} \vee \{\mu\text{-null sets of } C\}.$$

Then $(W(t), F_t^W, t \geq 0)$ is a Wiener martingale. Define:

$$\Xi \triangleq \{\xi: [0, \infty) \times E \to R^d | \xi(t) \text{ is } F_t^W\text{-measurable}, \xi(\cdot) \text{ is continuous on } [0, \infty) \text{ for each point in } E, \text{ and for each } T > 0, E^\mu[\sup_{t \leq T}\|\xi(t)\|^2] < \infty\}.$$

Define a metric ζ on Ξ as follows:

$$\zeta(\xi_1, \xi_2) \triangleq \sum_{N=1}^{\infty} 2^{-N}\min\left\{1, \left(E^\mu\left[\sup_{t \leq N}\|\xi_1(t) - \xi_2(t)\|^2\right]\right)^{1/2}\right\}.$$

PROPOSITION 2. *(Ξ, ζ) is a complete metric space.*

PROOF. The fact that ζ is a metric is easy to verify. The completeness of ζ is a direct consequence of the Borel–Cantelli theorem and the fact that the μ-null sets of C are contained in F_t^W, for each $t \geq 0$. □

Henceforth, a strong solution of the stochastic differential equation corresponding to the pair (σ, b) in Y and an initial data point (x, s) in $R^d \times [0, \infty)$ will be regarded as an element of the set Ξ, which satisfies the following two requirements:

(i) $\xi(t) = x$ for all $0 \leq t \leq s$, and
(ii) for each $t \geq s$,

$$\xi(t) = x + \int_s^t \sigma(\xi(u), u)\, dW(u) + \int_s^t b(\xi(u), u)\, du \quad \text{a.s. } [\mu].$$

The next proposition is a continuous dependence result:

PROPOSITION 3. *Let $\{(\sigma_n, b_n)\}$ and $\{(x_n, s_n)\}$ be sequences in (Y, ρ_y) and $R^d \times [0, \infty)$ converging to the limits $(\{\sigma, b)\}$, in Y^L, and (x, s), respectively. For each n assume that the stochastic differential equation with coefficients σ_n and b_n, from the initial data point (x_n, s_n) has a (not necessarily unique) strong solution ξ_n in Ξ, and let ξ be the unique solution in Ξ to the stochastic differential equation with coefficients σ and b, from the initial point (x, s). Then $\lim_{n \to \infty}\zeta(\xi_n, \xi) = 0$.*

PROOF. Fix $T > 0$. For each natural number N define

$$\tau_n^N \triangleq \inf\{t \geq 0 \,|\, \|\xi(t)\| \geq N \text{ or } \|\xi_n(t)\| \geq N\}. \tag{2.1}$$

Clearly, for each N and n,

$$\begin{aligned} E^{\mu}\Big[\sup_{t\leq T}\|\xi_n(t) - \xi(t)\|^2\Big] &\leq 3E^{\mu}\Big[\sup_{t\leq T}\|\xi_n(t) - \xi_n(t\wedge\tau_n^N)\|^2\Big] \\ &\quad + 3E^{\mu}\Big[\sup_{t\leq T}\|\xi_n(t\wedge\tau_n^N) - \xi(t\wedge\tau_n^N)\|^2\Big] \\ &\quad + 3E^{\mu}\Big[\sup_{t\leq T}\|\xi(t\wedge\tau_n^N) - \xi(t)\|^2\Big]. \end{aligned} \tag{2.2}$$

Now in view of the uniform bound M on the coefficients σ_n and b_n, and the boundedness of the sequence $\{(x_n, s_n)\}$, it follows that $\lim_{N\to\infty}\mu[\sup_{t\leq T}\|\xi_n(t)\| \geq N] = 0$ uniformly with respect to n, whence,

$$\lim_{N\to\infty} \mu\big[\tau_n^N > T\big] = 1 \tag{2.3}$$

uniformly with respect to n.

Now fix an $\varepsilon > 0$, and let $N(\varepsilon)$ be a natural number such that $N(\varepsilon) > T$ and

$$\mu\big[\tau_n^{N(\varepsilon)} > T\big] > (1 - \varepsilon)$$

for all n.

To lighten the notation, let τ_n denote $\tau_n^{N(\varepsilon)}$—no confusion will result as ε is henceforth held fixed. Considering the first term on the right of (2.2) with $N = N(\varepsilon)$:

$$\begin{aligned} E^{\mu}\Big[\sup_{t\leq T}\|\xi_n(t) - \xi_n(t\wedge\tau_n)\|^2\Big] &\leq 2E^{\mu}\Big[\sup_{t\leq T}\Big\|\int_{t\wedge\tau_n}^{t} b_n(\xi_n(u), u)\,du\Big\|^2\Big] \\ &\quad + 2E^{\mu}\Big[\sup_{t\leq T}\Big\|\int_{t\wedge\tau_n}^{t} \sigma_n(\xi_n(u), u)\,dW(u)\Big\|^2\Big] \end{aligned} \tag{2.4}$$

for all n.

Now in view of the definition of $N(\varepsilon)$, the first term on the right of (2.4) can be bounded as follows:

$$E^{\mu}\Big[\sup_{t\leq T}\Big\|\int_{t\wedge\tau_n}^{t} b_n(\xi_n(u), u)\,du\Big\|^2\Big] \leq TM^2\varepsilon$$

for all n, while for the second term Doob's inequality gives

$$\begin{aligned} &E^{\mu}\Big[\sup_{t\leq T}\Big\|\int_{t\wedge\tau_n}^{t} \sigma_n(\xi_n(u), u)\,dW(u)\Big\|^2\Big] \\ &= E^{\mu}\Big[\sup_{t\leq T}\Big\|\int_0^t \chi_{[\tau_n,\infty)}(u)\sigma_n(\xi_n(u), u)\,dW(u)\Big\|^2\Big] \\ &\leq 4E^{\mu}\Big[\int_0^T \chi_{[\tau_n,\infty)}(u)\operatorname{Trace}(\sigma_n\sigma_n^T(\xi_n(u), u))\,du\Big] \\ &\leq 4TM^2\varepsilon \end{aligned}$$

for all n, whence the first term on the right of (2.2) can be bounded from above as follows:

$$E^{\mu}\left[\sup_{t\le T}\|\xi_n(t)-\xi_n(t\wedge\tau_n)\|^2\right]\le 10TM^2\varepsilon \tag{2.5}$$

for all n. An identical assertion holds for the third term on the right of (2.2).

Considering the second term on the right of (2.2) [again with $N=N(\varepsilon)$] and taking $v\le T$, gives:

$$\begin{aligned}
&E^{\mu}\left[\sup_{t\le v}\|\xi_n(t\wedge\tau_n)-\xi(t\wedge\tau_n)\|^2\right]\\
&\quad\le 5\|x_n-x\|^2+5E^{\mu}\left[\sup_{t\le v}\left\|\int_s^{t\wedge\tau_n}b_n(\xi_n(u),u)-b(\xi(u),u)\,du\right\|^2\right]\\
&\qquad+5E^{\mu}\left[\left\|\int_{s_n}^{s}b_n(\xi_n(u),u)\,du\right\|^2\right]\\
&\qquad+5E^{\mu}\left[\sup_{t\le v}\left\|\int_s^{t\wedge\tau_n}\sigma_n(\xi_n(u),u)-\sigma(\xi(u),u)\,dW(u)\right\|^2\right]\\
&\qquad+5E^{\mu}\left[\left\|\int_{s_n}^{s}\sigma_n(\xi_n(u),u)\,dW(u)\right\|^2\right].
\end{aligned}\tag{2.6}$$

In view of [3] (Chapter 4, Theorem 4.7),

$$\begin{aligned}
&\sup_{t\le v}\left\|\int_s^{t\wedge\tau_n}\sigma_n(\xi_n(u),u)-\sigma(\xi(u),u)\,dW(u)\right\|^2\\
&\quad\le 2\sup_{t\le v}\left\|\int_s^{t\wedge\tau_n}\sigma_n(\xi_n(u),u)-\sigma(\xi_n(u\wedge\tau_n),u)\,dW(u)\right\|^2\\
&\qquad+2\sup_{t\le v}\left\|\int_s^{t}\sigma(\xi_n(u\wedge\tau_n),u)-\sigma(\xi(u\wedge\tau_n),u)\,dW(u)\right\|^2.
\end{aligned}\tag{2.7}$$

Now an application of Doob's inequality followed by use of the Krylov inequality ([6], Chapter 2, Theorem 2.4) and the fact that $N(\varepsilon)\ge T$ shows that the expectation of the first term on the right of (2.7) may be bounded from above by:

$$8E^{\mu}\left[\int_0^{T\wedge\tau_n}\Delta_n(\xi_n(u),u)\,du\right]\le 8K\left[\int_{S_{N(\varepsilon)}}\Delta_n(x,t)^{d+1}\,dx\,dt\right]^{1/d+1},$$

where

$$\Delta_n(x,t)\triangleq\operatorname{Trace}(\sigma_n-\sigma)(\sigma_n-\sigma)^T(x,t)$$

and the constant K depends only on d, $N(\varepsilon)$, and m (it does not depend on n). Taking expectations of the two sides of (2.7) and also applying Doob's inequality to the second term on the right, therefore gives:

$$(2.8)\quad \begin{aligned} &E^{\mu}\left[\sup_{t\le v}\left\|\int_s^{t\wedge\tau_n}\sigma_n(\xi_n(u),u) - \sigma(\xi(u),u)\,dW(u)\right\|^2\right] \\ &\le 8K\left[\int_{S_{N(\varepsilon)}}\Delta_n(x,t)^{d+1}\,dx\,dt\right]^{1/d+1} \\ &\quad + 8(L^{\sigma})^2\int_s^v E^{\mu}\left[\sup_{w\le u}\|\xi_n(w\wedge\tau_n) - \xi(w\wedge\tau_n)\|^2\right]du, \end{aligned}$$

where L^{σ} is the Lipschitz constant for σ.

Calculating an upper bound for the second term on the right-hand side of (2.6) in a similar manner, and using this and (2.8) to bound the right-hand side of (2.6) gives, after some simplification:

$$(2.9)\quad \begin{aligned} &E^{\mu}\left[\sup_{t\le v}\|\xi_n(t\wedge\tau_n) - \xi(t\wedge\tau_n)\|^2\right] \\ &\le \eta_n + \left[40(L^{\sigma})^2 + 10(L^{b})^2\right] \\ &\quad \times\int_s^v E^{\mu}\left[\sup_{w\le u}\|\xi_n(w\wedge\tau_n) - \xi(w\wedge\tau_n)\|^2\right]du. \end{aligned}$$

Here L^b is the Lipschitz constant for $b(\cdot)$, and η_n is the sum of the first, third, and fifth terms on the right of (2.6), the first term on the right of (2.8), and a corresponding term in the upper bound of the second term on the right of (2.6). In view of the convergence of x_n to x and (σ_n, b_n) to (σ, b), it follows that $\lim_{n\to\infty}\eta_n = 0$, and therefore an application of the Gronwall inequality to (2.9) shows that

$$\lim_{n\to\infty} E^{\mu}\left[\sup_{t\le T}\|\xi_n(t\wedge\tau_n) - \xi(t\wedge\tau_n)\|^2\right] = 0.$$

In view of (2.2) [with $N = N(\varepsilon)$], (2.5), and the above, it follows that

$$\limsup_{n\to\infty} E^{\mu}\left[\sup_{t\le T}\|\xi_n(t) - \xi(t)\|^2\right] \le 60TM^2\varepsilon.$$

Since ε and T are arbitrary, the proposition follows. □

Henceforth let $\xi(\sigma, b; x, s)(\cdot)$ denote the (unique) strong solution in Ξ of the stochastic differential equation corresponding to the pair (σ, b) in Y^L from the initial data-point (x, s). Define the oscillation function D_1: $Y\times R^d\times[0,\infty)\to[0,\infty)$ as follows:

$$D_1(\sigma, b; x, s) \triangleq \lim_{\delta\to 0}\Big[\sup\big\{\zeta(\xi(\sigma_1, b_1; x, s), \xi(\sigma_2, b_2; x, s))\,\big|\, (\sigma_i, b_i)\in Y^L, \text{ and } \rho_y((\sigma, b),(\sigma_i, b_i)) < \delta \quad \text{for } i = 1,2\big\}\Big].$$

The next claim is a direct consequence of the Ito existence and uniqueness theorem and Proposition 3.

PROPOSITION 4. *For each* (σ, b) *in* Y^L *and* (x, s) *in* $R^d \times [0, \infty)$, *it follows that* $D_1(\sigma, b; x, s) = 0$.

PROPOSITION 5. $D_1(\cdot)$ *is upper semicontinuous at each point of* $Y^L \times R^d \times [0, \infty)$.

PROOF. Fix sequences $\{(\sigma_n, b_n)\}$ and $\{(x_n, s_n)\}$ in Y and $R^d \times [0, \infty)$, converging to limits (σ, b), in Y^L and (x, s), respectively. It must be shown that $\lim_{n \to \infty} D_1(\sigma_n, b_n; x_n, s_n) = 0$. Assuming the contrary, there exists some $\varepsilon > 0$, and a subsequence $\{n_k\}$ such that, for each k there exist $(\sigma^i_{n_k}, b^i_{n_k})$ in Y^L, $i = 1, 2$, which satisfy:

$$\rho_y\big((\sigma_{n_k}, b_{n_k}), (\sigma^i_{n_k}, b^i_{n_k})\big) < 1/n_k \tag{2.10}$$

and

$$\zeta\big(\xi(\sigma^1_{n_k}, b^1_{n_k}; x_{n_k}, s_{n_k}), \xi(\sigma^2_{n_k}, b^2_{n_k}; x_{n_k}, s_{n_k})\big) > \varepsilon/2. \tag{2.11}$$

Thus Proposition 3 and (2.10) imply that

$$\lim_{k \to \infty} \zeta\big(\xi(\sigma^1_{n_k}, b^1_{n_k}; x_{n_k}, s_{n_k}), \xi(\sigma^2_{n_k}, b^2_{n_k}; x_{n_k}, s_{n_k})\big) = 0.$$

Since this contradicts (2.11), the claim follows. □

PROPOSITION 6. *If* $D_1(\sigma, b; x, s) = 0$ *for a pair* (σ, b) *in* Y *and an initial data-point* (x, s), *then the stochastic differential equation with coefficients* σ *and* b *has at least one strong solution in* Ξ *from the initial point* (x, s).

PROOF. For each natural number n there exists a real $\varepsilon_n > 0$ such that $\{\varepsilon_n\}$ decreases monotonically to zero and

$$\sup\big\{\zeta(\xi(\sigma_1, b_1; x, s), \xi(\sigma_2, b_2; x, s)) \,\big|\, (\sigma_i, b_i) \in Y^L, \text{and } \rho_y((\sigma, b), (\sigma_i, b_i)) < \varepsilon_n \quad \text{for } i = 1, 2\big\} < 1/n. \tag{2.12}$$

Now for each n there exists a (σ_n, b_n) in Y^L such that

$$\rho_y((\sigma, b), (\sigma_n, b_n)) < \varepsilon_n,$$

i.e.,

$$\rho_y((\sigma, b), (\sigma_{n+m}, b_{n+m})) < \varepsilon_{n+m} < \varepsilon_n,$$

whence, from (2.12), it follows that

$$\zeta(\xi(\sigma_n, b_n; x, s), \xi(\sigma_{n+m}, b_{n+m}; x, s)) < 1/n$$

for all natural n and m.

Thus by Proposition 2 there exists an $\hat{\xi}$ in Ξ which is the ζ-limit of the Cauchy sequence $\{\xi(\sigma_n, b_n; x, s)\}$. It will now be shown that $\hat{\xi}$ solves the equation from (x, s) with coefficients σ and b. Let θ: $R^d \to [0, \infty)$ be a fixed infinitely differentiable function whose integral on R^d is unity, and assuming a value of zero outside the unit ball of R^d, and for $\delta > 0$ define $\sigma^{(\delta)}(x, t) \triangleq \delta^{-d}\theta(x/\delta) * \sigma(x, t)$, where $*$ denotes convolution on R^d with respect to x. Now since

$$\lim_{N \to \infty} \mu\left[\sup_{u \le t} \|\xi(\sigma_n, b_n; x, s)(u)\| > N\right] = 0 \tag{2.13}$$

uniformly with respect to n, it follows from the continuity of $\sigma^{(\delta)}(x, t)$ in x, that $\sigma^{(\delta)}(\xi((\sigma_n, b_n; x, s))(t), t)$ converges in probability to $\sigma^{(\delta)}(\hat{\xi}(t), t)$ for each $t > s$, whence [10] (page 31, no. 5) gives that $\int_s^t \sigma^{(\delta)}(\xi((\sigma_n, b_n; x, s))(u), u)\, dW(u)$ converges in probability to $\int_s^t \sigma^{(\delta)}(\hat{\xi}(u), u)\, dW(u)$ for each $\delta > 0$ and $t > s$, as $n \to \infty$.

Moreover, an application of the Krylov inequality ([6], Chapter 2, Theorem 3.4) and (2.13) shows that as $\delta \to 0$, so $\int_s^t \sigma^{(\delta)}(\xi((\sigma_n, b_n; x, s))(u), u)\, dW(u)$ converges in probability to $\int_s^t \sigma(\xi((\sigma_n, b_n; x, s))(u), u)\, dW(u)$ uniformly with respect to n, while Fatou's theorem, Theorem 3.4 of [6] (Chapter 2) and the convergence of $\{\xi(\sigma_n, b_n; x, s)\}$ to $\hat{\xi}$ imply (cf. [6], page 90) that some $L > 0$ exists for which

$$E^{\mu}\left[\int_s^t |f(\hat{\xi}(u), u)|\, du\right] \le L\left(\int_{R^d \times [0, \infty)} |f(x, u)|^{d+1}\, dx\, du\right)^{1/d+1}$$

for all real Borel measurable $f(\cdot)$ defined on $R^d \times [0, \infty)$, whence clearly $\int_s^t \sigma^{(\delta)}(\hat{\xi}(u), u)\, dW(u)$ converges in probability to $\int_s^t \sigma(\hat{\xi}(u), u)\, dW(u)$ as $\delta \to 0$. Thus it follows that $\int_s^t \sigma(\xi((\sigma_n, b_n; x, s))(u), u)\, dW(u)$ converges in probability to $\int_s^t \sigma(\hat{\xi}(u), u)\, dW(u)$ as $n \to \infty$. Finally it is easily seen from another application of the Krylov inequality, along with the ρ_1 convergence of σ_n to σ and (2.13), that $\int_{s_n}^t \sigma_n(\xi((\sigma_n, b_n; x, s))(u), u)\, dW(u)$ converges in probability to $\int_s^t \sigma(\xi((\sigma_n, b_n; x, s))(u), u)\, dW(u)$ as $n \to \infty$. A similar claim holds for the integrals involving the drift terms, whence the proposition follows. □

PROPOSITION 7. *There exists a residual subset $\tilde{Y}$ of the Baire space (Y, ρ_y), of the second category of Baire, such that for each (σ, b) in $\tilde{Y}$, the stochastic differential equation with coefficients σ and b has a unique solution in Ξ from each initial data point (x, s) in $R^d \times [0, \infty)$.*

PROOF. Define

$$\hat{Y}(m, n) \triangleq \left\{(\sigma, b) \in Y | D_1(\sigma, b; x, s) < 1/n \text{ for all } \|x\| \le m, 0 \le s \le m\right\}$$

for each natural number n and m.

Now by Proposition 4, $Y^L \subset \hat{Y}(m, n)$, and so (Propositions 5 and 1), $\hat{Y}(m, n)$ contains a dense open subset of (Y, ρ_y).

Define

$$\hat{Y} \triangleq \bigcap_{m, n=1}^{\infty} \hat{Y}(m, n).$$

Clearly, if (σ, b) is in $\hat{Y}$, then by Proposition 6, the corresponding stochastic differential equation has at least one strong solution in Ξ for each initial data-point (x, s). To obtain uniqueness define

$$D_2: \hat{Y} \times R^d \times [0, \infty) \to [0, \infty)$$

as

$$D_2(\sigma, b; x, s) \triangleq \sup\{\zeta(\xi_1, \xi_2) | \xi_i \in \Xi, \text{ and } \xi_i \text{ is a strong solution to the stochastic differential equation from } (x, s) \text{ with coefficients } \sigma \text{ and } b, \text{ for } i = 1, 2\};$$

and

$$\tilde{Y}(m, n) \triangleq \{(\sigma, b) \in \hat{Y} | D_2(\sigma, b; x, s) < 1/n \text{ for all } \|x\| \leq m \text{ and } 0 \leq s \leq m\},$$

for each natural m and n, and let

$$\tilde{Y} \triangleq \bigcap_{m, n=1}^{\infty} \tilde{Y}(m, n).$$

Since (σ, b) in Y^L implies that $D_2(\sigma, b; x, s) = 0$ on $R^d \times [0, \infty)$, it is clear from Proposition 3 that each $\tilde{Y}(m, n)$ contains the intersection of $\hat{Y}$ and a dense and open subset of (Y, ρ_y), and thus $\tilde{Y}$ is a residual subset of (Y, ρ_y). Finally, if (σ, b) is in $\tilde{Y}$ then the corresponding stochastic differential equation has an unique solution in Ξ for each initial data point (x, s). □

3. Remarks.

1. The oscillation function $D_1(\cdot)$ defined immediately before Proposition 4 depends only on the joint distribution of $\xi(\sigma_1, b_1; x, s)$ and $\xi(\sigma_2, b_2; x, s)$ for (σ_i, b_i) in Y^L, $i = 1, 2$. Thus it is invariant with respect to the underlying probability triple (E, C, μ) and Wiener process $W(t)$. Therefore, the residual subset $\tilde{Y}$ in Proposition 7 is invariant with respect to the basic probability structure which drives the stochastic differential equation. It follows that the stochastic differential equation corresponding to a (σ, b) in $\tilde{Y}$ and an (x, s) in $R^d \times [0, \infty)$ admits a pathwise unique solution on any probability space carrying a Wiener process, the solution being a measurable functional of the Wiener process.

2. $\tilde{Y}$ is a residual subset in the product of two Baire spaces namely (Y_1, ρ_1) and (Y_2, ρ_2). The Kuratowski–Ulam category analogue of Fubini's theorem ([9], Theorem 15.1) gives the existence of a residual subset $\tilde{Y}_1$ of (Y_1, ρ_1) with the property that for each σ in $\tilde{Y}_1$ there is a residual subset $\tilde{Y}_2(\sigma)$ of (Y_2, ρ_2) such that if b is in $\tilde{Y}_2(\sigma)$ then the stochastic differential equation with coefficients (σ, b) has a unique strong solution from each initial data-point in $R^d \times [0, \infty)$.

3. In Barlow [2] a class of continuous functions σ: $R \to R$ is constructed such that $0 < \delta < \sigma(x) < K$ for all x, and the stochastic differential equation $d\xi(t) = \sigma(\xi(t))\, dW(t)$ has no strong solution. This collection of stochastic differential equations is therefore a subset of the complement of $\tilde{Y}$ occurring in Proposition 7.

4. The equations corresponding to points in Y have Borel measurable coefficients and nondegenerate diffusions. The nondegeneracy of the diffusion ensures that the stochastic differential equation has at least one weak solution. The set Y may be redefined by discarding the nondegeneracy condition and replacing it

with the requirement that the coefficients $\sigma(x, t)$ and $b(x, t)$ in Y_1 and Y_2 be continuous with respect to x for each t. This is also adequate to ensure the existence of at least one weak solution from each initial point (see [11], 6.1.7 and 8.1.1). If the metric ρ_1 on Y_1 is now defined as follows:

$$\rho_1(\sigma_1, \sigma_2) \triangleq \sum_{N=1}^{\infty} 2^{-N} \min\left\{1, \left[\int_0^N \max_{\|x\| \le N} \|\sigma_1(x, u) - \sigma_2(x, u)\|^2 \, du\right]^{1/2}\right\}$$

and a similar metric is defined on Y_2 (with b replacing σ), then the resulting product metric ρ_y on Y turns it into a complete metric space. It is easily verified that with (Y, ρ_y) thus redefined, all of the preceding propositions as well as remarks (1) and (2) remain valid.

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