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THE LOWER LIMIT OF A NORMALIZED RANDOM WALK

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Let $\{S_n\}$ be a random walk with underlying distribution function $F(x)$ and $\{\gamma_n\}$ be a sequence of constants such that γ_n/n is nondecreasing. A universal integral test is given which determines the lower limit of S_n/γ_n up to a constant scale for $\limsup \gamma_{2n}/\gamma_n < \infty$. The generalized LIL is obtained which contains the main result of Fristedt–Pruitt (1971). The rapidly growing random walks and the limit points of $\{S_n/\gamma_n\}$ are also studied.

1. Introduction. Let $\{X, X_n, n \geq 1\}$ be a sequence of independent random variables with common distribution function $F(x)$ and $\{\gamma_n, n \geq 1\}$ be a sequence of positive constants such that γ_n/n is nondecreasing in n. Put $S_n = X_1 + \cdots + X_n$, $n \geq 1$. We shall study the lower limit of the normalized random walk $\{S_n/\gamma_n\}$. Feller (1946) found an integral test which determines the upper limit of $\{|S_n/\gamma_n|\}$. The limit points of $\{S_n/\gamma_n\}$ have also been studied by Derman–Robbins (1955), Kesten (1970), Erickson–Kesten (1974), Erickson (1976), Pruitt (1981b), Mijnheer (1982), and Griffin (1983). We shall also study the generalized law of the iterated logarithm for asymmetric random variables which has been considered by Fristedt–Pruitt (1971), Klass (1976, 1977), Klass–Teicher (1977), and Pruitt (1981a). The relationship between the results in this paper and the previous papers will be discussed after the statement of the main theorems.

Chow–Zhang (1984) gave an integral test which determines the infinite limit points of $\{S_n/\gamma_n\}$. And it follows from Theorem 2 there that $P\{\liminf S_n/\gamma_n > -\infty\} = 1$ and $E|X| = \infty$ imply

$$P\left\{S_n/\gamma_n = \left(\sum_{i=1}^{n} X_i^+/\gamma_n\right)(1 + o(1)) + o(1)\right\} = 1,$$

where $x^+ = \max(x, 0)$. Therefore, as far as the lower limit is concerned, we only have to deal with the case where $F(0-) = 0$. Unless otherwise stated, the condition $F(0-) = 0$ will be assumed in the sequel.

Set

$$m(x) = E\min(X, x) = \int_0^x (1 - F(t))\,dt, \qquad x \geq 0, \tag{1.1}$$

$$\beta(x) = \gamma(x)/x, \qquad x > 0, \tag{1.2}$$

$$\beta^{-1}(x) = \inf\{y: \beta(y) \geq x\}, \qquad \inf \varnothing = \infty, \qquad x \geq 0, \tag{1.3}$$

$$I(\lambda) = \int_1^\infty x^{-1}\exp\left[-\beta^{-1}(m(x)/\lambda)m(x)/x\right] dx, \qquad \lambda > 0, \tag{1.4}$$

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and

$$(1.5) \qquad \theta = \theta(F, \gamma) = \inf\{\lambda: I(\lambda) = \infty\}, \qquad \inf \varnothing = \infty,$$

where $\gamma(n) = \gamma_n$, $n \geq 1$, $\gamma(0) = 0$, and $\gamma(x)/x$ is continuous and nondecreasing. The following theorem gives an integral test which determines the lower limit of $\{S_n/\gamma_n\}$ up to a scale constant.

THEOREM 1. *Suppose that*

$$(1.6) \qquad \limsup \gamma_{2n}/\gamma_n = \rho < \infty.$$

Then

$$(1.7) \qquad \rho^{-1}\max(1.17, 4/\rho)\theta(F,\gamma) \leq \liminf S_n/\gamma_n \leq \rho e \theta(F,\gamma) \quad a.s.,$$

where $e = 2.7182818\ldots$.

The proof of Theorem 1 is contained in Section 2. Consider the inequalities

$$(1.8) \qquad c_1\theta \leq \liminf S_n/\gamma_n \leq c_2\theta \quad \text{a.s.}$$

In (1.7) c_1 and c_2 are two constants not depending on F (note that there is no assumption on the distribution F in Theorem 1). But c_1 and c_2 are actually functions of ρ and the ratio c_2/c_1 is larger than one. Under various conditions on $\{\gamma_n\}$ and F, (1.8) holds for larger c_1 and smaller c_2. These improvements of (1.7) are put at the end of Section 2.

THEOREM 2. *Suppose that $m(x)$ is a slowly varying function and that*

$$(1.9) \qquad \inf\left\{ \limsup_{n<k<wn} \gamma_k/\gamma_n : w > 1 \right\} = 1.$$

Then $P\{\liminf S_n/\gamma_n = \theta\} = 1$.

COROLLARY 1. *Suppose that $m(x)$ is a slowly varying function. Then*

$$(1.10) \qquad \liminf S_n/b_n = 1 \quad a.s.,$$

where the constants b_n, $n \geq 1$, are defined by

$$(1.11) \quad b(x) = \inf\{xm(y): (y/m(y))\log_2 y \geq x\}, \qquad \log_2 y = \left(\log(\log y)^+\right)^+,$$

and

$$b_n = b(n), \qquad n \geq 1.$$

REMARK. (1.9) is the condition (2.9) in Kesten (1970, Theorem 2).

(1.12) $\{S_n\}$ is said to obey a generalized law of the iterated logarithm if there exist constants $\{b_n, n \geq 1\}$, $0 < b_n \to \infty$ such that $\liminf S_n/b_n = 1$ a.s.

In Corollary 1 these constants $\{b_n, n \geq 1\}$ are found for the case where $m(x)$ is slowly varying. The following theorem shows that (1.12) holds under a quite general condition.

THEOREM 3. *Suppose that* $P\{\lim S_n = \infty\} = 1$. *Then at least one of the following two conditions is satisfied*:

(i) $P\{|X| > x\}$ *is a slowly varying function as* x *tends to infinity*;
(ii) (1.12) *holds for some* $\{b_n, n \geq 1\}$ *such that* b_n/n *is nondecreasing.*

REMARKS. 1. Theorem 3 holds without the assumption that $F(0-) = 0$. 2. $P\{\lim S_n = \infty\} = 1$ is necessary for (1.12). 3. If $E|X| = \infty$ and b_n/n is nondecreasing, $P\{\limsup S_n/b_n = 1\} = 0$ by the SLLN of Feller (1946).

The proof of Theorem 2 is contained in Section 2, and the proof of Theorem 3 is contained in Section 3.

Following Theorem 6 of Kesten (1970), the integral test of Erickson (1973) determines the lower limit of $\{S_n/n\}$. Complete information about $\liminf n^{-1/\alpha}S_n$ may also be obtained by the integral test [equivalent to (1.4)] of Mijnheer (1982) under the condition that $x^\alpha(1 - F(x))$ is slowly varying and monotone for $0 < \alpha < 1$. Pruitt (1981b) pointed out that $\liminf n^{-1/\alpha}S_n$ is either zero or infinity except for the critical power $\alpha = \delta$, but no information about $\liminf n^{-1/\delta}S_n$ is given in his paper. The interest of Theorem 1 is that it assumes no condition on the distribution function $F(x)$ and a very general condition (1.6) on the normalizing constants, and that the test integral (1.4) is simple. The disadvantage of Theorem 1 is that the lower limit is not as fully determined as in Theorem 2.1 of Mijnheer (1982) for the special case. However, our integral test does give the complete information about $\liminf S_n/\gamma_n$ in Theorem 2 under the condition that $m(x)$ is slowly varying which covers a case left open by the previous authors. The generalized LIL derived from Theorem 2 (Corollary 1) is equivalent to Theorem 9.2 of Pruitt (1981a) which contains Theorem 4 of Klass–Teicher (1977) (in the sense that the ratio of our normalizing constants and theirs tends to a positive constant). For the generalized LIL, finding necessary and sufficient conditions for (1.12) was stated as an open problem in Pruitt (1981a). Fristedt–Pruitt (1971) proved the validity of (1.12) under the condition that $F(0-) = 0$ and $EX^p < \infty$ for some $p > 0$. Assume that $F(0-) = 0$. It follows from (i) of Theorem 3 that $EX^p \geq \lim x^p(1 - F(x)) = \infty$ for any $p > 0$. Therefore, the condition for (1.12) in Theorem 3 is weaker than that in Theorem 4 of Fristedt–Pruitt (1971). It is an interesting fact, by Theorem 3, that the validity of (1.12) is not completely determined by the "fatness" of the tail $1 - F(x)$. In fact, for any distribution function $F(x)$, an $F^*(x)$ with "fatter" tail $(F^*(x) \leq F(x))$ may be constructed such that (ii) of Theorem 3 holds for the random walk with underlying distribution function $F^*(x)$. In Section 3 Theorem 3 is restated more precisely as Theorem 3* where the sequence b_n, $n \geq 1$, is defined and a lower bound $\theta_* > 0$ and an upper bound $\theta^* < \infty$ are found for the lower limit. The lower bound θ_* and the upper bound θ^* are both equal to one (Corollary 2, Section 3) for a special case which is not covered by the previous studies. A random walk $\{S_n\}$ is said to be rapidly growing if the underlying distribution function satisfies (i) of Theorem 3. Some results about rapidly growing random walks are presented in Section 4. Conditions are given under which $\lim S_n/\max_{j\le n}X_j = 1$, a.s. and/or $\{S_n\}$ does not obey the generalized LIL (1.12). Further discussions about this problem are also included in Section 4. A brief discussion about the limit points of the normalized random walk is contained in Section 5.

2. The integral test. Let $X, X_1, X_2, \ldots$ be a sequence of independent nonnegative random variables with common distribution function $F(x)$ and $\gamma(x)$ be a positive function such that $\gamma(x)/x$ is nondecreasing. We shall assume that $\gamma(x)$ is continuous in x, $\lim_{x\to 0}\gamma(x)/x = 0$, $\lim_{x\to\infty}\gamma(x)/x = \infty$, $EX = \infty$, and

$$\gamma(2x)/\gamma(x) \le M < \infty. \tag{2.1}$$

Set

$$\beta(x) = \gamma(x)/x \quad \text{and} \quad \beta^{-1}(x) = \inf\{y: \beta(y) = x\}, \tag{2.2}$$

$$m(x) = \int_0^x P\{X > t\}\,dt. \tag{2.3}$$

And define the functions $h(x, \lambda)$ and constants u_k, v_k, $k = 1, 2, \ldots$ by

$$m(h(x,\lambda)) = \int_0^{h(x,\lambda)} P\{X > t\}\,dt = \lambda\beta(x), \qquad \lambda > 0, \tag{2.4}$$

$$\beta(u_k) = w^k \quad \text{and} \quad \gamma(v_k) = w^k, \qquad w > 1. \tag{2.5}$$

We shall keep the above notations and assumptions in this section.

Lemma 1. *Let $Y, Y_1, \ldots, Y_n$ be i.i.d. random variables and $\{f_1, \ldots, f_N\}$ be a submartingale with $f_N = Y_1 + \cdots + Y_n$. Then for any positive t and real x*

$$P\Big\{\max_{1\le j\le N} f_j \ge x\Big\} \le e^{-tx}(Ee^{tY})^n. \tag{2.6}$$

In addition, if $EY = 0$, $EY^2 = \sigma^2 < \infty$, $P\{Y \le C\} = 1$, and $x > 0$, then

$$P\Big\{\max_{1\le j\le N} f_j \ge x\Big\} \le \exp\big[-xC^{-1}G(xC/(n\sigma^2))\big], \tag{2.7}$$

where $G(y) = \int_0^y \log(1 + t)\,dt/y$ for any $y > 0$.

This lemma is the combination of Example 7.4.7 and Lemma 10.2.1 of Chow and Teicher (1978, pages 244 and 338). We take the constant b to minimize the right-hand side of Lemma 10.2.1 there. We shall use this lemma to obtain the bounds of probabilities.

Lemma 2. *If $P\{S_n/\gamma_n \le M\lambda \text{ i.o.}\} = 0$, then*

$$\sum_{k=1}^{\infty} P\{S_n/\gamma_n \le \lambda, \text{ for some } v_k \le n < v_{k+1}\} < \infty. \tag{2.8}$$

Remark. One can trace an analogy between this lemma and Theorem 2 of Kesten (1970).

PROOF. Let $p(x) = P\{S_n/\gamma_n > M\lambda, \text{ all } n \geq x\}$ and m, K be two integers such that $w^{m-1} \geq M$ and $p(x) \geq \frac{1}{2}$ for all $x \geq v_K$. Then by (2.1) and (2.5)

$$x - v_{k+1} \geq x/2 \geq v_{k+1}$$

and

$$M\gamma(x - v_{k+1})/\gamma(x) \geq \gamma(x - v_{k+1})/\gamma(x/2) \geq 1 \quad \text{for } x \geq v_{k+m},$$

$$\begin{aligned}
&\frac{1}{2}\sum_{k=K}^{\infty} P\{S_n/\gamma_n \leq \lambda \text{ for some } v_k \leq n < v_{k+1}\} \\
&\quad \leq \sum_{k=K}^{\infty} P\{S_n/\gamma_n \leq \lambda \text{ some } v_k \leq n < v_{k+1}\} p(v_{k+m} - v_{k+1}) \\
&\quad \leq \sum_{k=1}^{\infty} P\big\{S_n/\gamma_n \leq \lambda \text{ some } v_k \leq n < v_{k+1} \text{ and} \\
&\qquad\qquad S_j/\gamma_j > M\lambda\gamma(j - v_{k+1})/\gamma(j) \text{ for all } j \geq v_{k+m}\big\} \\
&\quad \leq \sum_{k=1}^{\infty} P\big\{S_n/\gamma_n \leq \lambda \text{ some } v_k \leq n < v_{k+1} \text{ and} \\
&\qquad\qquad S_j/\gamma_j > \lambda \text{ for all } j \geq v_{k+m}\big\} \\
&\quad = \sum_{k=1}^{\infty} q_k \text{ (say)} = \sum_{i=1}^{m}\sum_{k=0}^{\infty} q_{km+i} \leq m < \infty. \square
\end{aligned}$$

LEMMA 3. *Let c^* be the constant satisfying $\int_0^{c^*}\log(1 + t)\,dt = c^*G(c^*) = 1$. Suppose that* (2.8) *holds. Then*

$$\int_1^{\infty} x^{-1}\exp\left[-\beta^{-1}(m(x)/(c\lambda))m(x)/x\right] dx < \infty \quad \textit{for } c < (1 + c^*)^{-1}.$$

REMARK. $c^* = e - 1$ and $(1 + c^*)^{-1} = 1/e$ ($e = 2.7182818\ldots$).

PROOF. Let $\delta < 1/e$ and set $X' = \min(X, h(n, \delta\lambda))$ and $S_n' = \sum_{i=1}^{n}\min(X_i, h(n, \delta\lambda))$. Then

$$\begin{aligned}
P\{S_n/\gamma_n > \lambda\} &\leq (\lambda\gamma_n)^{-1}ES_n' + P\{S_n \neq S_n'\} \\
&= \delta + P\{S_n \neq S_n'\}.
\end{aligned} \tag{2.9}$$

Set $p = P\{X > h(n, \delta\lambda)\}$, we may assume that $p < \frac{1}{2}$,

$$\begin{aligned}
P\{S_n = S_n'\} &= P\{X_i \leq h(n, \delta\lambda), i = 1, \ldots, n\} \\
&\geq \exp[-np(1 + p)] \\
&\geq \exp[-(1 + p)nm(h(n, \delta\lambda))/h(n, \delta\lambda)] \\
&= \exp[-(1 + p)\delta\lambda\gamma(n)/h(n, \delta\lambda)].
\end{aligned} \tag{2.10}$$

On the other hand, it follows from Lemma 1 that

$$P\{S'_n/\gamma_n \geq \lambda\} = P\{S'_n - ES'_n \geq (1-\delta)\lambda\gamma_n\}$$
$$\leq \exp\Big[-(1-\delta)\lambda\gamma_n(h(n,\delta\lambda))^{-1}$$
$$\times G\big((1-\delta)\lambda\gamma_n h(n,\delta\lambda)/\big(nE(X')^2\big)\big)\Big].$$

Since $G(\cdot)$ is increasing and $E(X')^2 \leq h(n,\delta\lambda)\delta\lambda\beta(n)$, $n\beta(n) = \gamma_n$,

$$P\{S'_n/\gamma_n \geq \lambda\} \leq \exp\big[-(1-\delta)\lambda\gamma_n(h(n,\delta\lambda))^{-1}G((1-\delta)/\delta)\big].$$

It follows from the assumption $\delta < (1+c^*)^{-1}$ that

$$(1-\delta)/\delta > c^*$$

and

$$(1-\delta)\delta^{-1}G((1-\delta)/\delta) > c^*G(c^*) = 1.$$

Set $\varepsilon = (1-\delta)\delta^{-1}G((1-\delta)/\delta) - 1 > 0$. Then

$$P\{S'_n/\gamma_n \geq \lambda\} \leq \exp[-(1+\varepsilon)\delta\lambda\gamma_n/h(n,\delta\lambda)],$$

$$\begin{aligned} P\{S_n/\gamma_n \leq \lambda\} &\geq P\{S_n = S'_n\} - P\{S'_n/\gamma_n \geq \lambda\} \\ &\geq \exp[-(1+p)\delta\lambda\gamma_n/h(n,\delta\lambda)] - \exp[-(1+\varepsilon)\delta\lambda\gamma_n/h(n,\delta\lambda)] \\ &= \exp[-(1+p)\delta\lambda\gamma_n/h(n,\delta\lambda)] \\ &\quad \times(1 - \exp[-(\varepsilon-p)\delta\lambda\gamma_n/h(n,\delta\lambda)]). \end{aligned} \tag{2.11}$$

Since $P\{S_n/\gamma_n \leq \lambda\} = o(1)$ as $n \to \infty$, either $\liminf \gamma_n/h(n,\delta\lambda) = 0$ or $\lim \gamma_n/h(n,\delta\lambda) = \infty$. Assume that $\gamma_{n'}/h(n',\delta\lambda) \to 0$ for $n' \to \infty$. Then

$$\begin{aligned} P\{S_{n'}/\gamma_{n'} \geq \lambda\} &\leq P\{S'_{n'}/\gamma_{n'} \geq \lambda\} + P\{S_{n'} \neq S'_{n'}\} \\ &\leq \delta + o(1) \quad \text{by (2.9) and (2.10)}. \end{aligned}$$

This is a contradiction to $P\{S_n/\gamma_n \leq \lambda\} = o(1)$. Therefore,

$$P\{S_n/\gamma_n \leq \lambda\} \geq 2^{-1}\exp[-(1+p)\delta\lambda\gamma_n/h(n,\delta\lambda)] \quad \text{for large } n. \tag{2.12}$$

Let n be the smallest integer in $[v_k, \infty)$, $n - 1 < v_k \leq n$.

$$\begin{aligned} &v_k m(h(v_k,\delta\lambda))/h(v_k,\delta\lambda) \\ &\quad \geq v_k m(h(n,\delta\lambda))/h(n,\delta\lambda) \quad [m(x)/x \text{ is nonincreasing}] \\ &\quad \geq nm(h(n,\delta\lambda))/h(n,\delta\lambda) - m(h(n,\delta\lambda))/h(n,\delta\lambda). \end{aligned} \tag{2.13}$$

Since $\gamma(2^k) \leq M^k\gamma(1)$, $\lim_k k^{-1}\log v_k > 0$ and $v_k > 2k^2$ for large k. Let $h_n = h(n,\delta\lambda)$. If $(np+1)m(h_n)/h_n > \log 2$, then $n(m(h_n)/h_n)^2 > npm(h_n)/h_n > \frac{1}{2}$

and $nm(h_n)/h_n > (n/2)^{1/2} > k$ for large k. By (2.10), (2.12), and (2.13)

$$\begin{aligned}\exp[-\delta\lambda\gamma(v_k)/h(v_k,\delta\lambda)] &= \exp[-v_k m(h(v_k,\delta\lambda))/h(v_k,\delta\lambda)]\\ &\le \exp[-(n-1)m(h_n)/h_n]\\ &= \exp[-(1+p)nm(h_n)/h_n + (np+1)m(h_n)/h_n]\\ &= \exp[-(1+p)\delta\lambda\gamma_n/h(n,\delta\lambda)\\ &\qquad +(np+1)m(h_n)/h_n]\\ &\le \exp[-(1+p)\delta\lambda\gamma_n/h(n,\delta\lambda)+\log 2]\\ &\quad +\exp[-k+m(h_n)/h_n]\\ &\le 4P\{S_n/\gamma_n \le \lambda\} + 2e^{-k} \quad \text{for large } k.\end{aligned} \tag{2.14}$$

Let m be an integer such that $w^m \ge M$. $v_{k+m} \ge 2v_k \ge v_k + 1$ for large k, since $\gamma(2x)/\gamma(x) \le M$. By (2.14) and (2.8)

$$\begin{aligned}&\sum_{k=K}^{\infty} \exp[-\delta\lambda\gamma(v_k)/h(v_k,\delta\lambda)]\\ &\quad\le \sum_k 4P\{S_n/\gamma_n \le \lambda \text{ for some } v_k \le n < v_{k+m}\} + \sum_k 2e^{-k}\\ &\quad\le 4m\sum_k P\{S_n/\gamma_n \le \lambda \text{ for some } v_k \le n < v_{k+1}\} + 2(e-1)^{-1}\\ &\quad< \infty.\end{aligned} \tag{2.15}$$

Set $x_k = h(v_k, \delta\lambda)$. Then $m(x_k) = \delta\lambda\beta(v_k) \le wm(x_{k-1})$ for $k \ge 2$.

$$v_k \le \beta^{-1}(wm(x_k)/(\delta\lambda)),$$

$$\delta\lambda\gamma(v_k)/h(v_k,\delta\lambda) \le \beta^{-1}(wm(x_k)/(\delta\lambda))m(x_k)/x_k,$$

$$\begin{aligned}&\int_{y/K}^{y} x^{-1}\exp[-\beta^{-1}(w^2m(x)/(\delta\lambda))m(x)/x]\,dx\\ &\quad\le \int_{y/K}^{y} x^{-1}\exp[-\beta^{-1}(wm(x_k)/(\delta\lambda))m(y)/y]\,dx\\ &\quad\le (\log K)\exp[-\beta^{-1}(wm(x_k)/(\delta\lambda))m(y)/y]\\ &\qquad\qquad \text{for } x_{k-1} \le y/K < y \le x_k,\end{aligned} \tag{2.16}$$

$$\begin{aligned}&\int_{x_{k-1}}^{x_k} x^{-1}\exp[-\beta^{-1}(w^2m(x)/(\delta\lambda))m(x)/x]\,dx\\ &\quad\le \sum_{j\ge 0} (\log K)\exp\left[-\beta^{-1}(wm(x_k)/(\delta\lambda))(K/w)^j m(x_k)/x_k\right]\\ &\quad\le (\log K)\exp[-\beta^{-1}(wm(x_k)/(\delta\lambda))m(x_k)/x_k](1-o(1))^{-1}\\ &\qquad\qquad \text{for } K > w.\end{aligned} \tag{2.17}$$

Therefore, by (2.15), (2.16), and (2.17)

$$\int_1^{\infty} x^{-1}\exp[-\beta^{-1}(w^2m(x)/(\delta\lambda))m(x)/x]\,dx < \infty.$$

The proof is finished by choosing w and δ such that $\delta < 1/e$, $w > 1$, and $c < \delta/w^2 < 1/e$. □

LEMMA 4. *Suppose that*

$$\int_1^{\infty} x^{-1}\exp\left[-\beta^{-1}(m(x)/\lambda)m(x)/x\right]dx < \infty. \tag{2.18}$$

Then for every constant $c < \max((2/M)^2/3.415, (2/M)^3/2)$,

$$P\{S_n/\gamma_n < c\lambda \ i.o.\} = 0.$$

PROOF. Consider the case where $n - 1 < u_{k-1} \le n \le u_k$ for some integer n. Set $X' = \min(X, h(n, Kb))$ and $S_j' = \sum_{i=1}^{j}\min(X_i, h(n, Kb))$, $K > 1$. Then $ES_j'/j = Kb\gamma_n/n$ and $\{S_j/j,\ j = \cdots, n+1, n\}$ is a martingale. By Lemma 1

$$\begin{aligned} &P\{S_j/\gamma_j < b/w \text{ for some } u_{k-1} \le j < u_k\} \\ &\quad \le P\{S_j'(n/j) < b\gamma_n \text{ for some } n \le j < u_k\} \\ &\quad = P\{n(ES_j' - S_j')/j > (K-1)b\gamma_n \text{ for some } n \le j < u_k\} \\ &\quad \le \exp\left[-(K-1)b\gamma_n(EX')^{-1}G((K-1)b\gamma_n EX'/(nh(n,Kb)EX'))\right] \\ &\quad = \exp\left[-(K-1)b\gamma_n(Kb\gamma_n/n)^{-1}G((K-1)EX'/(Kh(n,Kb)))\right] \\ &\quad = \exp\left[-(K-1)K^{-1}nG((K-1)K^{-1}m(h(n,Kb))/h(n,Kb))\right]. \end{aligned}$$

Since $1 \ge m(x)/x \to 0$ as $x \to \infty$ and $G(x)/x \to \frac{1}{2}$ as $x \to 0$,

$$\begin{aligned} &P\{S_j/\gamma_j < b/w \text{ for some } u_{k-1} \le j < u_k\} \\ &\quad \le \exp\left[-(K-1)^2K^{-2}nm(h(n,Kb))/(h(n,Kb)2w)\right] \quad \text{for large } k. \end{aligned} \tag{2.19}$$

Let $x_k = h(u_k, Kb)$. $u_{k-1} \ge \beta^{-1}(m(x_{k-1})/(Kb)) = \beta^{-1}(m(x_k)/(wKb))$. Since $\beta(2x) \le (M/2)\beta(x)$, $\beta^{-1}((2/M)x) \le \beta^{-1}(x)/2$. Therefore,

$$\begin{aligned} &(K-1)^2K^{-2}(2w)^{-1}nm(h(n,Kb))/h(n,Kb) \\ &\quad \ge (K-1)^2K^{-2}(2w)^{-1}u_{k-1}m(h(u_k,Kb))/h(u_k,Kb) \\ &\quad \ge (K-1)^2K^{-2}(2w)^{-1}\beta^{-1}(m(x_k)/(wKb))m(x_k)/x_k \\ &\quad \ge (1-1/K)^2(2/w)\beta^{-1}\left((2/M)^2m(x_k)/(wKb)\right)m(x_k)/x_k \\ &\quad \ge (1-1/K)^2(4/w)\beta^{-1}\left((2/M)^3m(x_k)/(wKb)\right)m(x_k)/x_k. \end{aligned} \tag{2.20}$$

Set $1 < w < 2$,

$$(1-1/K_1)^2(2/w) = 1, \qquad K_1 = K_1(w), \qquad \lim_{w\to 1} K_1(w) < 3.415, \tag{2.21}$$

$$(1-1/K_2)^2(4/w) = 1, \qquad K_2 = K_2(w), \qquad \lim_{w\to 1} K_2(w) = 2. \tag{2.22}$$

Choose $w > 1$ such that $w - 1$ is small enough and that

$$w^3 c < \max\big((2/M)^2/K_1(w), (2/M)^3/K_2(w)\big). \tag{2.23}$$

It follows from (2.19)–(2.23) that

$$\begin{aligned} &P\{S_j/\gamma_j < b/w \text{ for some } u_{k-1} \le j < u_k\} \\ &\quad \le \exp\big[-\beta^{-1}\big(w^2 cm(x_k)/b\big)m(x_k)/x_k\big]. \end{aligned} \tag{2.24}$$

Since $m(x_k) = Kbw^k$, $m(x_{k+1})/m(x_k) = w$.

$$\begin{aligned} &x_{k+1}/x_k \ge w, \\ &\int_{x_k}^{x_{k+1}} x^{-1}\exp\big[-\beta^{-1}(m(x)/\lambda)m(x)/x\big]\,dx \\ &\quad \ge \exp\big[-\beta^{-1}\big(wm(x_k)/\lambda\big)m(x_k)/x_k\big]\int_{x_k}^{x_{k+1}} x^{-1}\,dx \\ &\quad \ge \exp\big[-\beta^{-1}\big(wm(x_k)/\lambda\big)m(x_k)/x_k\big](\log w). \end{aligned} \tag{2.25}$$

Set $b/w = c\lambda$. It follows from (2.24), (2.25), and (2.18) that $P\{S_n/\gamma_n < c\lambda \text{ i.o.}\} = 0$. □

Proof of Theorem 1. Since the behavior of $\gamma(x)$ on a compact interval has no effect on the lower limit, we may assume that $M = \rho + \varepsilon$ for any $\varepsilon > 0$.

Suppose that $\theta = \theta(F, \gamma) > \lambda > 0$. Then $I(\lambda) < \infty$ and by Lemma 4

$$\liminf S_n/\gamma_n \ge \lambda \max\big((2/M)^2/3.415, (2/M)^3/2\big) \quad \text{a.s.} \tag{2.26}$$

Let $\lambda \to \theta$ and $\varepsilon \to 0$. We have

$$\liminf S_n/\gamma_n \ge \rho^{-2}\max(1.17, 4/\rho)\theta(F, \gamma) \quad \text{a.s.} \tag{2.27}$$

Suppose that $\theta < \lambda/e < \infty$. Then $I(c\lambda) = \infty$ for some $c < 1/e$. It follows from Lemma 2, Lemma 3, and Hewitt–Savage zero-one law that

$$\sum_k P\{S_n/\gamma_n \le \lambda \text{ for some } v_k \le n < v_{k+1}\} = \infty \tag{2.28}$$

and

$$\liminf S_n/\gamma_n \le M\lambda \quad \text{a.s.} \tag{2.29}$$

Let $\lambda \to e\theta$ and $\varepsilon \to 0$.

$$\liminf S_n/\gamma_n \le \rho e\theta \quad \text{a.s.} \tag{2.30}$$ □

Proof of Theorem 2. Since $m(x)$ is slowly varying,

$$\begin{aligned} \operatorname{Var}(\min(X, h)) &\le E(\min(X, h))^2 \le 2\int_0^h x(1 - F(x))\,dx \\ &= 2hm(h) - 2\int_0^h m(x)\,dx = o(1)hm(h) \quad \text{as } h \to \infty. \end{aligned} \tag{2.31}$$

Let $X' = \min(X, h)$ and $h = h(n, Kb)$.

$$\lim_n G\big((K-1)b\gamma_n EX'/(n\operatorname{Var}(X'))\big)/G\big((K-1)EX'/(Kh)\big) = \infty.$$

Therefore, the right-hand side of (2.19) may be replaced by $\exp[-nm(h)/h]$ and K may be taken close to one. Following the proof of Lemma 4 line by line, we have

$$I(\lambda) < \infty \text{ implies } P\{S_n/\gamma_n < c\lambda \text{ i.o.}\} = 0 \quad \text{for any } c < 1. \tag{2.32}$$

Similarly, by (2.31) and the method in the proof of Lemma 3, we have

$$(2.8) \text{ implies } I(c\lambda) < \infty \quad \text{for any } c < 1. \tag{2.33}$$

Also, we can make use of Theorem 2 of Kesten (1970) because of (1.9). The rest of the proof is obvious. □

Proof of Corollary 1. Assume that $EX = \infty$. Since $y/m(y)$ and $\log_2 y$ are increasing and continuous, $\beta((x/m(x))\log_2 x) = m(x)$, where $\beta(x) = b(x)/x$.

$$\beta^{-1}(m(x))m(x)/x = \log_2 x, \qquad x > 0. \tag{2.34}$$

It is easy to verify that $b(x)$ is a regular varying function with exponent one. Hence, (1.9) holds and $\theta(F, b) = 1$. □

Let us consider improving (1.7) under various conditions.

Theorem 4. *Let $\theta(F,\gamma)$ be defined by* (1.5) *and* $0 < \alpha < 1$.

(i) *Let* $\rho(w) = \limsup \gamma(wx)/\gamma(x)$, $w > 1$. *Then*

$$c_1\theta(F,\gamma) \le \liminf S_n/\gamma_n \quad a.s. \tag{2.35}$$

holds for $c_1 = \sup\{(w/\rho(w))^n/K_n(w)\colon w > 1 \text{ and } n = 1,2,\dots\}$, *where* $K_n(w)$ *is defined by* $(1 - 1/K_n(w))^2/2 = w^{-n}$.

(ii) *Suppose that* (1.9) *holds. Then*

$$\liminf S_n/\gamma_n \le c_2\theta(F,\gamma) \quad a.s. \tag{2.36}$$

holds for $c_2 = e = 2.7182818\dots$.

(iii) *Suppose that* $\gamma(x)$ *is regular varying with exponent* $1/\alpha$. *Then* (1.9) *holds and* (2.35) *holds for*

$$c_1 = 2^{(1-\alpha)/\alpha}[(1-\alpha)/(2-\alpha)]^{2(1-\alpha)/\alpha}[\alpha/(2-\alpha)].$$

(iv) *Suppose that* $1 - F(x)$ *is regular varying with exponent* $-\alpha$ *and that* $\gamma(x)$ *is regular varying with exponent* $1/\alpha$. *Then* (2.35) *holds for* $c_1 = [(1-\alpha)/(2-\alpha)]^{(1-\alpha)/\alpha}[\alpha/(2-\alpha)]$ *and* (2.36) *holds for*

$$c_2 = \left[\alpha(1-\alpha)(2-\alpha)^{-1}c^*(\alpha) + 1\right](1-\alpha)^{(1-\alpha)/\alpha}, \tag{2.37}$$

where $c^*(\alpha)G(c^*(\alpha)) = (2-\alpha)/\alpha$.

Remark. $\alpha c^*(\alpha)\log c^*(\alpha) \to 2$ as $\alpha \to 0$. It follows that in (2.37) $c_2 = e^{-1}(1 + o(1))$ as $\alpha \to 0$.

Proof. (i) Similar to the proof of Theorem 1. $K_1 = K_1(2)$ in (2.21) is replaced by $K_n(w)$ for some w and n such that $(w/\rho(w))^n/K_n(w)$ is close to the sup.

(ii) Use Theorem 2 of Kesten (1970) instead of Lemma 2.

(iii) $\beta^{-1}(x)$ is regular varying with exponent $\alpha/(1-\alpha)$. Choose K in the proof of Lemma 4 to be $(2-\alpha)/\alpha$ (optimal value).

(iv) The relations

$$\lim_x x(1 - F(x))/m(x) = 1 - \alpha$$

and

$$\lim_x xm(x)/\mathrm{Var}(XI\{X \le x\}) = (1-\alpha)^{-1}(2-\alpha)/\alpha$$

are used in the proof of Lemma 3. And the relation

$$\lim_x xm(x)/\mathrm{Var}(\min(X, x)) = (1-\alpha)^{-1}(2-\alpha)/2$$

is used in the proof of Lemma 4. The arguments in the proof of (ii) and (iii) are also used. □

3. The generalized law of the iterated logarithm. In this section we shall assume that $F(0-) = 0$, $EX = \infty$, and that

$$\liminf_x (1 - F(2x))/(1 - F(x)) < 1. \tag{3.1}$$

Define

$$\rho = \rho(w) = 1 - \liminf(1 - F(wx))/(1 - F(x)), \qquad w > 1, \tag{3.2}$$

$$\rho^* = \rho^*(w^*) = \inf\{\rho(w): w > w^*\}, \qquad w^* \ge 1, \tag{3.3}$$

$$\theta_* = \theta_*(\rho^*) \text{ such that } \theta_* - \theta_* \log\theta_* = \rho^*, \qquad \theta_* \le 1, \qquad 0 < \rho^* \le 1, \tag{3.4}$$

$$\begin{aligned}&\theta^* = \theta^*(w^*, \rho^*) \text{ such that } \theta^* \ge u, \text{ and}\\ &(\theta^*/u)\log(\theta^*/u) - (\theta^*/u) + 1 = w^*(1-\rho^*)/u,\end{aligned} \tag{3.5}$$

where

$$u = \max(1, v + w^*\rho^*) \tag{3.6}$$

and

$$\begin{aligned} v &= 2(1-\theta_*)^{-2}(1-\rho^*) - 1 \quad \text{if } \rho^* < 1,\\ &= 0 \quad \text{if } \rho^* = 1.\end{aligned} \tag{3.7}$$

For fixed w^* and ρ^*, choose constants $w_k, \rho_k, h_k, n_k, \theta_k, v_k, c_k$, $k \ge 1$, and function $b(x)$, $x \ge 1$, as follows:

$$\lim_k w_k = w^* \quad \text{and} \quad \lim_k \rho_k = \rho^*, \qquad \rho_k < 1, \tag{3.8}$$

$$1 - F(w_k h_k) \le (1-\rho_k)(1 - F(h_k)) \le 1 - F(w_k h_k -), \tag{3.9}$$

$$n_k(1 - F(h_k)) = (1-\rho_k)^{-1}\log k, \tag{3.10}$$

$$\theta_k - \theta_k \log\theta_k = \rho_k, \qquad \theta_k < 1, \tag{3.11}$$

$$v_k = 2(1-\theta_k)^{-2}(1-\rho_k) - 1, \qquad v_k > 0 \quad \text{since } \rho_k < 1, \tag{3.12}$$

$$\begin{aligned} c_k &= n_k h_k(1 - F(h_k)) \quad \text{if } EXI\{X \le h_k\} \le v_k h_k(1 - F(h_k))\\ &= n_k m(h_k), \quad \text{otherwise}, \qquad m(x) = \int_0^x (1 - F(t))\, dt,\end{aligned} \tag{3.13}$$

where h_{k+1} is chosen (large enough) such that

$$c_{k+1}/n_{k+1} \geq c_k/n_k, \qquad \lim_k h_k = \infty \tag{3.14}$$

and

$$(F(h_{k+1}))^{n_k} \geq \tfrac{1}{2}, \tag{3.15}$$

$$\begin{aligned} b(x) &= xc_k/n_k, \qquad n_k \leq x < n_{k+1}, \\ &= xc_1/n_1, \qquad 1 \leq x < n_1. \end{aligned} \tag{3.16}$$

By (3.11) and (3.12)

$$(1 - \theta_k)/3 \leq v_k \leq (1 - \theta_k), \qquad k \geq 1. \tag{3.17}$$

Therefore, $c_k/n_k \geq EXI\{X \leq h_k\} \to \infty$ and it is possible to choose $\{h_k\}$ satisfying (3.14) and (3.15). Also, it is clear that

(3.18) $\quad b(x)/x$ is nondecreasing and $\lim_x b(x)/x = \infty$ by (3.14).

We shall keep the above notations and assumptions throughout the section.

Now, we are able to restate Theorem 3 in accord with what there is in the proof.

THEOREM 3*. *Suppose that $\rho^*(w^*) > 0$ for some $w^* < \infty$. Then*

$$\theta_* \leq \liminf S_n/b_n \leq \theta^* \quad a.s., \tag{3.19}$$

where $b_n = b(n)$, $n \geq 1$.

REMARKS. 1. If $\rho^* > 0$ and $w^* < \infty$, then $0 < \theta_* \leq 1 \leq \theta^* < \infty$. 2. If $1 < w^* < \infty$ and (3.1) holds, then $0 < \rho^*(w^*) \leq 1$.

COROLLARY 2. *Suppose that $\rho^*(1) = 1$. Then*

$$\liminf S_n/b_n = 1 \quad a.s., \tag{3.20}$$

where $b_n = b(n)$, $n \geq 1$, and $b(x)$, $x \geq 1$, is given by (3.16).

REMARK. Here the exact sequence $\{b_n\}$ for which (3.20) holds is given for a case different from the cases where the exact sequences were given previously [see, for example, Klass–Teicher (1977), Pruitt (1981a), and Mijnheer (1982)]. Usually, the condition that $1 - F(x)$ is regular varying (smoothness) with exponent in $[-1, 0)$ (moment) is required. However, the condition $\rho^*(1) = 1$ means, to certain degree, that $1 - F(x)$ is extremely unsmooth and it does not imply $Ef(X) < \infty$ for any unbounded function $f(x)$.

An example will be given after the proof.

PROOF. (i) Let $0 < \lambda < \theta_*$ and $p_k = 1 - F(h_k)$. Suppose that

$$c_k = n_k h_k p_k \; [EXI\{X \leq h_k\} \leq v_k h_k p_k \; \text{ by } \; (3.13)]. \tag{3.21}$$

Set $X_n' = h_k I\{X_n > h_k\}$ and $S_n' = \sum_{i=1}^n X_i' \leq S_n$, $n \geq 1$. Then S_n'/h_k has a binomial distribution $b(n, p_k)$ and $\{S_n'/n, n = n_{k+1}, \ldots, 2, 1\}$ is a martingale. By

(2.6) of Lemma 1 and (3.16),

$$P\{S_n \le \lambda b_n \text{ for some } n_k \le n < n_{k+1}\}$$
$$\le P\{S_n'/(nh_k) \le \lambda p_k \text{ for some } n_k \le n < n_{k+1}\}$$
$$\le \exp[t\lambda p_k n_k](1 - p_k + p_k e^{-t})^{n_k}$$
$$\le \exp[-p_k n_k(1 - e^{-t} - t\lambda)].$$

Take $t = -\log \lambda$. By (3.10) and (3.11)

$$p_k n_k(1 - e^{-t} - t\lambda) = p_k n_k(1 - \lambda + \lambda \log \lambda)$$
$$= [(1 - \lambda + \lambda \log \lambda)/(1 - \rho_k)]\log k,$$
$$1 - \lambda + \lambda \log \lambda > 1 - \rho^* \quad \text{by (3.4)}.$$

It follows from (3.8) that there exists an $\varepsilon > 0$ such that

$$P\{S_n \le \lambda b_n \text{ for some } n_k \le n < n_{k+1}\} \le k^{-(1+\varepsilon)} \quad \text{for large } k. \tag{3.22}$$

Consider the other case where

$$c_k = m(h_k)n_k \quad [EXI\{X \le h_k\} > v_k h_k p_k \text{ by (3.13)}]. \tag{3.23}$$

Set $X_n' = \min(X_n, h_k)$ and $S_n' = \sum_{i=1}^n X_i' \le S_n$, $n \ge 1$. Then

$$ES_n' = b_n = nc_k/n_k, \qquad n_k \le n < n_{k+1},$$
$$P\{S_n \le \lambda b_n \text{ for some } n_k \le n < n_{k+1}\}$$
$$\le P\{(ES_n' - S_n')/n \ge (1 - \lambda)c_k/n_k \text{ for some } n_k \le n < n_{k+1}\}$$
$$\le \exp\Big[-(1 - \lambda)m(h_k)(m(h_k)/n_k)^{-1} \times G((1 - \lambda)m(h_k)m(h_k)/\mathrm{Var}(X'))\Big] \tag{3.24}$$
$$\le \exp[-(1 - \lambda)n_k G((1 - \lambda)m(h_k)/h_k)] \quad \text{by (2.7) of Lemma 1}.$$

Since $m(h_k)/h_k \to 0$ as $k \to \infty$ and $G(x)/x \to \frac{1}{2}$ as $x \to 0$,

$$(1 - \lambda)n_k G((1 - \lambda)m(h_k)/h_k) = (1 + o(1))(1 - \lambda)^2 n_k m(h_k)/(2h_k),$$
$$m(h_k) > (1 + v_k)h_k p_k \quad \text{by (3.23)},$$
$$n_k m(h_k)/h_k > (1 + v_k)n_k p_k = (1 + v_k)(1 - \rho_k)^{-1}\log k$$
$$= 2(1 - \theta_k)^{-2}\log k \quad \text{by (3.23) and (3.12)}.$$

By (3.24)

$$P\{S_n \le \lambda b_n \text{ for some } n_k \le n < n_{k+1}\}$$
$$\le \exp\Big[-(1 + o(1))(1 - \lambda)^2(1 - \theta_k)^{-2}\log k\Big].$$

By (3.4), (3.8), and (3.11)

$$\lim_k \theta_k = \theta_*. \tag{3.25}$$

Since $\lambda < \theta_*$ (3.22) holds for both cases. Therefore,

$$P\{S_n \le \lambda b_n \text{ i.o.}\} = 0 \quad \text{for any } \lambda < \theta_*. \tag{3.26}$$

(ii) Let $p_k = (1 - F(h_k))$, $k \ge 1$, and for each $k \ge 1$, let $X(k)$, $X(1, k)$, $X(2, k), \ldots$ be i.i.d. random variables such that

$$\{X_n < w_k h_k\} \subset \{X_n = X(n, k)\} \subset \{X_n \le w_k h_k\} \tag{3.27}$$

and

$$\begin{aligned} P\{X_n \ne X(n, k)\} &= (1 - \rho_k)(1 - F(h_k)) \\ &= P\{X_n \ne X(n, k) \text{ and } X(n, k) = 0\}. \end{aligned} \tag{3.28}$$

Set

$$T_k = X(1, k) + \cdots + X(n_k, k), \qquad k = 1, 2, \ldots, \tag{3.29}$$

and

$$A_k = \{T_k = S_{n_k}\}, \qquad k = 1, 2, \ldots. \tag{3.30}$$

It follows from (3.10), (3.15), and (3.28) that

$$\sum_{k=1}^{\infty} P\{A_k\} = \sum_{k=1}^{\infty} \left(1 - n_k^{-1} \log k\right)^{n_k} = \infty,$$

$$\begin{aligned} P\left\{A_{k+1} \bigcup_{j=m}^{k} A_j\right\} &= P\left\{[X_i = X(i, k+1) \text{ all } n_k < i \le n_{k+1}] \bigcup_{j=m}^{k} A_j\right\} \\ &\le P\left\{\bigcup_{j=m}^{k} A_j\right\} \exp[(n_{k+1} - n_k) \log P\{X = X(k+1)\}] \\ &\le 2P\left\{\bigcup_{j=m}^{k} A_j\right\} P\{A_{k+1}\} \quad \text{by (3.15) and (3.28).} \end{aligned} \tag{3.31}$$

By (3.31), Hewitt–Savage zero-one law, and Lemma 4.2.4 of Chow–Teicher (1978),

$$P\{A_k \text{ i.o.}\} = P\{T_k = S_{n_k} \text{ i.o.}\} = 1. \tag{3.32}$$

Let

$$\begin{aligned} &\theta^* < \lambda < \infty \quad \text{and} \quad p_k = 1 - F(h_k), \\ &u_k = \max(1, v_k + w_k \rho_k), \qquad k \ge 1. \end{aligned} \tag{3.33}$$

If $c_k/n_k = h_k p_k$ $[EXI\{X \le h_k\} \le v_k h_k p_k$ by (3.13)], then

$$\begin{aligned} EX(k) &\le EXI\{X \le h_k\} + w_k h_k (p_k - P\{X \ne X(k)\}) \\ &\le v_k h_k p_k + w_k h_k p_k \rho_k \le u_k h_k p_k = u_k c_k/n_k. \end{aligned}$$

Otherwise, $c_k/n_k = m(h_k)$ $[EXI\{X \le h_k\} > v_k h_k p_k$ by (3.13)],

$$EX(k) \le \max(1, w_k \rho_k) m(h_k) \le u_k c_k/n_k.$$

Therefore, in both cases,

$$EX(k) \le u_k c_k/n_k \quad \text{and} \quad ET_k \le u_k c_k = u_k b(n_k), \qquad k \ge 1,$$

$$(y-1)G(y-1) = \int_0^{y-1} \log(1+t)\,dt = y\log y - y + 1, \qquad y \ge 1,$$

$$\begin{aligned} &P\{T_k \ge b(n_k)\} \\ &\quad \le P\{T_k - ET_k \ge (\lambda - u_k)c_k\} \quad \text{for large } k \\ &\quad \le \exp\left[-(\lambda-u_k)c_k(w_k h_k)^{-1} G((\lambda-u_k)c_k w_k h_k/(n_k w_k h_k u_k c_k/n_k))\right] \\ &\quad = \exp\left[-c_k(w_k h_k)^{-1}(\lambda-u_k)G((\lambda-u_k)/u_k)\right] \quad \text{by (2.7) of Lemma 1,} \end{aligned}$$

$$\begin{aligned} &c_k(w_k h_k)^{-1}(\lambda-u_k)G((\lambda-u_k)/u_k) \\ &\quad \ge n_k p_k w_k^{-1} u_k(\lambda/u_k - 1)G(\lambda/u_k - 1) \quad \text{by (3.13)} \\ &\quad \ge (\log k)(1-\rho_k)^{-1} w_k^{-1} u_k\left[(\lambda/u_k)\log(\lambda/u_k) - (\lambda/u_k) + 1\right] \quad \text{by (3.10)} \\ &\quad \ge (1+\varepsilon)\log k \end{aligned}$$

for large k by (3.5)–(3.8), (3.11), and (3.33), where ε is some positive constant.

Hence, it follows from (3.32) that

$$P\{T_k \ge \lambda b(n_k)\} \le k^{-(1+\varepsilon)} \quad \text{for some } \varepsilon > 0$$

and

$$P\{T_k \ge \lambda b(n_k) \text{ i.o.}\} = 0 = 1 - P\{S_n < \lambda b_n \text{ i.o.}\}.$$

(iii) Proof of Corollary 2. Since $\rho^* = w^* = 1$, by (3.4)–(3.7) $v = 0$, $u = 1$, and $\theta^* = \theta_* = 1$. □

Example 1. Let $x_k = \exp[k^2]$ and $P\{X = x_k\} = k/(k+1)!$, $k \ge 1$. Then

$$\liminf S_n/b_n = 1 \quad \text{a.s.,}$$

where

$$b_n = n\exp[k^2]/k! \quad \text{for } (k+1)!\log k \le n < (k+2)!\log(k+1).$$

Also, we have $EX^p = \infty$ for any $p > 0$.

Remark. The condition of Corollary 2 is satisfied $[\rho^*(1) = 1]$ if $P\{X = x_k\} = P\{X \ge x_k\}(1 + o(1))$ as $k \to \infty$ for an unbounded sequence of positive constants $\{x_k\}$.

Proof. Set $h_k = x_k - 1$, $w_k = x_k/h_k$, $\rho_k = k/(k+1)$, $n_k = (k+1)!\log k$. Then (3.8)–(3.10) hold and $c_k/n_k = h_k/k! = (1+o(1))x_k/k!$ by (3.11)–(3.13). It is clear that (3.14) holds and Theorem 3* still holds if (3.15) is replaced by $n_k(1 - F(h_{k+2})) = o(1)$ which is satisfied for the constants and F defined above. □

4. Rapidly growing random walks. Let $\{S_n = \sum_{i=1}^n X_i, n \ge 1\}$ be a random walk with underlying distribution function $F(x)$. In this section we shall put on the restrictions that $F(0-) = 0$ and that $1 - F$ is slowly varying as x tends to infinity. A random walk with such a distribution is called a rapidly growing random walk. It will be demonstrated in this section that the following statement (4.1) is true for rapidly growing random walks under quite general conditions.

(4.1) There do not exist constants $b_n \to \infty$ such that $\liminf S_n/b_n = 1$ a.s.

The argument is composed of two steps. Theorem 5 gives conditions under which

$$\lim S_n/X_n^* = 1 \quad \text{a.s.,} \quad X_n^* = \max_{1 \le i \le n} X_i. \tag{4.2}$$

Theorem 6 gives conditions under which (4.1) holds if S_n there is replaced by X_n^*. An example will be given below for which

$$\liminf X_n^*/b_n = 1 \quad \text{a.s. for some } b_n \to \infty. \tag{4.3}$$

THEOREM 5. *Let* $L(x) = (1 - F(x))^{-1}$. *Suppose that*

$$\int_0^\infty \int_0^1 P\{ux < X \le x\}\, du\, dL(x) < \infty. \tag{4.4}$$

Then (4.2) *holds.*

THEOREM 6. *Let* $L(x) = (1 - F(x))^{-1}$. *Suppose that*

$$\limsup(\log L(x))(L(2x) - L(x))/L(x) < \infty. \tag{4.5}$$

Then (4.3) *does not hold.*

We shall put the proofs at the end of the section. It is clear that the random walk does not obey the generalized law of the iterated logarithm [i.e., (4.1) holds] if the underlying distribution function satisfies (4.4) and (4.5). Darling (1952) proved that $\lim E(S_n/X_n^*) = 1$ for rapidly growing random walks. Pruitt (1981a) constructed an example satisfying (4.1) by using the integral test of Erickson (1973). The works of Teicher (1979) and Klass (1984) are also related to the almost sure properties of rapidly growing random walks. While this paper was being revised, the paper of Maller–Resnick (1984) appeared which contains Theorem 5 as part of their Theorem 3.1. The proof of Theorem 5 here is different from theirs and very simple. The following example shows that (4.3) holds for some rapidly growing random walks.

EXAMPLE 2. Let $1 - F(x) = \exp[-\log x \log_3 x/\log_2 x]$, $x \ge e^2$. Define the normalizing function $b(x)$ by

$$n(x)(1 - F(b(n(x)))) = \log x, \qquad x \ge e, \tag{4.6}$$

where $n(x) = \exp[x/(2\log x)]$.

Then $1 - F(x)$ is slowly varying as x tends to infinity and

$$\liminf X_n^*/b(n) = e^{-1} \quad \text{a.s.,} \tag{4.7}$$

where $X_n^* = \max_{1 \le i \le n} X_i$, $n \ge 1$, and $e = 2.7182818\ldots$.

PROOF. Clearly,

$$\lim_x (1 - F(vx))/(1 - F(x)) = 1 \quad \text{for any } v > 0, \tag{4.8}$$

$$\lim n(x+1)/n(x) = 1, \qquad dn(x)/dx = n(x)(2\log x)^{-1}(1 + o(1)), \tag{4.9}$$

$$\lim(n(x+1) - n(x))(2\log x)/n(x) = 1. \tag{4.10}$$

It follows from (4.6), (4.8), and (4.10) that

$$\lim(n(x+1) - n(x))(1 - F(vb(n(x)))) = \tfrac{1}{2} \quad \text{for any } v > 0. \tag{4.11}$$

Therefore, by Theorem 2 of Klass (1984), $P\{X_n^* \le vb(n) \text{ i.o.}\} = 0$ iff

$$\sum_{k=1}^{\infty} \exp[-n(k)(1 - F(vb(n(k))))] < \infty,$$

$$\begin{aligned} dF(x)/dx &= (1 - F(x))(x\log_2 x)^{-1}(\log_3 x)(1 + o(1)), \\ \log_2 b(n(x)) &= (1 + o(1))\log x, \\ \log_3 b(n(x)) &= \log_2 x + o(1), \end{aligned} \tag{4.12}$$

$$\begin{aligned} &(1 - F(vb(n(x)))) - (1 - F(b(n(x)))) \\ &\quad = (-\log v)(1 - F(b(n(x))))(\log x)^{-1}(\log_2 x)(1 + o(1)). \end{aligned}$$

Hence,

$$n(k)(1 - F(vb(n(k)))) = \log k - (\log v \log_2 k)(1 + o(1)). \square$$

PROOF OF THEOREM 5. We assume that $F(x)$ is continuous without loss of generality. By Theorem 3.2 of Darling (1952),

$$\begin{aligned} E(S_n/X_n^* - 1) &= n\int_0^{\infty}(1 - F(x))\int_0^1 (L(x) - L(ux))/L(ux)\, du\, dF^{n-1}(x) \\ &= \int_0^{\infty} n(n-1)F^{n-2}(x)\int_0^1 P\{ux < X \le x\}\, du\, dF(x). \end{aligned} \tag{4.13}$$

Since

$$\begin{aligned} \sum_{k=1}^{\infty} 4^k \exp[2^k \log p] &\le 4\int_0^{\infty} 4^x \exp[2^x \log p]\, dx \\ &\le 4(-\log p)^{-2}\int_0^{\infty} ye^{-y}\, dy/\log 2 \\ &\le 6(1-p)^{-2}, \qquad 0 \le p \le 1, \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{\infty} E\left(S_{2^k}/X_{2^k}^* - 1\right) &\le 6\int_0^{\infty} L^2(x)\int_0^1 P\{ux < X \le x\}\, du\, dF(x) \\ &= 6\int_0^{\infty}\int_0^1 P\{ux < X \le x\}\, du\, dL(x) \\ &< \infty \quad \text{by (4.4)}. \end{aligned} \tag{4.14}$$

(Actually, $\sum_{k=1}^{\infty} E(S_{2^k}/X_{2^k}^* - 1) < \infty$ iff (4.4) holds.)

Let G_n be the σ-algebra generated by all the random variables $h = h(X_1, X_2, \dots)$ such that the function $h(x_1, x_2, \dots)$ is symmetric to the permutation of the first n variables $x_1, \dots, x_n$. Set

$$S_n^{(i)} = \sum_{j=1,\, j \neq i}^{n} X_j, \qquad i \le n, \qquad n \ge 2,$$

$$X_n^{(i)} = \max(X_j\colon 1 \le j \le n,\, j \neq i), \qquad i \le n, \qquad n \ge 2,$$

and

$$X_n^0 = \max(X_j\colon 1 \le j \le n,\, X_j \neq X_n^*), \qquad n \ge 2.$$

(X_n^0 is the second largest among $X_1, \dots, X_n$.)

We have

$$\begin{aligned} E[S_{n-1}/X_{n-1}^* | G_n] &= n^{-1} E\left[\sum_{i=1}^{n} S_n^{(i)}/X_n^{(i)} | G_n\right] \\ &= n^{-1}(n-1) S_n/X_n^* + (S_n - X_n^*)(1/X_n^0 - 1/X_n^*)/n \\ &= S_n/X_n^* + n^{-1}\big((S_n - X_n^* - X_n^0)/X_n^0 \\ &\qquad - 2(S_n - X_n^*)/X_n^*\big). \end{aligned}$$

Therefore,

$$E\left[(S_{n-1}/X_{n-1}^* - 1) | G_n\right] \ge n^{-1}(n-2)(S_n/X_n^* - 1). \tag{4.15}$$

Set

$$f_j(n) = E\left[(S_n/X_n^* - 1) | G_j\right], \qquad j \ge n.$$

It follows from (4.15) that for $3 \le n \le j < 2n$

$$S_j/X_j^* - 1 \le j(j-1) n^{-1}(n-1)^{-1} f_j(n) \le 5 f_j(n) \quad \text{a.s.} \tag{4.16}$$

By the Doob inequality and (4.14),

$$\begin{aligned} &P\{\sup(S_n/X_n^* - 1\colon n \ge 2^m) > 5u\} \\ &\quad \le \sum_{k=m}^{\infty} P\{\max(f_n(2^k)\colon 2^k \le n < 2^{k+1}) \ge u\} \quad \text{for } m \ge 3 \\ &\quad \le \sum_{k=m}^{\infty} u^{-1} E f_{2^k}(2^k) \to 0 \quad \text{as } m \to \infty \text{ for any } u \ge 0. \end{aligned}$$

This finishes the proof. □

Proof of Theorem 6. We shall prove by contradiction. Assume that

$$P\{\liminf X_n^*/b_n = 1.5\} = 1 \quad \text{for some } b_n \to \infty. \tag{4.17}$$

Since X_n^* is nondecreasing in n, we can also assume that b_n is nondecreasing in n and that $\lim b_n = \infty$.

Let $p_n = 1 - F(b_n)$ and $\delta_n = F(2b_n) - F(b_n)$.

$$P\{X_n^* \le 2b_n\} \le \exp[-n(1 - F(2b_n))] = \exp[-np_n + n\delta_n].$$

By (4.5) there exists a constant $K \ge 4$ such that for large n

$$\delta_n = (L(2b_n) - L(b_n))/(L(2b_n)L(b_n)) \le Kp_n(-\log p_n)^{-1}.$$

For $n\delta_n > K^2$ and large n

$$np_n(-\log p_n)^{-1} > K \ge 4 \quad \text{and} \quad np_n > 3\log n.$$

Therefore, there exists a constant K^* such that for large n

$$\begin{aligned} P\{X_n^* \le 2b_n\} &\le \exp[-np_n + K^2] + \exp[-2\log n] \\ &\le K^* P\{X_n^* \le b_n\} + n^{-2}. \end{aligned}$$

And

$$\begin{aligned} &P\left\{\bigcup_{k=n}^{m} [X_k^* \le 2b_k]\right\} \\ &\quad = \sum_{k=n}^{m-1} P\{X_k^* \le 2b_k \text{ and } X_j^* > 2b_j \text{ for all } k < j \le m\} \\ &\qquad + P\{X_m^* \le 2b_m\} \\ &\quad = \sum_{k=n}^{m-1} P\{X_k^* \le 2b_k\} P\left\{\max_{k<i\le j} X_i > 2b_j \text{ for all } k < j \le m\right\} \\ &\qquad + P\{X_m^* \le 2b_m\} \\ &\quad \le \sum_{k=n}^{m-1} K^* P\{X_k^* \le b_k\} P\left\{\max_{k<i\le j} X_i > b_j \text{ for all } k < j \le m\right\} \\ &\qquad + K^* P\{X_m^* \le b_m\} + \sum_{k=n}^{m} k^{-2} \\ &\quad \le K^* P\left\{\bigcup_{k=n}^{m} [X_k^* \le b_k]\right\} + 2n^{-1} \quad \text{for large } n \text{ and any } m \ge n. \end{aligned}$$

It follows from (4.17) that $\lim_n P\{\bigcup_{k=n}^{\infty}[X_k^* \le b_k]\} = 0$. Therefore,

$$\begin{aligned} &\lim_n P\left\{\bigcup_{k=n}^{\infty} [X_k^* \le 2b_k]\right\} \\ &\quad \le \lim_n K^* P\left\{\bigcup_{k=n}^{\infty} [X_k^* \le b_k]\right\} + \lim_n 2n^{-1} = 0. \end{aligned} \tag{4.18}$$

The proof is finished since (4.18) is contrary to (4.17) which is equivalent to (4.3). □

5. The limit points. Let $\{Z(n), n \ge 1\}$ be a sequence of random variables and $\{\gamma(n), n \ge 1\}$ be a sequence of positive constants. The accumulation points of $\{Z(n)/\gamma(n)\}$ are

$$A(Z, \gamma) = \bigcap_m \overline{\{Z(n)/\gamma(n)\colon n \geq m\}}, \tag{5.1}$$

where the bar indicates the closure in the topology of $\overline{R} = [-\infty, \infty]$.

Define

$$B(F, \gamma) = \Big\{u \in \overline{R}\colon P\{S_{n_k}/\gamma(n_k) \to u\} = P\{n_k \to \infty\} = 1, \text{ for some random variables } n_k, k \geq 1\Big\}. \tag{5.2}$$

By Theorem 1 of Kesten (1970),

$$P\{B(F, \gamma) = A(S, \gamma)\} = 1, \qquad (S(n) = S_n, n \geq 1), \tag{5.3}$$

provided that $\lim \gamma(n) = \infty$. In this section we consider the limit points of the normalized random walk for which $-\infty \notin B(F, \gamma)$ or $\infty \notin B(F, \gamma)$.

Let us start with a simple pointwise argument.

PROPOSITION 1. *Let* $\{z(n)\}$ *and* $\{\gamma(n)\}$ *be two sequences of constants with* $\lim \gamma(n) = \infty$. *Set* $z_n = z(n)$ *and* $\gamma_n = \gamma(n)$, $n \geq 1$.

(i) *If* $\limsup(z_n - z_{n-1}^+)/\gamma_n \leq 0$ *and* $\limsup \gamma_n/\gamma_{n+1} \leq 1$, *then*

$$[0, \infty] \cap A(z, \gamma) = \left[(\liminf z_n/\gamma_n)^+, \limsup z_n/\gamma_n\right],$$

where $A(z, \gamma)$ *is defined by* (5.1) *and* $[a, b] = \varnothing$ *if* $b < a$.

(ii) *If* $\limsup(z_n - z_{n-1})/\gamma_n \leq 0$ *and* $\liminf \gamma_n/\gamma_{n+1} \geq 1$, *then*

$$[-\infty, 0] \cap A(z, \gamma) = \left[\liminf z_n/\gamma_n, -(\limsup z_n/\gamma_n)^-\right].$$

PROOF. (i) For $b > 0$ and large n,

$$\begin{aligned}(z_n/\gamma_n - z_{n-1}^+/\gamma_{n-1}) & I\{z_{n-1}/\gamma_{n-1} \leq b\} \\ & \leq (z_n - z_{n-1}^+)/\gamma_n + b(\gamma_{n-1}/\gamma_n - 1)^+ = o(1).\end{aligned}$$

Therefore, $b \in A(z, \gamma)$ if $b \geq 0$ and $\{z_n/\gamma_n\}$ crosses over b "up hill" infinitely many times.

(ii) For $z_{n-1} \leq 0$ and n large,

$$z_n/\gamma_n \leq (z_n - z_{n-1})/\gamma_n + z_{n-1}/\gamma_n \leq (1 + o(1))z_{n-1}/\gamma_{n-1} + o(1). \square$$

COROLLARY 3. *Suppose that* γ_n/n *is nondecreasing and* $\lim \gamma_n/\gamma_{n-1} = 1$. *If* $P\{\liminf S_n/\gamma_n > -\infty\} = 1$ *or* $P\{\limsup S_n/\gamma_n < \infty\} = 1$, *then the set* $B(F, \gamma)$ *in* (5.2) *is a closed interval in* $\overline{R}$.

REMARK. This corollary reduces the problem of finding $B(F, \gamma)$ to the problem of finding the lower limit of the normalized random walk for the cases where $-\infty \notin B(F, \gamma)$ and (1.9) holds [in particular, the cases in Mijnheer (1982), Examples 2 and 5 in Erickson–Kesten (1974), and Example 1 of Erickson (1976)].

PROOF. Use Theorem 2 of Chow–Zhang (1984) and (ii) of Proposition 1. □

COROLLARY 4. *Let $\{b_n\}$ and $\{m_n\}$ be two sequences of constants. Suppose that, for some $\varepsilon > 0$ and $\pi > 0$,*

$$-\infty < \liminf(S_n - m_n)/b_n < \limsup(S_n - m_n)/b_n < \infty \quad a.s., \tag{5.4}$$

$$P\{S_n \le m_n\} \ge \pi, \qquad P\{S_n \ge m_n\} \ge \pi \quad \text{for large } n, \tag{5.5}$$

and

$$b_n/n^{\varepsilon} \text{ is increasing in } n. \tag{5.6}$$

Then there exist constants a and b for which

$$P\{A(S_n - m_n, b_n) = [a, b]\} = 1, \tag{5.7}$$

where $A((S - m), b)$ is defined by (5.1), $(S - m)(n) = S_n - m_n$, *and* $b(n) = b_n$.

REMARK. This corollary is related to Theorem 6 and Problem 5 of Kesten (1972).

PROOF. Clearly, (5.5) still holds if m_n is replaced by m_{n-1} + median(X) and π is replaced by $\pi/2$. By Theorem 6 of Kesten (1972),

$$\lim(m_n - m_{n-1})/b_n = 0. \tag{5.8}$$

By (5.4)

$$\limsup |X_n|/b_n < \infty \quad \text{a.s.}$$

It follows from (5.6) that

$$\lim X_n/b_n = 0 \quad \text{a.s.} \tag{5.9}$$

The corollary follows from (5.6), (5.8), (5.9), and (ii) of Proposition 1. □

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