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HOW SMALL ARE THE INCREMENTS OF THE LOCAL TIME OF A WIENER PROCESS?

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Let $W(t)$ be a standard Wiener process with local time $L(x, t)$. Put $L(t) = L(0, t)$ and $L^*(t) = \sup_{-\infty < x < \infty} L(x, t)$. We study the almost sure behaviour of small increments of $L(t)$ and also, the joint behaviour of $L(t)$ and the last excursion, $U(t)$. The increment problem of $L(x, t)$ are also studied uniformly in x. This implies a liminf-type law of the iterated logarithm for $L^*(t)$ due to Kesten (1965), in which case the exact constant, not known before, is also determined.

Introduction. Let $\{W(t), t \geq 0\}$ be a Wiener process and let $L(x, t)$ $(-\infty < x < \infty, 0 \leq t)$ be its local time which is jointly continuous a.s. Csáki, Csörgő, Földes, and Révész (1983) investigated the big increments of the local time and proved the following results:

THEOREM A. *Let $0 < a_T \leq T$ be a nondecreasing function of T. Assume that a_T/T is nonincreasing. Then*

$$\limsup_{T \to \infty} \gamma_T Y(T) = 1 \quad a.s. \tag{1}$$

and

$$\limsup_{T \to \infty} \beta_T X(T) = 1 \quad a.s., \tag{2}$$

where

$$Y(T) = a_T^{-1/2} \sup_{0 \leq s \leq T - a_T} (L(0, s + a_T) - L(0, s)), \tag{3}$$

$$X(T) = a_T^{-1/2} \sup_{0 \leq s \leq T - a_T} \sup_{-\infty < x < \infty} (L(x, s + a_T) - L(x, s)), \tag{4}$$

$$\gamma_T = (\log(T/a_T) + 2 \log \log T)^{-1/2}, \tag{5}$$

and

$$\beta_T = (2 \log(T/a_T) + 2 \log \log T)^{-1/2}. \tag{6}$$

If we also assume that

$$\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = \infty, \tag{7}$$

then

$$\lim_{T \to \infty} \gamma_T Y(T) = 1 \quad a.s. \tag{8}$$

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and

(9) $$\lim_{T\to\infty} \beta_T X(T) = 1 \quad a.s.$$

On the other hand, investigating the small increments of the Wiener process, Csörgő and Révész (1979, 1981) proved the following results:

THEOREM B. *Let a_T be a nondecreasing function of T for which*

(i) $$0 < a_T \le T \qquad (T \ge 0),$$

(ii) $$a_T/T \quad \text{is nonincreasing}.$$

Then

(10) $$\liminf_{T\to\infty} \delta_T I(T) = 1 \quad a.s.,$$

where

$$I(T) = \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |W(t+s) - W(t)|$$

and

$$\delta_T = \left(\frac{8(\log(T/a_T) + \log\log T)}{\pi^2 a_T} \right)^{1/2}.$$

If we also have

(iii) $$\frac{\log(T/a_T)}{\log\log T} \uparrow +\infty$$

then

(11) $$\lim_{T\to\infty} \delta_T I(T) = 1 \quad a.s.$$

In this paper we study the small increments of the local time. More precisely we investigate the lower classes of

(12) $$R(T, a_T) = \inf_{a_T\le t\le T} (L(0,t) - L(0, t-a_T))$$

in terms of certain integral tests.

This problem also suggests study of the joint behaviour of $L(t)$ and $U(t)$, where $U(t)$, the last excursion, is defined by

(13) $$U(T) = T - \sup\{s\colon W(s) = 0, 0 \le s \le T\}.$$

In Section 2 we give the analogue of Theorem B for the increments

(14) $$Q(T) = \inf_{0\le t\le T-a_T} \sup_{-\infty<x<\infty} (L(x, t+a_t) - L(x,t)).$$

This implies the case when $a_T = T$, i.e., $Q(T) = L^*(T) = \sup_{-\infty<x<\infty} L(x,T)$. For $L^*(T)$, Kesten (1965) proved

(15) $$\liminf_{T\to\infty} (\log\log T/T)^{1/2} L^*(T) = \gamma \quad \text{a.s.},$$

where $j_1/2 \le \gamma \le \sqrt{3}j_1^2(2j_1^2 - 4)^{-1/2}$, and j_1 is the first positive root of the Bessel function $J_0(x)$, but the exact value of γ is unknown.

By using a recent result of Borodin (1982) concerning the Laplace transform of the distribution of $L^*(T)$, we are able to give an expression for the exact and asymptotic distributions of $L^*(T)$. As a consequence, the exact value of γ in (15) turns out to be equal to $j_1\sqrt{2}$.

1. The increments of $L(t)$. In this section we consider the local time of the Wiener process at zero.

Let $L(0, t) = L(t)$. Moreover denote by $J(t)$ the length of the longest excursion of $W(t)$ (longest zero-free interval) up to the point t.

A theorem of Chung and Erdös (1952) (originally formulated for the time spent by the simple symmetric random walk on the positive side) reads as follows.

THEOREM C. *Let $f_1(x)$ be a nondecreasing function for which*

$$\lim_{x \to \infty} f_1(x) = +\infty,$$

$x/f_1(x)$ is nondecreasing and $\lim_{x\to\infty} x/f_1(x) = +\infty$, and let

$$I_1(f_1) = \int_1^\infty \frac{dx}{x\sqrt{f_1(x)}}. \tag{1.1}$$

If $I_1(f_1) = \infty$, then with probability 1 there exists a random sequence $T_1 < T_2 < \cdots < T_n < \cdots \to \infty$ such that

$$J(T_i) \ge T_i\left(1 - \frac{1}{f_1(T_i)}\right), \qquad i = 1, 2, \dots .$$

If $I_1(f_1) < \infty$ then with probability 1 there exists a random T_0 such that

$$J(T) < T\left(1 - \frac{1}{f_1(T)}\right) \quad \textit{for } T \ge T_0.$$

We remark that Theorem C remains true if we replace $J(t)$ by $U(t)$, where $U(t)$ denotes the length of the last excursion up to the point t defined by (13).

Another result of Chung and Hunt (1949) deals with the local time $L(t)$ (again originally formulated for simple symmetric random walk).

THEOREM D. *Let $f_2(y)$ be a nonincreasing function and $\sqrt{y}f_2(y)$ be nondecreasing for which $\lim_{y\to\infty}\sqrt{y}f_2(y) = +\infty$, $\lim_{y\to\infty} f_2(y) = 0$, and*

$$I_2(f_2) = \int_1^\infty \frac{f_2(y)}{y}\, dy. \tag{1.2}$$

If $I_2(f_2) = \infty$, then with probability 1 there exists a random sequence $T_1 < T_2 < \cdots < T_n \cdots \to \infty$ such that for all i

$$L(T_i) \le \sqrt{T_i}\, f_2(T_i).$$

If $I_2(f_2) < \infty$ *then with probability* 1 *there exists a random* T_0 *such that*

$$L(T) > \sqrt{T} f_2(T) \quad \text{for } T > T_0.$$

Clearly

$$\inf_{a_T \le t < T} (L(t) - L(t - a_T)) = 0$$

if a_T is shorter than $J(T)$.

This suggests that the problem, how small are the increments of $L(t)$, should be formulated in the following manner:

Let the function a_T be defined by

$$a_T = T\left(1 - \frac{1}{f_1(T)}\right) \tag{1.3}$$

and put

$$R(T, a_T) = \inf_{a_T \le t \le T} (L(t) - L(t - a_T)). \tag{1.4}$$

How can we characterize the small values of $R(T, a_T)$ with the behaviour of the integrals I_1 and I_2? As an answer we prove the following result:

THEOREM 1.1. *Let* $f_1(x)$ *be a nondecreasing function for which* $x/f_1(x)$ *is also nondecreasing and* $\lim_{x\to\infty} x/f_1(x) = +\infty$, $\lim_{x\to\infty} f_1(x) = +\infty$, *and put*

$$I_1(f_1) = \int_1^\infty \frac{dx}{x\sqrt{f_1(x)}}.$$

Let $f_2(y)$ *be a nonincreasing function for which* $\lim_{y\to\infty} f_2(y) = 0$, $\sqrt{y} f_2(y)$ *is nondecreasing, and* $\lim_{y\to\infty} \sqrt{y} f_2(y) = +\infty$ *and put*

$$I_2(f_2) = \int_1^\infty \frac{f_2(y)}{y}\, dy.$$

Set a_T *and* $R(T, a_T)$ *as above.*

(i) *If* $I_1(f_1) = \infty$ *or* $I_2(f_2) = \infty$ *then with probability* 1 *there exists a random sequence* $T_1 < T_2 < \cdots < T_n < \cdots \to \infty$ *such that for all i*

$$R(T_i, a_{T_i}) \le \sqrt{T_i} f_2(T_i).$$

(ii) *If* $I_1(f_1) < \infty$ *and* $I_2(f_2) < \infty$ *then with probability* 1 *there exists a random* T_0 *such that*

$$R(T, a_T) > \sqrt{T} f_2(T) \quad \text{for } T > T_0.$$

An important tool for the proof of Theorem 1.1 is

LEMMA 1.1. *For any positive* U, V, *and* u

$$P\left(L(U+V) - L(U) \le u\sqrt{V}\right) \le \frac{4}{\pi}\sqrt{\frac{U}{U+V}} + \sqrt{\frac{2}{\pi}}\, u \sqrt{\frac{V}{U+V}}. \tag{1.5}$$

Proof of Lemma 1.1. Using the exact distribution of the local time due to Lévy, and a simple conditioning argument,

$$
\begin{aligned}
&P(L(U+V)-L(U)\le u\sqrt{V})\\
&= 1-\int_{-\infty}^{\infty} P\{L(U+V)-L(U)>u\sqrt{V}\mid W(U)=z\}\,dP(W(U)\le z) \\
&= 1-2\int_{-\infty}^{\infty}\left[1-\Phi\left(\frac{|z|}{\sqrt{V}}+u\right)\right]d_z\Phi\left(\frac{z}{\sqrt{U}}\right).
\end{aligned}
\tag{1.6}
$$

Now we have from the Taylor expansion that for some $0\le u^*\le u$

$$
\begin{aligned}
\Phi\left(\frac{|z|}{\sqrt{V}}+u\right) &= \Phi\left(\frac{|z|}{\sqrt{V}}\right)+\varphi\left(\frac{|z|}{\sqrt{V}}\right)u+\varphi'\left(\frac{|z|}{\sqrt{V}}+u^*\right)\frac{u^2}{2}\\
&\le \Phi\left(\frac{|z|}{\sqrt{V}}\right)+\varphi\left(\frac{|z|}{\sqrt{V}}\right)u.
\end{aligned}
\tag{1.7}
$$

(1.6) and (1.7) and simple calculation lead to

$$
\begin{aligned}
&P(L(U+V)-L(U)\le u\sqrt{V})\\
&\le 1-2\int_{-\infty}^{\infty}\left[1-\Phi\left(\frac{|z|}{\sqrt{V}}\right)-\varphi\left(\frac{|z|}{\sqrt{V}}\right)u\right]d_z\Phi\left(\frac{z}{\sqrt{U}}\right)\\
&= 1-2\int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{|z|}{\sqrt{V}}\right)\right)d_z\Phi\left(\frac{z}{\sqrt{U}}\right)+2u\int_{-\infty}^{\infty}\varphi\left(\frac{|z|}{\sqrt{V}}\right)d_z\Phi\left(\frac{z}{\sqrt{U}}\right)\\
&= P(L(U+V)-L(U)=0)+u\sqrt{\frac{2}{\pi}}\sqrt{\frac{V}{U+V}}\,.
\end{aligned}
\tag{1.8}
$$

For the first probability we have (see Lévy (1948) Theorem 44.4) that

$$
P(L(U+V)-L(U)=0)=\frac{2}{\pi}\arcsin\sqrt{\frac{U}{U+V}}\le\frac{4}{\pi}\sqrt{\frac{U}{U+V}}\,. \qquad \square
\tag{1.9}
$$

Proof of Theorem 1.1. Part (i) is a simple consequence of Theorems C and D.

Proof of (ii). Suppose now that $I_1(f_1)<\infty$, and $I_2(f_2)<\infty$. Start with the following observation:

$$
L(a_T)-L(T-a_T)\le R(T,a_T).
\tag{1.10}
$$

First we show, that it is enough to prove, that for some sequence $T_1<T_2<\cdots<T_k<\cdots\to\infty$ the following inequality holds with probability 1:

$$
L(a_{T_k})-L(T_{k+1}-a_{T_{k+1}})>\sqrt{T_{k+1}}\,f_2(T_{k+1})\quad\text{for } k>k_0\ (=k_0(\omega)).
\tag{1.11}
$$

To see this, observe that from the condition of the theorem for $f_1(\cdot)$ the functions T, a_T, and $T-a_T$ are all nondecreasing. Consequently for any $T_k\le T<T_{k+1}$ we have

$$
T-a_T\le T_{k+1}-a_{T_{k+1}},\qquad a_{T_k}\le a_T;
$$

hence by (1.10)

$$L(a_{T_k}) - L(T_{k+1} - a_{T_{k+1}}) \le L(a_T) - L(T - a_T) \le R(T, a_T). \tag{1.12}$$

Thus if $k > k_0$, then from (1.11), (1.12), and the nondecreasingness of $\sqrt{x} f_2(x)$ for any $T_k \le T < T_{k+1}$

$$\sqrt{T} f_2(T) \le \sqrt{T_{k+1}} f_2(T_{k+1}) \le L(a_{T_k}) - L(T_{k+1} - a_{T_{k+1}}) \le R(T, a_T), \tag{1.13}$$

i.e., our statement follows from (1.11).

To see (1.11), let $T_k = \rho^k$ with $\rho > 1$ and estimate the probability of

$$A_k = \left\{L(a_{T_k}) - L(T_{k+1} - a_{T_{k+1}}) \le \sqrt{T_{k+1}} f_2(T_{k+1})\right\}$$

from our Lemma 1.1. (We may suppose that $a_{T_k} > T_{k+1} - a_{T_{k+1}}$ otherwise $P(A_k) = 0$.)

$$\begin{aligned} P(A_k) &\le \frac{4}{\pi}\sqrt{\frac{T_{k+1} - a_{T_{k+1}}}{a_{T_k}}} + \sqrt{\frac{2}{\pi}} \frac{f_2(T_{k+1})\sqrt{T_{k+1}}}{\sqrt{a_{T_k} - (T_{k+1} - a_{T_{k+1}})}} \\ &\quad \times \sqrt{\frac{a_{T_k} - (T_{k+1} - a_{T_{k+1}})}{a_{T_k}}} \\ &= \frac{4}{\pi}\sqrt{\frac{1}{f_1(T_{k+1})}} \cdot \sqrt{\frac{T_{k+1}}{a_{T_k}}} + \sqrt{\frac{2}{\pi}} f_2(T_{k+1})\sqrt{\frac{T_{k+1}}{a_{T_k}}}. \end{aligned} \tag{1.14}$$

Observe that as $f_1(x) \to \infty$, there exist an integer k_1 and a $0 < \alpha < 1$ such that for $k \ge k_1$

$$a_{T_k} = T_k\left(1 - \frac{1}{f_1(T_k)}\right) > \alpha T_k = \alpha\rho^k. \tag{1.15}$$

Hence

$$P(A_k) \le \frac{4}{\pi}\sqrt{\frac{1}{f_1(\rho^{k+1})}} \cdot \sqrt{\frac{\rho}{\alpha}} + \sqrt{\frac{2}{\pi}} f_2(\rho^{k+1})\sqrt{\frac{\rho}{\alpha}}.$$

Now $\Sigma P(A_k) < \infty$ clearly follows from the assumption that both of the integrals $I_1(f_1)$ and $I_2(f_2)$ are convergent, due to the following well-known:

REMARK. For any nonincreasing function $g(x)$ for which $\lim_{x\to\infty} g(x) = 0$

$$\sum_{k=1}^{\infty} g(\rho^k) \quad \text{and} \quad \int_1^{\infty} \frac{g(x)}{x}\,dx \qquad (\rho > 1) \tag{1.16}$$

are equiconvergent.

This implies the result via Borel–Cantelli Lemma. □

Our next theorem treats a somewhat related problem, the joint behaviour of $L(t)$ and $U(t)$ (defined by (13)).

Theorem 1.2. *Let $f_1(x)$ and $f_2(x)$ be functions satisfying the conditions of Theorem* 1.1.

If

$$I_3(f_1, f_2) = \int_1^{\infty} \min\left(f_2(x), \frac{1}{\sqrt{f_1(x)}} \right) \frac{dx}{x} < \infty,$$

then with probability 1 *there exists a random T_0 such that*

$$L(t) \le f_2(t)\sqrt{t}, \qquad U(t) \ge \left(1 - \frac{1}{f_1(t)}\right)t$$

do not hold simultaneously for any $t \ge T_0$.

If $I_3(f_1, f_2) = \infty$ then with probability 1 *there exists a random sequence $T_1 < T_2 < \cdots < T_n < \cdots \to \infty$ such that*

$$L(T_k) \le f_2(T_k)\sqrt{T_k}, \qquad U(T_k) \ge \left(1 - \frac{1}{f_1(T_k)}\right)T_k.$$

Proof of Theorem 1.2. The proof will be sketched briefly; the details of the calculations are omitted.

The joint distribution of $L(t)$ and $U(t)$ can be obtained from Lévy (1939, 1948):

$$(1.17) \quad P\left(L(t) \le u\sqrt{t}, U(t) \ge zt\right) = \int_0^{1-z} \left(1 - e^{-u^2/2a}\right) \frac{1}{\pi\sqrt{a(1-a)}}\, da,$$

from which the following estimations are easily obtained:

$$(1.18) \quad \begin{aligned} c_1 \min(u, \sqrt{1-z}) &\le P\left(L(t) \le u\sqrt{t}, U(t) > zt\right) \\ &\le c_2 \min(u, \sqrt{1-z}) \qquad (0 < u, 0 < z < 1) \end{aligned}$$

with positive constants c_1 and c_2.

Now let $t_k = \rho^k$, $k = 1, 2, \ldots, \rho > 1$, and define the events A_k and B_k by

$$A_k = \left\{ L(t_k) \le f_2(t_k)\sqrt{t_k}, U(t_k) \ge \left(1 - \frac{1}{f_1(t_k)}\right)t_k \right\},$$

$$B_k = \left\{ L(t_k) \le f_2(t_{k+1})\sqrt{t_{k+1}}, U(t_k) \ge \left(1 - \frac{t_{k+1}}{t_k f_1(t_{k+1})}\right)t_k \right\}.$$

If $I_3(f_1, f_2) < \infty$, then by the remark at the end of the proof of Theorem 1.1, using the inequality (1.18), it can be seen that $\Sigma_k P(B_k) < \infty$, which in turn implies the first part of Theorem 1.2.

To show the second part, first observe that $I_3(f_1, f_2) = \infty$ and (1.18) imply $\sum_k P(A_k) = \infty$ and this together with

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n \sum_{l=1}^n P(A_k A_l)}{\left(\sum_{k=1}^n P(A_k)\right)^2} \le 1 \tag{1.19}$$

is a sufficient condition of $P(A_k \text{ i.o.}) = 1$ (see Erdös–Rényi (1959) or Rényi (1962), etc.).

To verify (1.19), by simple but somewhat tedious calculation we have for $k < l$,

$$\begin{aligned}
P(A_k A_l) &\le P\left(L(t_k) \le u_k\sqrt{t_k}, U(t_k) > z_k t_k, L(t_l - t_k) \le u_l\sqrt{t_l}, U(t_l) \ge z_l t_l\right) \\
&= \int_{a=0}^{t_k(1-z_k)} \int_{b=t_k}^{t_l(1-z_l)} \int_{y=-\infty}^{\infty} \left(1 - \exp\left(-\frac{u_k^2 t_k}{2a}\right)\right) \frac{1}{\sqrt{2\pi a}} \frac{|y|}{\sqrt{2\pi}} \\
&\quad \times \exp\left(-\frac{y^2}{2(t_k - a)}\right) \cdot \frac{1}{(t_k - a)^{3/2}} \frac{1}{\sqrt{2\pi(b - t_k)}} \\
&\quad \times \left\{\exp\left(-\frac{y^2}{2(b - t_k)}\right) - \exp\left(-\frac{1}{2}\left(\frac{|y|}{\sqrt{b - t_k}} + u_l\sqrt{\frac{t_l}{b - t_k}}\right)^2\right)\right\} \\
&\quad \times \frac{2\,dy\,db\,da}{\sqrt{2\pi(t_l - b)}} + \int_0^{t_k(1-z_k)} \left(1 - \exp\left(-\frac{u_k^2 t_k}{2a}\right)\right) \frac{2}{\sqrt{2\pi(t_l - a)}} \frac{da}{\sqrt{2\pi a}},
\end{aligned}$$

where $u_n = f_2(t_n)$, $z_n = 1 - 1/f_1(t_n)$, and from this we obtain for $\varepsilon > 0$ and k big enough,

$$P(A_k A_l) \le P(A_k)\left((1 + \varepsilon)P(A_l) + c\rho^{-(l-k)/2}\right) \tag{1.20}$$

provided that $l > l_0(k, \varepsilon) = \max\{l\colon t_k/t_l((1 + \varepsilon)^2 - 1)\}$.

On the other hand,

$$\sum_{l=k}^{l_0} P(A_k A_l) \le cP(A_k). \tag{1.21}$$

Now (1.19) follows from (1.20) and (1.21); hence the second part of Theorem 1.2. □

2. The increments $\sup_x(L(x,t) - L(x,t - a_t))$. Our results in this section are heavily based on a result of Borodin (1982) concerning the Laplace transform of the distribution of $L^*(t) = \sup_x L(x, t)$. Borodin's result states

$$\lambda \int_0^\infty e^{-\lambda t} P(L^*(t) > z)\, dt = \frac{4z\sqrt{2\lambda}\, e^{z\sqrt{2\lambda}} I_1\left(z\sqrt{\lambda/2}\right)}{\left(e^{z\sqrt{2\lambda}} - 1\right)^2 I_0\left(z\sqrt{\lambda/2}\right)}, \tag{2.1}$$

where I_0 and I_1 are modified Bessel functions defined by

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{x}{2}\right)^{2k},$$

$$I_1(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}\left(\frac{x}{2}\right)^{2k+1}.$$

From the scale change of the Wiener process it follows that $t^{-1/2}L^*(t)$ has the same distribution as $L^*(1)$ and by putting $z = 1$, (2.1) is easily seen to be equivalent to

$$(2.2) \qquad \int_0^{\infty} e^{-\lambda t} P(L^*(1) < 1/\sqrt{t})\, dt = \frac{1}{\lambda} - \sqrt{\frac{2}{\lambda}}\, \frac{I_1(\sqrt{\lambda/2})}{I_0(\sqrt{\lambda/2})(\mathrm{sh}\sqrt{\lambda/2})^2}.$$

Our first result in this section concerns the inversion of (2.2) and its asymptotics.

THEOREM 2.1. *The following relations hold for* $0 < z$:

$$(2.3) \quad P(L^*(1) < z) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{2j_n^2}{z^2}\right) + \sum_{k=1}^{\infty}\left(b_k + \frac{c_k}{z^2}\right)\exp\left(-\frac{2k^2\pi^2}{z^2}\right),$$

where $0 < j_1 < j_2 < \cdots$ *are the positive zeros of the Bessel function* $J_0(x) = I_0(ix)$,

$$a_n = \frac{4}{\sin^2 j_n} \qquad (n = 1,2,\dots),$$

$$b_k = 4\left(-1 + \frac{J_1(k\pi)}{k\pi J_0(k\pi)} - \left(\frac{J_1(k\pi)}{J_0(k\pi)}\right)^2\right),$$

$$c_k = 16k\pi\frac{J_1(k\pi)}{J_0(k\pi)} \qquad (k = 1,2,\dots),$$

$$J_1(x) = \frac{1}{i} I_1(ix).$$

Furthermore

$$(2.4) \qquad P(L^*(1) < z) \sim a_1 \exp\left\{-\frac{2j_1^2}{z^2}\right\} \quad \text{as } z \to 0.$$

PROOF OF THEOREM 2.1. From the identity (Abramowitz and Stegun (1970), 9.5.10)

$$(2.5) \qquad J_0(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{j_n^2}\right)$$

we get by logarithmic derivation that

$$(2.6)\qquad -\frac{J_1(z)}{zJ_0(z)}=\sum_{n=1}^{\infty}\frac{2}{z^2-j_n^2}\quad\text{and}$$

$$\frac{1}{z}\frac{I_1(z)}{I_0(z)}=\sum_{n=1}^{\infty}\frac{2}{z^2+j_n^2}.$$

On the other hand, from the partial fraction expansion of cot z (Abramowitz and Stegun (1970), 4.3.91),

$$(2.7)\qquad \cot z=\frac{1}{z}+2z\sum_{k=1}^{\infty}\frac{1}{z^2-k^2\pi^2}.$$

By simple differentiation it follows that

$$(2.8)\qquad \frac{1}{\sin^2 z}=\frac{1}{z^2}-2\sum_{k=1}^{\infty}\frac{1}{z^2+k^2\pi^2}+4z^2\sum_{k=1}^{\infty}\frac{1}{(z^2+k^2\pi^2)^2}.$$

Applying (2.6) and (2.8) we easily get from (2.2)

$$f(\lambda)=\frac{1}{\lambda}-\sqrt{\frac{2}{\lambda}}\frac{I_1(\sqrt{\lambda/2})}{I_0(\sqrt{\lambda/2})(\operatorname{sh}(\sqrt{\lambda/2}))^2}$$

$$(2.9)\qquad =\frac{1}{\lambda}-4\left(\sum_{n=1}^{\infty}\frac{1}{\lambda+2j_n^2}\right)$$

$$\times\left(\frac{2}{\lambda}+4\sum_{k=1}^{\infty}\frac{1}{\lambda+2k^2\pi^2}-16\sum_{k=1}^{\infty}\frac{k^2\pi^2}{(\lambda+2k^2\pi^2)^2}\right).$$

Taking into account that (Watson (1966), 15.51)

$$(2.10)\qquad \sum_{n=1}^{\infty}\frac{1}{j_n^2}=\frac{1}{4},$$

we arrive at

$$(2.11)\qquad f(\lambda)=4\sum_{n=1}^{\infty}\frac{1}{j_n^2(\lambda+2j_n^2)}+4\left(\sum_{n=1}^{\infty}\frac{1}{\lambda+2j_n^2}\right)$$

$$\times\left(4\sum_{k=1}^{\infty}\frac{1}{\lambda+2k^2\pi^2}-16\sum_{k=1}^{\infty}\frac{k^2\pi^2}{(\lambda+2k^2\pi^2)^2}\right).$$

Observe that $f(\lambda)$ is analytic on the complex plane except for $\lambda=-2j_n^2$ $(n=1,2\dots)$ which are simple poles, and $\lambda=-2k^2\pi^2$ $(k=1,2\dots)$ which are poles of order two. Moreover the four series in (2.11) are uniformly absolute convergent for $\operatorname{Re}\lambda\geq 0$ (where (2.10) is used again).

Hence $f(\lambda)$ might be written in the form of partial fractions:

$$(2.12)\qquad \begin{aligned} f(\lambda) = {} & \sum_{n=1}^{\infty} a_n \frac{1}{\lambda + 2j_n^2} + \sum_{k=1}^{\infty} b_k \frac{1}{\lambda + 2k^2\pi^2} \\ & + \sum_{k=1}^{\infty} c_k \frac{1}{(\lambda + 2k^2\pi^2)^2}, \end{aligned}$$

where clearly

$$a_n = \lim_{\lambda \to -2j_n^2} (\lambda + 2j_n^2) f(\lambda),$$

$$b_k = \lim_{\lambda \to -2k^2\pi^2} (\lambda + 2k^2\pi^2) f(\lambda),$$

and

$$c_k = \lim_{\lambda \to -2k^2\pi^2} \frac{d}{d\lambda}\left((\lambda + 2k^2\pi^2)^2 f(\lambda)\right).$$

By simple but tedious calculation using that (Watson (1966))

$$I_0'(z) = I_1(z) \quad \text{and} \quad I_1'(z) = I_0(z) - \frac{I_1(z)}{z}$$

we get that a_n, b_k, and c_k are the coefficients given in our theorem. Now we get the inverse transform of $f(\lambda)$ by termwise inversion (see Doetsch (1950), Theorem 2, page 305). □

COROLLARY 2.1. *Let* $u(t) > 0$ *be a nonincreasing function such that* $\lim_{t\to\infty} u(t) = 0$, $u(t)t^{1/2}$ *is nondecreasing, and* $\lim_{t\to\infty} u(t)t^{1/2} = \infty$. *Let furthermore*

$$(2.13)\qquad I = \int_1^{\infty} \frac{1}{tu^2(t)} \exp\left(-\frac{2j_1^2}{u^2(t)}\right) dt.$$

If $I = \infty$, *then with probability* 1 *there exists a random sequence* $T_1 < T_2 < \cdots < T_n < \cdots \to \infty$ *such that*

$$L^*(T_i) < T_i^{1/2} u(T_i), \qquad i = 1, 2, \dots .$$

If $I < \infty$, *then with probability* 1 *there exists a random* T_0 *such that*

$$L^*(T) \geq T^{1/2} u(T) \quad \text{for } T \geq T_0.$$

PROOF. The proof of Corollary 2.1 is standard, therefore we give a brief outline only. As usual, without loss of generality, it may be assumed that

$$(2.14)\qquad \frac{c^*}{\log\log t} \leq u^2(t) \leq \frac{c^{**}}{\log\log t}$$

with certain conveniently small enough positive c^* and big enough c^{**} constants. Now put $t_1 = 1$, $t_k = \exp\{k/\log k\}$ $(k = 2, 3, \dots)$. Then it is readily seen that the integral I given by (2.13) and the sum $\Sigma_k \exp\{-2j_1^2/u_k^2\}$ converge or diverge together, where $u_k = u(t_k)$. Define the events A_k and B_k by

$$A_k = \{L^*(t_k) < t_k^{1/2} u_k\}, \tag{2.15}$$

$$B_k = \{L^*(t_k) < t_{k+1}^{1/2} u_{k+1}\}. \tag{2.16}$$

Then $I < \infty$ implies $\Sigma_k P(B_k) < \infty$ which in turn implies the second part of Corollary 2.1 by Borel–Cantelli lemma.

On the other hand, $I = \infty$ implies $\Sigma_k P(A_k) = \infty$. To verify (1.19), we note that $L^*(t_k, t_l) = \sup_{-\infty < x < \infty}(L(x, t_l) - L(x, t_k))$ $(k < l)$ is independent of $L^*(t_k)$ and has the same distribution as $L^*(t_l - t_k)$. Since $L^*(t)$ is nondecreasing, we have

$$P(A_k A_l) \le P(A_k) P\big(L^*(t_l - t_k) < t_l^{1/2} u_l\big). \tag{2.17}$$

Now for fixed k, split the indices l $(k < l \le n)$ into three parts:

$$\begin{aligned} L_1 &= \{l: 0 < l - k \le \log l\}, \\ L_2 &= \{l: \log l < l - k \le \log^2 l\}, \\ L_3 &= \{l: \log^2 l < l - k\}. \end{aligned} \tag{2.18}$$

Then from (2.3) and (2.4) one can verify that for any $\varepsilon > 0$ and k large enough

$$P\big(L^*(t_l - t_k) < t_l^{1/2} u_l\big) \le C \exp\left\{ -\frac{2j_1^2}{u_l^2} \frac{t_l - t_k}{t_l} \right\}, \tag{2.19}$$

where C is a suitable constant for $l \in L_1$ or $l \in L_2$ and $C = a_1(1 + \varepsilon)$ for $l \in L_3$.

By using the inequality

$$\log \frac{t_l}{t_k} \ge \frac{l - k}{2 \log l}$$

if k is large enough, one can obtain the estimations

$$\frac{2j_1^2}{u_l^2} \frac{t_l - t_k}{t_l} \ge c_1(l - k), \qquad l \in L_1,$$

$$\frac{2j_1^2}{u_l^2} \frac{t_l - t_k}{t_l} \ge c_2 \log l, \qquad l \in L_2,$$

$$\frac{2j_1^2}{u_l^2} \frac{t_k}{t_l} \le \varepsilon, \qquad l \in L_3,$$

with some constants c_1 and c_2 and all $\varepsilon > 0$, and hence one can verify that

$$\sum_{l \in L_i} P(A_k A_l) \le c P(A_k) \qquad (i = 1, 2) \tag{2.20}$$

and

$$P(A_k A_l) \le (1 + \varepsilon) P(A_k) P(A_l), \qquad l \in L_3. \tag{2.21}$$

Now (1.19) follows and this proves the first part of Corollary 2.1 by the already mentioned sufficient condition of $P(A_k \text{ i.o.}) = 1$ (see the proof of Theorem 1.2).

The technique of this proof has a long history. It goes back to the well-known Kolmogorov–Erdös–Feller–Petrovski integral test and was developed further in many papers and books. The interested reader may find a detailed version of this technique, e.g., in Csáki, Erdös, and Révész (1985).

It follows that $\gamma = j_1\sqrt{2}$ in Kesten's law of the iterated logarithm (see (15) in the Introduction). We can also give the analogues of Theorems 1.6.1 and 1.7.1 in Csörgő and Révész (1981). □

COROLLARY 2.2.

$$\lim_{h \to 0} \inf_{0 \le s \le 1-h} \sup_{-\infty < x < \infty} \sqrt{\frac{\log h^{-1}}{2 j_1^2 h}} |L(x, s+h) - L(x, s)| = 1 \ a.s. \tag{2.22}$$

COROLLARY 2.3. *Let a_T be a nondecreasing function of T for which*

(i) $$0 < a_T \le T,$$

(ii) $$a_T/T \quad \textit{is nonincreasing.}$$

Then

$$\liminf_{T \to \infty} \rho_T Q(T) = 1 \quad a.s., \tag{2.23}$$

where

$$Q(T) = \inf_{0 \le t \le T - a_T} \sup_{-\infty < x < \infty} (L(x, t + a_T) - L(x, t)) \tag{2.24}$$

and

$$\rho_T = \left(\frac{\log(T/a_T) + \log\log T}{2 j_1^2 a_T} \right)^{1/2}. \tag{2.25}$$

If we also have

(iii) $$\frac{\log(T/a_T)}{\log\log T} \uparrow +\infty \qquad (T \to \infty),$$

then

$$\lim_{T \to \infty} \rho_T Q(T) = 1 \quad a.s. \tag{2.26}$$

The proofs of Corollaries 2.2 and 2.3 are the same as those given by Csörgő and Révész (1981) with the slight modification that (2.4) should be used in place of their Lemma 1.6.1, and their Theorem 1.2.1 should be replaced by Theorem 3 from Csáki, Csörgő, Földes, and Révész (1983). Therefore we omit the proofs.

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