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DUALITY FOR GENERAL ATTRACTIVE SPIN SYSTEMS WITH APPLICATIONS IN ONE DIMENSION[1]

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A duality theory is developed which works for general Markovian spin–flip systems with attractive rates. This theory is applied to one-dimensional nearest neighbor translation invariant systems to extend results which were first proved for the contact process by Durrett and Griffeath (1983). In particular, exponential convergence to equilibrium starting from all 1's is shown for noncritical nonergodic systems (Theorem 2). As a consequence, two different definitions of the critical value are shown to be equivalent (Theorem 5). In the course of the proof of Theorem 2, a new result concerning the distribution of the system near edges is obtained (Theorem 4).

1. Introduction. This paper has two main purposes. The first is to develop a theory of dual processes which applies to general spin systems with attractive transition rates. Duality has been a powerful tool in recent research, but only for certain special systems. We have found a way to construct dual processes which does not rely on the restrictive conditions that have always been imposed.

The second main purpose is to illustrate the applicability of our dual processes by using them to generalize some results of Durrett and Griffeath (1983) which were proved (using duality) for the nearest neighbor additive contact process in one dimension. They showed that, except at critical parameter values, the process converges exponentially fast starting from all 1s to an equilibrium measure which has exponentially decaying correlation functions. Results of this type should hold for a large class of translation invariant models with finite range interactions (for an exception, see Fröhlich and Spencer (1981)), but the author is aware of only two other special cases in which some type of exponential mixing is known at *all* noncritical parameter values: the classical nearest neighbor two-dimensional Ising model (a proof can be constructed using high-temperature/low-temperature duality, as found in Benettin et al. (1973) and the fact that exponential decay of correlations occurs for all $T > T_c$ (see Lebowitz (1972))); and certain two-dimensional percolation models (as can be shown using the methods of Chapter 6 in Kesten (1982)). The result of Durrett and Griffeath is the only one involving dynamical spin systems. We will extend this result to the class of all attractive one-dimensional nearest neighbor spin systems which exhibit critical behavior. While this class is not nearly as general as we would like, it is much larger than the class dealt with by Durrett and Griffeath and is sufficient to illustrate the potential usefulness of our dual processes.

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Here is a brief outline of the paper: In Section 2 we construct dual processes using a graphical representation of spin systems that goes back to Harris (1978). Some general properties are developed and examples are given. In Section 3, we give a brief description of the situation in the one-dimensional nearest neighbor case and discuss the results of Durrett and Griffeath. In Section 4 we begin the proof of the theorem on exponential convergence (Theorem 2) by carrying out a modification of the percolation argument used by Durrett and Griffeath. The remainder of the proof is contained in Section 5 and is based on a completely new result (Theorem 4), which states roughly that if the process starts with the sites in $(-\infty, 0]$ occupied, then the distribution of occupied sites is thickest near the edge of the process. Section 5 also contains a proof that the critical value for the system starting from a finite set of occupied sites is the same as it is for the system which starts with all sites occupied.

2. Dual processes. Although our entire construction can be carried out in considerable generality for attractive spin systems, we will restrict our attention to translation invariant systems with finite range interactions. The reader who is interested in greater generality can get a start by looking at the discussion of graphical representations given in Gray (1982).

Let Ξ be the set of all subsets of $\mathbb{Z}^d$, endowed with the usual topology (i.e., $A_n \to A$ iff for each $x \in \mathbb{Z}^d$, $A_n \cap \{x\} = A \cap \{x\}$ for all sufficiently large n.) For each $x \in \mathbb{Z}^d$, let $\beta_x: \Xi \to [0, \infty)$ and $\delta_x: \Xi \to [0, \infty)$ satisfy

(1) $A \supset B \Rightarrow \beta_x(A) \geqq \beta_x(B)$ and $\delta_x(A) \leqq \delta_x(B)$;

(2) $\beta_x(A + x) = \beta_0(A)$ and $\delta_x(A + x) = \delta_0(A)$ for all $A \in \Xi$, where $A + x = \{y: y - x \in A\}$;

(3) there exists a finite set $N_x \subset \mathbb{Z}^d \setminus \{x\}$ such that $\beta_x(A) = \beta_x(A \cap N_x)$ and $\delta_x(A) = \delta_x(A \cap N_x)$ for all $A \in \Xi$.

The functions β_x and δ_x are called the *birth* and *death rates at* x, respectively. Conditions (1), (2), and (3) say that these rates are attractive, translation invariant, and finite range. By (2), we can let $N_x = N_0 + x$ for all x. The birth rate $\beta_x(A)$ is the infinitesimal rate at which a transition is made from A to $A \cup \{x\}$, while $\delta_x(A)$ is the rate at which the system goes from A to $A \setminus \{x\}$.

We will now describe a graphical construction of the spin system with rates β_x and δ_x. We will state many of its properties without proof. See Gray (1982) for the technical details. Generally speaking, a graphical representation is a means of constructing a system $(\xi(s, t, A); s \leqq t, A \in \Xi)$ of Ξ-valued random variables, jointly defined on some probability space $(\Omega, \mathscr{F}, P)$, in such a way that

(4) for each fixed $s \in \mathbb{R}$ and $A \in \Xi$, the process $(\xi(s, t, A); t \in [s, \infty))$ is a spin system with initial state A, initial time s, and rates β_x and δ_x;

(5) $\xi(s, u, A) = \xi(t, u, \xi(s, t, A))$ for all $s \leqq t \leqq u$ and $A \in \Xi$;

(6) $A \supset B \Rightarrow \xi(s, t, A) \supset \xi(s, t, B)$ (this condition is only found in representations of attractive systems);

(7) the shifted system $(\xi(s + u, t + u, A); s \leqq t, A \in \Xi)$ has the same probability law as the original system for all $u \in \mathbb{R}$.

There are many different graphical representations (see Griffeath (1979), where different representations are chosen to represent different systems). We will use only one in this paper. It gives what is known as the "basic coupling" of the processes described in (4). Our construction of dual processes works equally well with other graphical representations, although different representations result in different dual processes.

The construction is based on certain random point locations in the space–time graph $\mathbb{Z}^d \times \mathbb{R}$. The points in those random locations are called *transition points*: If (x, t) is a transition point, then depending on the state of the process just before time t, a birth or death may occur at x at time t. To define these random locations, let $0 < b_1 < b_2 < \cdots < b_i$ and $0 < d_1 < d_2 < \cdots < d_j$ be the distinct nonzero values taken by the birth and death rates, respectively. There are only finitely many such values by (2) and (3). Let $b_0 = d_0 = 0$. For $x \in \mathbb{Z}^d$ and $1 \leqq k \leqq i$, let $B(x, k)$ be a Poisson point location contained in the line $\{x\} \times \mathbb{R}$ with density parameter $b_k - b_{k-1}$. (The line $\{x\} \times \mathbb{R}$ is thought of as a subset of the space–time graph $\mathbb{Z}^d \times \mathbb{R}$.) Similarly, let $D(x, l)$ be a Poisson point location in $\{x\} \times \mathbb{R}$ with parameter $d_l - d_{l-1}$ for $x \in \mathbb{Z}^d$ and $1 \leqq l \leqq j$. (Recall that in a one-dimensional Poisson point location with parameter λ, the intervals between points are iid exponentially distributed random variables with mean $1/\lambda$.) We choose all these Poisson point locations to be mutually independent and let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. The points in each $B(x, k)$ are called *birth points* and those in each $D(x, l)$ are called *death points*. We can assume (after removing a null set if necessary) that

(8) for each $t \in \mathbb{R}$ there is at most one $x \in \mathbb{Z}^d$ such that (x, t) is a transition point;

(9) each bounded subset of $\mathbb{Z}^d \times \mathbb{R}$ contains at most finitely many transition points.

It can be shown that there exists a collection $(\xi(s, t, A); s \leqq t, A \in \Xi)$ of Ξ-valued random variables defined on $(\Omega, \mathscr{F}, P)$ which satisfies (4)–(7). After possibly removing a second null set, it can be shown that this collection satisfies and is uniquely determined by the following conditions:

(10) right continuity and the existence of left limits (i.e., for all $\omega \in \Omega$, $\lim_{t \downarrow u} \xi(s, t, A)(\omega) = \xi(s, u, A)(\omega)$ when $u \geqq s$, and $\xi(s, u^-, A)(\omega) = \lim_{t \uparrow u} \xi(s, t, A)(\omega)$ exists when $u > s$);

(11) a birth occurs at time $t > s$ iff $x \notin \xi(s, t^-, A)$, $(x, t) \in B(x, k)$ for some k and $\beta_x(\xi(s, t^-, A)) \geqq b_k$;

(12) a death occurs at time $t > s$ iff $x \in \xi(s, t^-, A)$, $(x, t) \in D(x, l)$ for some l and $\delta_x(\xi(s, t^-, A)) \geqq d_l$.

(Note that by right continuity, births or deaths cannot occur at the initial time s. Also, (8), (11), and (12) imply that births or deaths cannot occur simultaneously at two different sites.)

We are now ready to define dual processes. In doing so, we will first define certain functions which we call "$[s, t]$-paths." These play essentially the same role here that paths have always played in graphical duality theory. However, unlike the paths in previous versions of duality, they do not live in $\mathbb{Z}^d$. Instead, they live in

$$\Xi_f = \text{the collection of finite subsets of } \mathbb{Z}^d \quad \text{(including the empty set)}.$$

DEFINITION 1. Fix $\omega \in \Omega$. A function $\pi: [s, t] \to \Xi_f$ is called an *$[s, t]$-path from A to B* (for ω) if:

(13) π is right continuous with left limits;

(14) $\pi(s) = A$ and $\pi(t) = B$;

(15) if $s \leqq u \leqq v \leqq t$, then $\xi(u, v, \pi(u)) \supset \pi(v)$;

(16) π is *minimal* in the sense that if $\tilde{\pi}: [s, t] \to \Xi_f$ satisfies (13)–(15) and if $\tilde{\pi}(u) \subset \pi(u)$ for all $u \in [s, t]$, then $\pi = \tilde{\pi}$.

We say that π is an $[s, t]$-path to B *out of* A if π is an $[s, t]$-path to B from A' for some $A' \subset A$. We will omit the words "for ω" in referring to $[s, t]$-paths, even though they are always implied.

DEFINITION 2. For each $B \in \Xi_f$ and $t \in \mathbb{R}$, the *dual process* with terminal time t and terminal state B is the collection $(\zeta(s, t, B), s \in (-\infty, t])$ defined by

$$\zeta(s, t, B) = \{A \in \Xi_f : \text{there exists an } [s, t]\text{-path from } A \text{ to } B\}.$$

For dual processes, time is thought of as running backwards from the terminal time. It is easy to check that $\zeta(t, t, B) = \{B\}$, so it is only a slight abuse of terminology to call B the terminal state. The following theorem gives the exact relationship between the system of dual processes and the original spin system. During the course of the proof, we will obtain much useful information about $[s, t]$-paths. Since the proof is rather long, the reader may wish to first study the statements of the theorem and the lemmas used in the proof and then skip ahead to the discussion and examples at the end of the section before reading all the details. In order for Theorem 1 to be true for all ω as stated, one last null set may need to be removed from Ω. We will explain during the proof how this is done.

THEOREM 1. *If $A \in \Xi$, $B \in \Xi_f$, and $s \leqq t$, then*

(17) $B \subset \xi(s, t, A)$ *iff* $A' \in \zeta(s, t, B)$ *for some* $A' \subset A$; *or equivalently,*

(18) $B \subset \xi(s, t, A)$ *iff there is an* $[s, t]$*-path to* B *out of* A.

PROOF. The equivalence between (17) and (18) is immediate from the definitions. We will prove (18). First, let π be an $[s, t]$-path to B from A' for some $A' \subset A$. By (14), $\pi(s) = A'$ and $\pi(t) = B$. Therefore, (15) implies that $\xi(s, t, A') \supset B$. It follows from (6) that $\xi(s, t, A) \supset B$, completing the "if" part of the proof.

Now assume that $\xi(s, t, A) \supset B$. Define $\tilde{\pi}(u) = \xi(s, u, A)$ for $u \in [s, t]$. Then $\tilde{\pi}$ satisfies (13) by (10) and it satisfies (15) by (5). However, in general, $\tilde{\pi}$ is not

Ξ_f-valued and does not satisfy (16), so $\tilde{\pi}$ is generally not an $[s, t]$-path. We need to "thin out" $\tilde{\pi}$ until it is Ξ_f-valued and does satisfy (16). The steps of this procedure will be organized into the following lemmas:

LEMMA 1. *If* $\tilde{\pi}: [s, t] \to \Xi$ *satisfies* (13) *and* (15), *then for each* $u \in [s, t]$ *and* $x \in \tilde{\pi}(u)$, *the following three conditions hold*:

(19) *if* (x, u) *is not a transition point, then* $x \in \tilde{\pi}(u^-)$;

(20) *if* $(x, u) \in B(x, k)$, *then either* $x \in \tilde{\pi}(u^-)$ *or* $\beta_x(\pi(u^-)) \geq b_k$;

(21) *if* $(x, u) \in D(x, l)$, *then* $x \in \tilde{\pi}(u^-)$ *and* $\delta_x(\tilde{\pi}(u^-)) < d_l$.

LEMMA 2. *If* $\tilde{\pi}: [s, t] \to \Xi$ *satisfies* (13) *and* (19)–(21), *then for any* $B \subset \tilde{\pi}(t)$, *there is a function* $\pi: [s, t] \to \Xi$ *such that* $\pi(u) \subset \tilde{\pi}(u)$ *for all* $u \in [s, t]$ *and such that* π *is a minimal function which satisfies* (13), (19)–(21), *and the condition* $\pi(t) \supset B$.

LEMMA 3. *If* $\pi: [s, t] \to \Xi$ *is a minimal function which satisfies* (13), (19)–(21), *and the condition* $\pi(t) \supset B$ *for some* $B \in \Xi_f$, *then* $\pi(u) \in \Xi_f$ *for all* $u \in [s, t]$.

LEMMA 4. *If* $\pi: [s, t] \to \Xi_f$ *is a minimal function which satisfies* (13), (19)–(21), *and the condition* $\pi(t) \supset B$ *for some* $B \in \Xi_f$, *then* π *is an* $[s, t]$-*path to* B.

Note that if we apply Lemmas 1–4 to the function $\tilde{\pi}(\cdot) = \xi(s, \cdot, A)$ defined above, then we can obtain an $[s, t]$-path to B out of A, so the "only if" part of Theorem 1 follows from Lemmas 1–4.

PROOF OF LEMMA 1. Let $\pi: [s, t] \to \Xi$ satisfy (13) and (15). Choose $u \in [s, t]$ and $x \in \tilde{\pi}(u)$. By (9) there exists $u' \in (s, u)$ such that the space–time set $(N_x \cup \{x\}) \times [u', u)$ contains no transition points. By (11) and (12), no births or deaths can occur on the set $N_x \cup \{x\}$ during $[u', u)$, so

$$\xi(u', u^-, \tilde{\pi}(u')) \cap (N_x \cup \{x\}) = \tilde{\pi}(u') \cap (N_x \cup \{x\}).$$

By (13) we can assume that u' was chosen so that $\tilde{\pi}(u') \cap (N_x \cup \{x\}) = \tilde{\pi}(u^-) \cap (N_x \cup \{x\})$, so

$$\xi(u', u^-, \tilde{\pi}(u')) \cap (N_x \cup \{x\}) = \tilde{\pi}(u^-) \cap (N_x \cup \{x\}). \tag{22}$$

By (3) and (22),

$$\begin{aligned} &\beta_x(\xi(u', u^-, \tilde{\pi}(u'))) = \beta_x(\tilde{\pi}(u^-)) \quad \text{and} \\ &\delta_x(\xi(u', u^-, \tilde{\pi}(u'))) = \delta_x(\tilde{\pi}(u^-)). \end{aligned} \tag{23}$$

By (15), $\tilde{\pi}(u) \subset \xi(u', u, \tilde{\pi}(u'))$, so $x \in \xi(u', u, \tilde{\pi}(u'))$. If (x, u) is not a transition point, then (11) and (12) imply that $x \in \xi(u', u^-, \tilde{\pi}(u'))$, and so (19) follows from (22). If $(x, u) \in B(x, k)$, then (11) and (12) imply either $x \in \xi(u', u^-, \tilde{\pi}(u'))$ or $\beta_x(\xi(u', u^-, \tilde{\pi}(u'))) \geq b_k$, and (20) follows from (22) and (23). Similarly, (21) follows from (22) and (23). □

Proof of Lemma 2. Let π_n: $[s, t] \to \Xi$ be functions which satisfy (13), (19)–(21), and the condition $\pi_n(t) \supset B$ such that $\pi_{n+1}(u) \supset \pi_n(u)$ for all $u \in [s, t]$ and $n \geqq 1$. It is easily checked that $\pi = \lim_n \pi_n$ exists and satisfies (13), (19)–(21), and the condition $\pi(t) \supset B$. It follows that for each time $u \in [s, t]$ and each finite set $B' \subset \mathbb{Z}^d$, we can find a function π: $[s, t] \to \Xi$ which satisfies (19)–(21) and the condition $\pi(t) \supset B$ such that if π' is any other such function and if $\pi'(v) \subset \pi(v)$ for all $v \in [s, t]$, then $\pi'(u) \cap B' = \pi(u) \cap B'$. Since there are only countably many finite subsets of $\mathbb{Z}^d$, a standard diagonalization procedure implies that π can be chosen to satisfy this last condition for all rational $u \in [s, t]$ and all finite sets B'. It follows from the continuity assumptions that π satisfies this condition for all $u \in [s, t]$, so π has the desired minimality properties. □

Proof of Lemma 3. We will define a collection $(\phi(s, t, B);\ s \leqq t,\ B \in \Xi_f)$ of Ξ_f-valued processes and then prove that if π: $[s, t] \to \Xi$ is a minimal function satisfying (13), (19)–(21), and the condition $\pi(t) \supset B$ for some $B \in \Xi_f$, then $\pi(u) \subset \phi(u, t, B)$ for all $u \in [s, t]$.

Fix $B \in \Xi_f$ and $t \in \mathbb{R}$. Let $\tau_0 = t$. If (x, τ_0) is a transition point for some $x \in B$, let x_0 be the (unique) point such that (x_0, τ_0) is a transition point and let $B_0 = B \cup N_x$. Otherwise, let $B_0 = B$ and let x_0 be an arbitrary member of B. Now define τ_n, B_n, and x_n inductively for $n \geqq 1$ as follows:

$$\tau_n = \sup\{u < t\colon (x, u) \text{ is a transition point for some } x \in B_{n-1}\};$$

$$x_n = \text{the (unique) site } x \text{ such that } (x_n, \tau_n) \text{ is a transition point;}$$

$$B_n = B_{n-1} \cup N_{x_n}.$$

(Note that x_n exists by (8) and (9) and the fact that B_{n-1} is finite.) Let $\phi(t, t, B) = B_0$ and $\phi(u, t, B) = B_n$ for $u \in [\tau_{n+1}, \tau_n)$. Because of the way in which the positions of transition points were determined using Poisson point locations, it is easily checked that $|\phi(u, t, B)|$ is stochastically dominated by a branching process with initial size $|B_0|$ in which time runs backward from time t and in which each member of the population produces $|N_x|$ offspring at rate $\bar{\beta} + \bar{\delta}$, where $\bar{\beta}$ and $\bar{\delta}$ are the maximum birth and death rates, respectively. Thus, except on some null set $N \subset \Omega$, $\phi(u, t, B)$ is a well-defined finite set for all $u \leqq t$. Since Ξ_f is countable, this null set can be made to serve for all $B \in \Xi_f$ and all rational t. We claim that it works for all t. To see this, fix $t \in \mathbb{R}$. By (8) and (9), for each choice of the set $B \in \Xi_f$ there exists a rational number $t' < t$ such that $B_0 \times [t', t)$ contains no transition points. By definition, $\phi(u, t, B) = B_0 = \phi(t', t', B_0)$ for all $u \in [t', t]$. It follows from the construction that $\phi(u, t, B) = \phi(u, t', B_0)$ for all $u \leqq t'$, so $\phi(u, t, B)$ is a well-defined finite set for all $u \leqq t'$ except on the null set, as claimed. From now on, we assume that N has been removed from Ω, so that $\phi(s, t, B)$ is a well-defined member of Ξ_f for all $s \leqq t$, $B \in \Xi_f$ and $\omega \in \Omega$.

Now let π: $[s, t] \to \Xi$ satisfy the conditions of Lemma 3 for some $B \in \Xi_f$. We will show that $\pi(u) \subset \phi(u, t, B)$ for all $u \leqq t$, proving the lemma. Let

$$\theta = \inf\{u \in [s, t]\colon \pi(v) \subset \phi(v, t, B) \text{ for all } v \in [u, t]\}.$$

Note that by right continuity, $\pi(v) \subset \phi(v, t, B)$ for all $v \in [\theta, t]$, provided that $\theta \in [s, t]$ (or in other words, provided that $\theta \neq \infty$). We will prove eventually that $\theta = s$. First we will show that $\theta \leqq t$ by proving that $\pi(t) \subset \phi(t, t, B) = B_0$. Actually we will prove that

$$\pi(t) = B. \tag{24}$$

Define π': $[s, t] \to \Xi$ by letting $\pi'(t) = B$ and $\pi'(u) = \pi(u)$ for $s \leqq u < t$. Clearly π' satisfies (13), (19)–(21), and the condition $\pi'(t) \supset B$, so by minimality, $\pi' = \pi$, proving (24) and the claim that $\theta \leqq t$. For the purpose of obtaining a contradiction, assume that $s < \theta \leqq t$. As noted above, $\pi(\theta) \subset \phi(\theta, t, B)$. Choose n such that $\tau_{n+1} < \theta \leqq \tau_n$, and define π': $[s, t] \to \Xi$ by

$$\begin{aligned} \pi'(u) &= \pi(u), & u &\in [\theta, t] \cup [s, \tau_{n+1}), \\ &= \bigcap_{v \in [\tau_{n+1}, \theta)} (\pi(v) \cap B_n), & u &\in [\tau_{n+1}, \theta). \end{aligned}$$

Note that $\pi'(u) \subset \pi(u)$ for all $u \in [s, t]$, that $\pi'(t) = B$, and that π' satisfies (13). Clearly, π' satisfies (19)–(21) for $v \notin [\tau_{n+1}, \theta]$ since π does. Since $B_n \times (\tau_{n+1}, \theta)$ contains no transition points, it follows that π' also satisfies (19)–(21) for $v \in (\tau_{n+1}, \theta)$. Furthermore, (19) implies that $\pi(u) \cap B_n$ must be decreasing on (τ_{n+1}, θ), so $\pi(\theta^-) \cap B_n = \pi'(\theta^-)$. If (x, θ) is a transition point, then $x = x_n$ and $\{x\} \cup N_x \subset B_n$, so it follows from (3) that $\beta_x(\pi(\theta^-)) = \beta_x(\pi'(\theta^-))$ and $\delta_x(\pi(\theta^-)) = \delta_x(\pi'(\theta^-))$. It is now easily checked that π' satisfies (19)–(21) for $v = \theta$ since π satisfies (19)–(21). Finally, since $\pi'(\tau_{n+1}) \subset \pi(\tau_{n+1})$, it follows that π' also satisfies (19)–(21) at $v = \tau_{n+1}$. Thus, π' satisfies (19)–(21) on $[s, t]$. By the minimality of π, $\pi = \pi'$, from which it follows that $\pi(u) \subset B_n = \phi(u, t, B)$ for $u \in [\tau_{n+1}, \theta)$, contradicting the definition of θ. □

Note that in the above argument, π' (and hence π) is constant on the interval $[\tau_{n+1}, \theta)$. The argument works for any $\theta \leqq \tau_n$, so we have also proved that

(25) π can only have discontinuities at the times τ_n, and $\pi(u) \subset B_n$ for all $u \in [\tau_{n+1}, \tau_n)$.

Proof of Lemma 4. Let π be as in the statement of Lemma 4. By (24), $\pi(t) = B$. Thus to show that π is an $[s, t]$-path to B, it is enough to prove (15) and (16). Let τ_n be as in the proof of Lemma 3 for $n \geqq 0$. We will prove by induction on $k \geqq 0$ that for all $n \geqq 0$,

(26) if $\tau_{n+k} > s$, then $\xi(u, v, \pi(u)) \supset \pi(v)$ for all $u \in [\tau_{n+k+1} \vee s, \tau_{n+k}]$ and $v \in [\tau_{n+1} \vee s, \tau_n]$.

Since $\tau_n \to -\infty$ as $n \to \infty$, (15) follows from (26). We first take the case $k = 0$. By (25), π is constant on $[\tau_{n+1}, \tau_n)$ and contained in B_n. Since $B_n \times (\tau_{n+1}, \tau_n)$ contains no transition points, it follows as in the proof of (22) that

$$\xi(u, v, \pi(u)) \cap B_n = \pi(u) \cap B_n = \pi(v) \cap B_n = \pi(v) = \pi(\tau_n^-)$$

whenever $\tau_{n+1} \leqq u \leqq v < \tau_n$, which proves (26) for $v < \tau_n$. It also shows that

$$\xi(u, \tau_n^-, \pi(u)) \cap B_n = \pi(\tau_n^-) \quad \text{for } u \in [\tau_{n+1}, \tau_n). \tag{27}$$

We will use (27) to prove (26) for $v = \tau_n$. Choose $x \in \pi(\tau_n)$ and $u \in [\tau_{n+1}, \tau_n)$. We must show that $x \in \xi(u, \tau_n, \pi(u))$. We know that $x \in B_{n-1} \subset B_n$ since $\pi(\tau_n) \subset B_{n-1}$. If (x, τ_n) is not a transition point, then (19) implies that $x \in \pi(\tau_n^-)$, and thus $x \in \xi(u, \tau_n^-, \pi(u))$ by (27). It follows from (11) and (12) that $x \in \xi(u, \tau_n, \pi(u))$. If (x, τ_n) is a transition point, then since $x = x_n$ by definition, $B_n \supset N_x$. Therefore by (27), $\beta_x(\xi(u, \tau_n^-, \pi(u))) = \beta_x(\pi(\tau_n^-))$ and $\delta_x(\xi(u, \tau_n^-, \pi(u))) = \delta_x(\pi(\tau_n^-))$. Now follow an argument similar to the one used in the case when (x, τ_n) was not a transition point, this time using (20) or (21) instead of (19), to conclude that $x \in \xi(u, \tau_n, \pi(u))$.

The inductive step is easy. Choose $k \geq 1$, $u \in [\tau_{n+k+1} \vee s, \tau_{n+k})$ and $v \in [\tau_{n+1}, \tau_n]$. By the inductive hypothesis (applied twice), $\xi(u, \tau_{n+1}, \pi(u)) \supset \pi(\tau_{n+1})$ and $\xi(\tau_{n+1}, v, \pi(\tau_{n+1})) \supset \pi(v)$. Now apply (5) and (6) to get (26).

It is also easy to show that π satisfies (16). Let π': $[s, t] \to \Xi_f$ satisfy (13), (15), and the condition $\pi'(t) = B$, and assume that $\pi'(u) \subset \pi(u)$ for all $u \in [s, t]$. By Lemma 1, π' satisfies (19)–(21), so $\pi = \pi'$ by the minimality of π. It follows that π satisfies (16). □

In proving Theorem 1, we have proved several other useful facts along the way which we restate in the following lemmas for later use:

LEMMA 5.

(a) *If* $\tilde{\pi}$: $[s, t] \to \Xi$ *satisfies* (13) *and* (19)–(21), *then for all* $B \in \Xi_f$ *such that* $B \subset \tilde{\pi}(t)$ *there exists an* $[s, t]$*-path* π *to* B *such that* $\pi(u) \subset \tilde{\pi}(u)$ *for all* $u \in [s, t,]$.
(b) *If* π: $[s, t] \to \Xi$ *satisfies* (13) *and* (15), *then the conclusion is the same as in* (a).
(c) *If* $B \subset \xi(s, t, A)$ *for some* $B \in \Xi_f$, *then there exists an* $[s, t]$*-path to* B *out of* A *such that* $\pi(u) \subset \xi(s, u, A)$ *for all* $u \in [s, t,]$.

PROOF.

(a) Follows from Lemmas 2–4.
(b) Follows from (a) and Lemma 1.
(c) This is what we actually proved in Theorem 1. □

LEMMA 6. *Choose* $B \in \Xi_f$. *A function* π *is an* $[s, t]$*-path to* B *iff* π *is a minimal function satisfying* (13), (19)–(21), *and the condition* $\pi(t) \supset B$.

PROOF. If π is an $[s, t]$-path to B, then π satisfies (13) and (15), so Lemmas 1–4 imply the existence of an $[s, t]$-path π' to B which is a minimal function satisfying (13), (19)–(21), and the condition $\pi'(t) \supset B$ such that $\pi'(u) \subset \pi(u)$ for all $u \in [s, t,]$. By (16), $\pi = \pi'$, proving the "only if" portion of Lemma 6. The "if" portion follows from Lemmas 3 and 4. □

LEMMA 7. *Choose $B \in \Xi_f$. If π is an $[s, t]$-path to B, then π satisfies* (25).

PROOF. From the proof of Theorem 1 we know that π satisfies (25) if π is a minimal function satisfying (13), (19)–(21), and the condition $\pi(t) \supset B$. The conclusion of Lemma 7 now follows from Lemma 6. □

LEMMA 8. *For any $B \in \Xi_f$, there are at most finitely many $[s, t]$-paths to B. In particular, $\zeta(s, t, B)$ is finite for all $s \leqq t$ and $B \in \Xi_f$.*

PROOF. There are at most finitely many functions which satisfy (25) for any fixed $s \leqq t$ and $B \in \Xi_f$, so Lemma 8 follows from Lemma 7. □

LEMMA 9. *A function π: $s \leq t \to \Xi_f$ which satisfies* (13) *and* (14) *is an $[s, t]$-path to B iff for each $u \in [s, t]$, $\pi(u^-)$ is a minimal set which satisfies* (19)–(21) *for all $x \in \pi(u)$.*

PROOF. This is easily derived from Lemma 6. □

It will be convenient to characterize the above condition that $\pi(u^-)$ is a minimal set which satisfies (19)–(21) for all $x \in \pi(u)$. It is easily seen to be equivalent to:

(19′) if $\pi(u) \times \{u\}$ contains no transition points, then $\pi(u^-) = \pi(u)$;

(20′) if $(x, u) \in B(x, k)$ for some $x \in \pi(u)$, then $\pi(u^-) = (\pi(u) \setminus \{x\}) \cup D$, where D is a minimal subset of $N_x \cup \{x\}$ such that either $x \in D$ or $\beta_x((\pi(u) \setminus \{x\}) \cup D) \geq b_k$;

(21′) if $(x, u) \in D(x, l)$ for some $x \in \pi(u)$, then $\pi(u^-) = \pi(u) \cup D'$ where D' is a minimal subset of N_x such that $\delta_x(\pi(u) \cup D') < d_l$.

We next give some examples to illustrate the relationship between our dual processes and those that have been previously defined:

EXAMPLE 1. The contact process. Take the dimension d to be 1 and let $N_x = \{x - 1, x + 1\}$. Define $\delta_x \equiv 1$ and

$$\begin{aligned} \beta_x(A) &= 0 && \text{if } A \cap N_x = \varnothing, \\ &= \lambda && \text{if } A \cap N_x = \{x-1\} \text{ or } \{x+1\}, \\ &= \theta\lambda && \text{if } A \cap N_x = N_x, \end{aligned}$$

where $\lambda > 0$ and $\theta \geqq 1$. This is the process studied by Durrett and Griffeath (1983). Depending on the value of θ, three different situations are possible:

(i) $\theta = 1$. In this case, it is not hard to check from Lemma 9 that if B is a singleton, then any $[s, t]$-path to B is singleton-valued for all $u \in [s, t,]$. Thus, $\zeta(s, t, \{x\})$ is always a (possibly empty) collection of singletons for $x \in \mathbb{Z}$ and $s \leq t$. Under the obvious identification, we could think of $\zeta(s, t, \{x\})$ as a set in Ξ_f rather than a collection of singletons. If we take this point of view, then the process $(\zeta(s, t, \{x\});\ s \in (-\infty, t])$ is equivalent to the usual dual process found in Durrett and Griffeath (1983) and in many other papers.

(ii) $1 < \theta \leqq 2$. In this case, an $[s, t]$-path π is not necessarily singleton-valued, even when $\pi(t)$ is a singleton. Consequently, our dual processes are not equivalent to dual processes that have been previously defined. The reason for this is that in previous treatments of graphical duality theory, a different graphical representation is used for each value of θ. The graphical representation that we use is independent of θ. If instead, we had also used different representations for different values of θ, then our definition of dual processes would lead to a duality theory which is equivalent to previous versions in those cases where previous versions existed.

(iii) $\theta > 2$. In this case, no previous version of duality exists. Our construction works no matter what value θ takes.

EXAMPLE 2. The voter model. Change the death rates in Example 1 so that $\delta_x(A) = \beta_x(A^c)$. Previous versions of duality exist for this model if $2 \leqq \theta \leqq 3$ (see Holley and Liggett (1978) and Holley and Stroock (1979)). The reader may find it a useful exercise to work out our version of duality for this example and compare it with others. Our dual process has admittedly more complex behavior, but it should be noted that even in this relatively simple example, the graphical representation that goes with previous versions requires some cleverness to construct (see Griffeath (1979)).

One of the main applications of duality is to the study of the asymptotic behavior of spin systems. We will conclude this section with a general discussion of the connection between duality and asymptotic behavior. The remainder of the paper will be concerned with a specific application.

As in previous versions of duality, there are two special situations that are often singled out. We will say that $\zeta(s, t, B)$ is *dead* if $\zeta(s, t, B) = \varnothing$ and we will call $\zeta(s, t, B)$ *immortal* if $\varnothing \in \zeta(s, t, B)$. (Note the distinction here: In the first case, $\zeta(s, t, B)$ is the empty collection of subsets of Ξ_f and no $[s, t]$-paths to B exist; in the second case, the collection $\zeta(s, t, B)$ contains the empty set, so there is an $[s, t]$-path to B from $\varnothing$.) It is easy to check that if $\zeta(s, t, B)$ is dead or immortal, then $\zeta(r, t, B)$ is dead or immortal, respectively, for all $r \leqq s$, so we can think of "dead" and "immortal" as two absorbing states in our dual processes, even though there is actually more than one state represented by the term "immortal." By Theorem 1,

$$\zeta(s, t, B) \text{ is dead iff } \xi(s, t, \mathbb{Z}^d) \text{ does not contain } B;$$

$$\zeta(s, t, B) \text{ is immortal iff } \xi(s, t, \varnothing) \supset B.$$

If we let $B = \{x\}$, it follows that $\xi(s, t, \varnothing)$ and $\xi(s, t, \mathbb{Z}^d)$ agree at a site x iff $\zeta(s, t, \{x\})$ is either dead or immortal, so

(28) $\xi(s, t, \mathbb{Z}^d) \cap B = \xi(s, t, \varnothing) \cap B$ iff for each $x \in B$, $\zeta(s, t, \{x\})$ is either dead or immortal.

It is well-known that for systems with attractive rates, $\xi(s, t, \varnothing)$ and $\xi(s, t, \mathbb{Z}^d)$ converge in distribution to equilibria μ_0 and μ_1, respectively, as $t \to \infty$. Furthermore, the system is ergodic (i.e., has a unique equilibrium) iff $\mu_0 = \mu_1$. These facts were first proved by Holley (1972). Since $\xi(s, t, \varnothing) \subset \xi(s, t, \mathbb{Z}^d)$ by (6), it follows from this and (28) that the system is ergodic iff for all $x \in \mathbb{Z}^d$,

$$\lim_{t \to \infty} P(\zeta(s, t, \{x\}) \text{ is dead or immortal}) = 1.$$

By the translation invariance in space and time of our graphical representation and the fact that "dead" and "immortal" are absorbing, it follows that

(29) there is a unique equilibrium iff $P(\zeta(s, 0, \{0\}))$ is dead or immortal for some $s \leqq 0) = 1$.

This criterion is analogous to the corresponding criterion in previous versions of duality.

One of the most commonly studied situations is when $\beta_x(\varnothing) = 0$. In this case, $\varnothing$ is an absorbing state for the spin system, so μ_0 is the point mass concentrated on $\varnothing$. It also follows in this case that $\zeta(s, t, B)$ cannot be immortal if $B \neq \varnothing$, so (29) becomes

(30) if $\beta_x(\varnothing) = 0$, then there is a unique equilibrium iff $P(\zeta(s, 0, \{0\})$ is dead for some $s \leqq 0) = 1$.

We will be concerned with this case in the sections that follow, and it will be convenient to restate some of the facts discussed here in terms of $[s, t]$-paths. We call a function $\pi: (-\infty, t] \to \Xi_f$ a $(-\infty, t]$-path to B if the restriction of π to $[s, t]$ is an $[s, t]$-path to B for all $s \in (-\infty, t]$. Clearly, if there exists a $(-\infty, t]$-path to B, then $\zeta(s, t, B)$ is not dead for any $s \in (-\infty, t]$. The implication also goes the other way: If $\zeta(s, t, B)$ is not dead for any $s \in (-\infty, t]$, then there exists a sequence of times $t > s_1 > s_2 > \cdots$ such that $s_n \to -\infty$ as $n \to \infty$ and a sequence of functions $\pi_1, \pi_2, \ldots$ such that π_n is an $[s_n, t]$-path to B for all n. For all $m \leqq n$, the restriction of π_n to $[s_m, t]$ is an $[s_m, t]$-path. (This follows from Lemma 9.) Since there are only finitely many $[s_m, t]$-paths to B by Lemma 8, it follows that along a subsequence n_k, the functions π_{n_k} agree on $[s_m, t]$. By taking further subsequences, we can find a function π which is an $(-\infty, t]$-path to B. This part of the discussion has not relied on the assumption that $\beta_x(\varnothing) = 0$, so we have proved the following analogue to Theorem 1:

THEOREM 1′. *Fix $t \in \mathbb{R}$. Then $B \subset \xi(s, t, \mathbb{Z}^d)$ for all $s \in (-\infty, t]$ iff $\zeta(s, t, B)$ is not dead for any $s \in (-\infty, t]$ iff there is a $(-\infty, t]$-path to B.*

Analogous definitions and versions of Theorem 1′ can be made for $(-\infty, \infty)$-paths and $[s, \infty)$-paths.

If we combine Theorem 1′ with (30), we get

(31) if $\beta_x(\varnothing) = 0$, then $\mu_1 \neq \mu_0$ iff P(there exists a $(-\infty, t]$-path to $\{0\}) > 0$.

Also, since

$$\begin{aligned} \mu_1(\xi: x \in \xi) &= \lim_{t \to \infty} P(x \in \xi(0, t, \mathbb{Z}^d)) \\ &= \lim_{s \to -\infty} P(\zeta(s, 0, \{x\}) \text{ is not dead}) \qquad (32) \\ &= P(\zeta(s, 0, \{x\}) \text{ is not dead for any } s \leqq 0), \end{aligned}$$

it follows from Theorems 1 and 1′ that

$$
\begin{aligned}
&|P(x \in \xi(0,t,\mathbb{Z}^d)) - \mu_1(\xi\colon x \in \xi)| \\
&\quad = P(\text{there exists a } [0,t]\text{-path to } \{x\}) \\
&\qquad - P(\text{there exists a } (-\infty,0]\text{-path to } \{x\}) \\
&\quad = P(\text{there exists a } [-t,0]\text{-path to } \{x\}) \\
&\qquad - P(\text{there exists a } (-\infty,0]\text{-path to } \{x\}),
\end{aligned} \tag{33}
$$

so for all $t \leqq 0$,

$$
\begin{aligned}
&|P(x \in \xi(0,t,\mathbb{Z}^d)) - \mu_1(\xi\colon x \in \xi)| \\
&\quad = P(\text{there exists a } [-t,0]\text{-path to } \{x\} \\
&\qquad \text{but not a } (-\infty,0]\text{-path to } \{x\}).
\end{aligned} \tag{34}
$$

We will use (34) to obtain the exponential convergence theorem mentioned in the introduction.

3. Nearest neighbor systems on $\mathbb{Z}$. In this section and throughout the rest of the paper we consider only the one-dimensional nearest neighbor case:

$$\Xi = \text{all subsets of } \mathbb{Z} \text{ and } N_x = \{x-1, x+1\} \text{ for all } x \in \mathbb{Z}. \tag{35}$$

We will find it convenient in this case to separate all the possible death rates into one-parameter families: Assume that $\delta_x(\mathbb{Z}) = 0$ for all x and let

$$\delta_x^\varepsilon = \delta_x + \varepsilon \quad \text{for } \varepsilon \geqq 0.$$

Because of the attractiveness assumption, any death rate can be written as δ_x^ε for some unique $\varepsilon \geqq 0$ and δ_x satisfying $\delta_x(\mathbb{Z}) = 0$. We will assume from now on that all death rates are written this way and that when we write δ_x without an ε as a superscript, then $\delta_x(\mathbb{Z}) = 0$. The following facts are known concerning the ergodic behavior of the family of systems with birth rates β_x and death rates δ_x^ε:

(36) if $\beta_x(\varnothing) > 0$, then the spin system with rates β_x and δ_x^ε is ergodic for all $\varepsilon > 0$;

(37) if $\beta_x(\varnothing) = 0$ and $\beta_x(\{x-1)\} + \beta_x(\{x+1\}) \leqq \delta_x(\{x-1\}) + \delta_x(\{x+1\})$, then the spin system with rates β_x and δ_x^ε is ergodic for all $\varepsilon > 0$ with unique equilibrium $\mu_1 = \mu_0 =$ the point mass at $\varnothing$;

(38) if $\beta_x(\varnothing) = 0$ and $\beta_x(\{x-1\}) + \beta_x(\{x+1\}) > \delta_x(\{x-1\}) + \delta_x(\{x+1\})$, then there exists $\varepsilon_c \in (0,\infty)$ such that the system with rates β_x and δ_x^ε is ergodic for $\varepsilon > \varepsilon_c$ and nonergodic for $\varepsilon < \varepsilon_c$;

These results are found in Gray (1982) and Gray and Griffeath (1982). (The case when $\varepsilon = 0$ is trivially nonergodic if $\beta_x(\varnothing) = 0$; otherwise it can be made equivalent to the cases in (37) and (38) by considering the family $((\xi(s,t,A)^c;\ s \leqq t,\ A \in \Xi))$.

In spite of these results, there remain some unanswered questions: What is the behavior for $\varepsilon = \varepsilon_c$? What is the exact value of ε_c? What is the speed of convergence to equilibrium? The first two questions have not been answered for any system. Any progress would be interesting. The third question has been answered for certain special cases. In (36), the convergence to equilibrium is known to be exponentially fast if ε and $\beta_x(\varnothing)$ are large enough (see Sullivan (1974)). In (37), the convergence is always exponentially fast (this was proved for the contact process by Griffeath (1981); the general case follows from the ideas in Durrett (1980)). If $\varepsilon > \varepsilon_c$ in (38), the result is the same as in (37). This leaves the case $0 < \varepsilon < \varepsilon_c$ in (38), which is the subject of the remainder of this paper. We will show that $\xi(0, t, \mathbb{Z})$ converges exponentially fast to μ_1 for all $\varepsilon \in [0, \varepsilon_c)$. Durrett and Griffeath (1983) proved this result for the contact process of Example 1 under the restriction that $\theta \leq 2$. As mentioned in Example 1, this restriction has been needed in the past in order for previous versions of duality to be applied. (Incidentally, the rates in Example 1 and in Durrett and Griffeath (1983) are not parameterized in the same way that they are in this section, and the reader should keep this in mind when comparing the results of Durrett and Griffeath to ours.)

THEOREM 2. *Let* $(\xi(s, t, A); s \leq t; A \in \Xi)$ *be a spin system with rates* β_x *and* δ_x^ε *as in* (38), *with* $0 \leq \varepsilon < \varepsilon_c$. *Then for all* $B \in \Xi_f$, *there exist constants* $a, b > 0$ *such that*

$$|P(\xi(s, t, \mathbb{Z}) \supset B) - \mu_1(\xi\colon \xi \supset B)| < ae^{b(s-t)}$$

for all $t > s$.

Theorem 2 has several consequences, the most important of which is that μ_1 must have exponentially decaying correlation functions under the above hypotheses. The interested reader should consult the paper of Durrett and Griffeath, where this and other results of this type are proved. All of these can either be derived from Theorem 2 or proved in a similar manner. See Theorem 5 below for another consequence of our proof.

The proof of Theorem 2, which is contained in the next two sections, follows along the lines of that of Durrett and Griffeath, with various changes being made to accommodate our version of duality. A key ingredient is the following result of Durrett (1980). For $A \in \Xi$, let

$$L(A) = \inf\{x\colon x \in A\}, \qquad R(A) = \sup\{x\colon x \in A\}.$$

THEOREM 3 (Durrett). *Let* β_x *and* δ_x^ε *be as in* (37) *or* (38). *For all* $\varepsilon \geq 0$, *there exist constants* λ *and* ρ *in* $[-\infty, \infty]$ *such that for all* $n \in \mathbb{Z}$,

$$L(\xi(0,t,[n,\infty)))/t \to \lambda \text{ and } R(\xi(0,t,(-\infty,n]))/t \to \rho \quad a.s. \text{ as } t \to \infty.$$

If $\varepsilon \leq \varepsilon_c$, *then* λ *and* ρ *are finite and* $\rho \geq \lambda$.

(We should point out that Durrett's original result is more general than we have stated it.)

In Section 4, we will show that Theorem 2 holds if we assume in addition that $\lambda < \rho$, and in Section 5 we will complete the proof of Theorem 2 by showing that this assumption always holds when $0 \le \varepsilon < \varepsilon_c$. This second step was quite easy in the special case considered by Durrett and Griffeath. For us it is somewhat harder. However, the extra work needed yields a bonus: Theorem 4 below is a new result which was not previously known even for the contact process.

4. The percolation argument. We will assume throughout this section that the rates β_x and δ_x^ε are as in (37) or (38). We will prove

(39) if $\rho > \lambda$, then there exist constants a and b such that P(there exists a $[-t, 0]$-path to $\{0\}$ but not a $(-\infty, 0]$-path to $\{0\}$) $\le ae^{-bt}$ for all $t \ge 0$.

Once we also prove that $\rho > \lambda$ when $\varepsilon < \varepsilon_c$ (as we will in Section 5), then Theorem 2 follows from (34) and the translation invariance in space and time of our spin system.

To prove (39), we will follow Durrett and Griffeath and find a certain percolation structure within our graphical representation. The results of Durrett and Griffeath imply certain exponential estimates for this percolation model which can then be transformed into the bound in (39).

The building blocks of our percolation model will be certain space–time sets which we call *tubes*:

DEFINITION 3. Choose $\delta > 0$, $T > 0$ and a space–time point (x, s). The sets

$$\{(y, t)\colon t \in [s, s + T] \text{ and } \lambda(t - s) - \delta T \le y - x \le \lambda(t - s) + \delta T\} \quad \text{and}$$

$$\{(y, t)\colon t \in [s, s + T] \text{ and } \rho(t - s) - \delta T \le y - x \le \rho(t - s) + \delta T\}$$

are called, respectively, the λ-*tube* and ρ-*tube* with *dimension* $\delta T \times T$ and *location* (x, s).

We imagine that there is a liquid that attempts to flow through these tubes. A tube with dimensions $\delta T \times T$ and location (x, s) is called *open* if there is a nontrivial $[s, s + T]$-path π (nontrivial means $\pi(s + T) \ne \varnothing$) whose graph lies in the tube. Such an $[s, s + T]$-path will connect the base to the top of the tube and will stay between the left and right edges of the tube. We only allow liquid to flow through open tubes. Furthermore, we restrict the direction of flow to be downward (i.e., in the direction of decreasing time). This corresponds with our point of view that time runs backward in dual processes.

Now consider the arrangement of tubes shown in Figure 1. In this picture, the λ-tubes are shown with negative slopes and the ρ-tubes with positive slopes, although in general, λ and ρ (and hence both slopes) may have the same sign. However, as long as $\rho > \lambda$, tubes with dimension $\delta T \times T$ can be arranged to criss-cross in the manner shown, provided δ is sufficiently small. Once such a value of δ is chosen, T is just an arbitrary scaling factor and does not affect the picture.

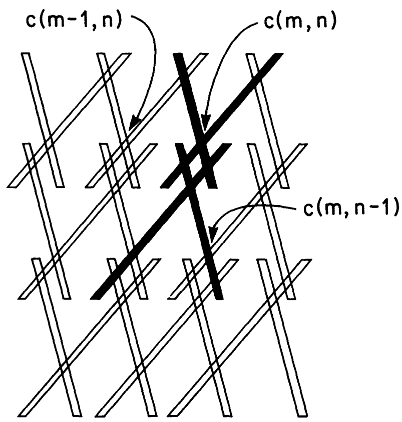

FIG. 1.

The basic unit of repetition in Figure 1 is shown shaded in. We call such a union of four tubes a *cell.* The entire structure is formed by repeating a cell periodically throughout the space–time plane. We label the cells as shown by $c(m, n)$, $m, n \in \mathbb{Z}$. The cells labeled $c(n, 0)$, $n \in \mathbb{Z}$, are centered on the line $t = 0$ and $c(0, 0)$ is centered at the origin. In our labeling scheme, $c(m, n)$ shares a tube with $c(m, n + 1)$, $c(m + 1, n + 1)$, $c(m - 1, n - 1)$, and $c(m, n - 1)$ as shown. These four cells, along with $c(m + 1, n)$ and $c(m - 1, n)$ are called the *neighbors* of $c(m, n)$. Note that $c(m, n)$ is disjoint from all other cells except its neighbors.

We call a cell open if all four of the tubes that form the cell are open. A sequence of cells $\{c(m_n, n)\}_{n \in \mathbb{Z}}$ is called a *percolation path* if each cell $c(m_n, n)$ is open and if each pair of cells $c(m_n, n)$ and $c(m_{n+1}, n + 1)$ are neighbors for all n. Note that if P is a percolation path, then there is an infinite connected route which moves downward through the open tubes that form the cells of P. We imagine that the presence of percolation paths allows liquid to flow or percolate down through the entire structure. This completes our description of the percolation model. We will assume from now on that $\delta > 0$ has been chosen to make the picture in Figure 1 correct. We will leave the scaling factor T undetermined until later. Define

$$p(T) = P(c(m, n) \text{ is open}).$$

Of course by translation invariance, $p(T)$ does not depend on m or n.

The model we have described is slightly different than the one used by Durrett and Griffeath. However, all the essential features are the same. The important thing is that we have an oriented percolation model in two dimensions (oriented because we have restricted the direction of flow) in which the events $\{c(m, n)$ is open$\}$ and $\{c(m', n')$ is open$\}$ are independent if $c(m, n)$ and $c(m', n')$ are not neighbors. (This independence is due to the independence built into the underlying graphical representation.) The version given here was developed by the author in cooperation with R. Durrett and T. Liggett at UCLA.

The important property possessed by our percolation model is:

LEMMA 10. *For $p(T)$ sufficiently close to* 1, *there exist positive constants a' and b' such that $P(A_N^+ \cap A_N^-) \geqq 1 - a'e^{-b'N}$ for all $N = 1, 2, 3, \dots,$ where*

$$A_N^+ = \{\text{there exists a percolation path } \{c(m_n, n)\}_{n \in \mathbb{Z}} \text{ such that } m_0 > 0 \text{ and } 3(N - m_n)/2 \geq -n \text{ for all } n \leq 0\},$$

$$A_N^- = \{\text{there exists a percolation path } \{c(m_n, n)\}_{n \in \mathbb{Z}} \text{ such that } m_0 < 0 \text{ and } 3(-N - m_n) \leq -n \text{ for all } n \leq 0\}.$$

This lemma is a variation of the percolation result found in the proof of Theorem 4 in Durrett and Griffeath (1983). We leave it to the reader to make the minor adjustments necessary to obtain our Lemma 10. The point is that if $p(T)$ is close to 1, it is possible to use contour arguments to estimate the probability of finding percolation paths with slopes which are close to the largest and smallest values allowed by the graph. In our case, $0 \leq m_0 - m_n \leq -n$ for $n \leq 0$ (since m_{n+1} is always equal to m_n or $m_n + 1$), so the extreme slopes are 1 and ∞. We have used $\frac{3}{2}$ and 3 for numbers close to these extremes.

In order to get from Lemma 10 to (39), we need the following results:

LEMMA 11. *$p(T) \to 1$ as $T \to \infty$.*

LEMMA 12. *If P is a percolation path, then there is a $(-\infty, \infty)$-path π whose graph is contained in the collection of tubes that make up the cells of P such that $\pi(t)$ is nonempty for all t.*

LEMMA 13. *For $i = 1, 2, 3$ let π_i be an $[s, t]$-path to B_i, where $L(B_1) \leqq L(B_2) \leqq R(B_2) \leqq R(B_3)$. Let*

$$u = \inf\{u' \in [s, t]: L(\pi_1(v)) < R(\pi_3(v)) \text{ for all } v \in [u', t]\}.$$

Then there exists an $[s, t]$-path π to B_2 such that $\pi(v) \subset \bigcup_{i=1}^3 \pi_i(v) \cap [L(\pi_1(v)), R(\pi_3(v))]$ for all $v \in [u, t]$. Moreover, if $u > s$ then π can be chosen either so that $\pi(v) \subset \pi_1(v)$ or so that $\pi(v) \subset \pi_3(v)$, for all $v \in [s, u)$.

Lemma 13 can be rephrased in a less precise but more descriptive manner as follows: Suppose π_1 and π_3 are $[s, t]$-paths which cross at some time $u \in (s, t)$ and suppose there is another $[s, t]$-path π_2 which is between π_1 and π_3 at time t. Then π_2 can be modified into an $[s, t]$-path which stays between π_1 and π_3 during $[u, t]$ and which is absorbed into either π_1 or π_3 during $[s, u)$. Lemma 13 will be used to prove Lemma 12. Before proving Lemmas 11–13, we will show how they can be used together with Lemma 10 to prove (39).

PROOF OF (39). Fix T sufficiently large so that $p(T)$ is close enough to 1 to satisfy the hypotheses of Lemma 10. This can be done by Lemma 11. Note that if A_N^+ and A_N^- occur, then there are percolation paths P^+ and P^- which pass the origin on the right and left, respectively and which cross at some level $n \geq -6N$ (i.e., if $c(m, n) \in P^+$ and $c(m', n) \in P^-$ and if $n < -6N$, then $m < m'$). It follows from Lemma 12 that there exist $(-\infty, \infty)$-paths π^+ and π^- such that $L(\pi^-(0)) < R(\pi^+(0))$ and such that $L(\pi^-(s)) > R(\pi^+(s))$ for $s < -MN$, where M is a constant which depends on our choice of T but not on N. Fix $s < -MN$ and let π_1 and π_3 be, respectively, the restrictions of π^- and π^+ to $[s, 0]$. Suppose that π_2 is an $[s,0]$-path to $\{0\}$. Then π_1, π_2, and π_3 satisfy the conditions of Lemma 13, so there exists an $[s,0]$-path π to $\{0\}$ such that $\pi(s) \subset \pi_1(s) = \pi^-(s)$. If we extend π to the interval $(-\infty, 0]$ by defining $\pi(u) = \pi_1(u)$ for $u \leqq s$, then it follows from Lemma 9 that π is a $[u, 0]$-path to $\{0\}$ for all $u < 0$, so π is a $(-\infty, 0]$-path to $\{0\}$. We have shown that if $A_N^+ \cap A_N^-$ occurs and if there exists a $[-t, 0]$-path to $\{0\}$ for some $t > MN$, then there exists a $(-\infty, 0]$-path to $\{0\}$. Lemma 10 now implies (39). □

It remains to prove Lemmas 11–13. We will prove them in reverse order.

PROOF OF LEMMA 13. For $j = 1$ or 3, define

$$\tilde{\pi}_j(v) = \bigcup_{i=1}^{3} \pi_i(v) \cap [L(\pi_1(v)), R(\pi_3(v))] \quad \text{if } v \in [u, t];$$
$$= \pi_j(v) \quad \text{if } v \in [s, u).$$

We will prove that $\tilde{\pi}_1$ and $\tilde{\pi}_3$ satisfy (19)–(21). Lemma 13 then follows immediately from Lemma 5(a). For the remainder of the proof, fix $j = 1$ or 3.

First note that since π_j satisfies (19)–(21), $\tilde{\pi}_j$ also satisfies (19)–(21) at least on the interval $[s, u)$. Thus, we can restrict our attention to $v \in [u, t]$. To save space, we let $L(v) = L(\pi_1(v))$ and $R(v) = R(\pi_3(v))$. We will need the following facts concerning $L(v)$ and $R(v)$:

(40) $$L(v) < R(v) \quad \text{for all } v \in [u, t];$$

(41) if $(L(v), v)$ is not a transition point, then $L(v^-) = L(v)$, otherwise $|L(v) - L(v^-)| \leqq 1$; similar statements hold for $R(v)$;

(42) L and R cannot be simultaneously discontinuous at any time $v \in [u, t]$, and $L(u^-) = R(u^-)$.

Right continuity of π_1 and π_3 and the definition of u imply (40). To prove (41), we use the fact that π_1 and π_3 satisfy (19′)–(21′). Thus, if $x \in \pi_1(v)$ and if (x, v) is not a transition point, then $x \in \pi_1(v^-)$, while if (x, v) is a transition point, then

$$\pi_1(v) \setminus \{x\} \subset \pi_1(v^-) \subset \pi_1(v) \cup N_x.$$

Furthermore, since $\beta_x(\varnothing) = 0$, $\pi_1(v^-) \cap (N_x \cup \{x\})$ is nonempty. Similar statements hold for π_3, and (41) can now be easily deduced, since $N_x = \{x - 1, x + 1\}$. Taken together, (40) and (41) imply (42).

We are now ready to prove that $\tilde{\pi}_j$ satisfies (19)–(21) on the interval $[u, t]$. Thus, fix $v \in [u, t]$ and $x \in \tilde{\pi}_j(v)$. There are many cases to check. The proof of (19) is easy (using (40)–(42)), and is left to the reader. We also leave the proof of

(21) to the reader and concentrate on the slightly more difficult proof of (20). Thus we assume that $(x, v) \in B(x, k)$ for some k. If $L(v) < x < R(v)$, then it is not too hard to prove (20) using (40)–(42) and the fact that π_1, π_2, and π_3 satisfy (19)–(21). If $x = L(v)$ and $v > u$, then (20) follows from (40)–(42) and from the fact that π_1 satisfies (20). The case $x = R(v)$ is similar. (We are leaving some tedious but straightforward checking here to the reader.) We finally come to the case in which $v = u$ and $x = L(u)$ or $R(u)$. We will assume $x = L(u)$. It follows from (40) and (42) that $R(u) = R(u^-) = L(u^-) = L(u) + 1 = x + 1$. Since $L(u^-) \in \pi_1(u^-)$ and $R(u^-) \in \pi_3(u^-)$, it follows from the definition of $\tilde{\pi}_j$ that $x + 1 \in \tilde{\pi}_j(u^-)$. Also, since π_1 satisfies (20), it must be that $b_k \leqq \beta_x(\pi_1(u^-) \cap N_x) = \beta_x(\{x + 1\})$, so $b_k \leqq \beta_x(\tilde{\pi}_j(u^-))$. Therefore $\tilde{\pi}_j$ satisfies (20) in this case as well. □

Proof of Lemma 12. Let S_1 and S_2 be any two tubes in Figure 1 which cross. Let (x_1, s_1) and (x_2, s_2) be their respective locations. We will show that

(43) if S_1 and S_2 are open, then there is a nontrivial $[s_1, s_2 + T]$-path whose graph is contained in $S_1 \cup S_2$.

Such a path will run from the base of S_1 to the top of S_2. We leave it to the reader to use this fact to form a proof by induction that if $c(m_1, n_1)$ and $c(m_2, n_2)$ are two (open) cells in some percolation path P with $m_1 < m_2$, then there is a nontrivial $[s_1, s_2 + T]$-path whose graph is contained in the tubes that make up the cells of P, where (x_1, s_1) and (x_2, s_2) are the locations of tubes in $c(m_1, n_1)$ and $c(m_2, n_2)$, respectively. We further leave it to the reader to derive Lemma 12 from this second fact. It remains to prove (43).

If S_1 and S_2 are open, there is a nontrivial $[s_1, s_1 + T]$-path π, and a nontrivial $[s_2, s_2 + T]$-path π_2 whose graphs are contained in S_1 and S_2, respectively. There are four cases, depending on whether $s_1 \leqq s_2$ or $s_1 > s_2$ and whether S_1 is a λ-tube or a ρ-tube. We will only consider the case in which $s_1 \leqq s_2$ and S_1 is a λ-tube, as shown in Figure 2. Let π_i' be the restriction of π_i to $[s_2, s_1 + T]$ for $i = 1, 2$ and let $\pi_3' = \pi_2'$. Then π_1', π_2', and π_3' satisfy the hypotheses of Lemma 13, so there exists an $[s_2, s_1 + T]$-path π to $\pi_2(s_1 + T)$ such that $\pi(v) \subset \pi_1(v) \cup \pi_2(v)$ for $v \in [s_2, s_1 + T]$ and such that $\pi(s_2) \subset \pi_1'(s_2)$. Since π_1 satisfies (15) on $[s_1, s_2]$, Lemma 5(b) implies that there exists an $[s_1, s_2]$-path π' to $\pi(s_2)$ out of $\pi_1(s_1)$ such that $\pi'(v) \subset \pi_1(v)$ for $v \in [s_1, s_2]$. Extend π to all of $[s_1, s_2 + T]$ by defining

$$\begin{aligned} \pi(v) &= \pi'(v), \qquad v \in [s_1, s_2], \\ &= \pi_2(v), \qquad v \in [s_1 + T, s_2 + T]. \end{aligned}$$

It follows from Lemma 9 that π is an $[s_1, s_2 + T]$-path whose graph is contained in $S_1 \cup S_2$. □

Proof of Lemma 11. Since a cell consists of four tubes, in order to show that $p(T) \to 1$ as $T \to \infty$, it is enough to show that the probability that a given λ- or ρ-tube with dimension $\delta T \times T$ is open can be made arbitrarily close to 1 by choosing T sufficiently large. We will prove this statement for ρ-tubes and leave

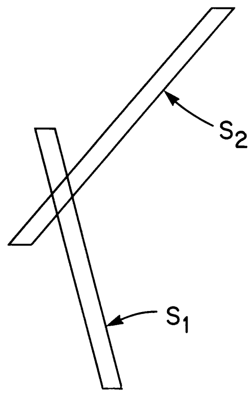

FIG. 2.

the analogous proof for λ-tubes to the reader. By translation invariance, it is sufficient to consider the ρ-tube with location $(0,0)$.

Let S be the ρ-tube with dimension $\delta T \times T$ and location $(0,0)$. Recall that S is open if there is a nontrivial $[0, T]$-path whose graph is contained in S. We will build a sort of a space–time "funnel" which will be designed to trap just such a path inside S. This device is shown in Figure 3, which depicts a collection of tubes with dimension $(\delta/10)T \times T$. This collection contains one ρ-tube and $N+2$ λ-tubes, where N is the largest integer less than $(5(\rho-\lambda)/\delta)-1$. (Our earlier choice of δ ensures that $\delta < (\rho-\lambda)/2$, so $N \geqq 9$.) We call the ρ-tube R. It has location $(0,0)$. The λ-tubes are called $S_{-1}, S_0, S_1, \dots, S_N$. The location of S_k is $(k\delta T/5, 0)$ for $-1 \leqq k \leqq N$. The larger tube S is indicated by dotted lines. The

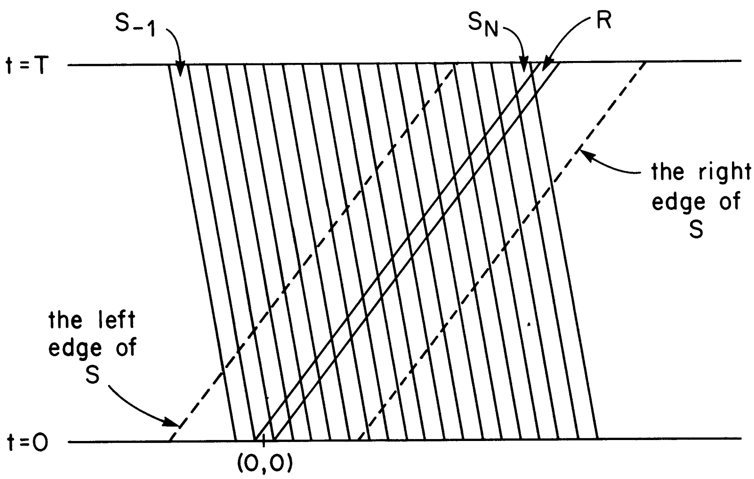

FIG. 3.

integer N has been chosen so that S_N crosses R and so that the top of S_N is contained in the top of S, as shown. We now define the following events:

$$E(T) = \{\text{the graph of } R(\xi(0, t, (-\infty, 0])) \text{ is contained in } R \text{ for } 0 \leqq t \leqq T\};$$

$$E_k(T) = \{\text{the graph of } L(\xi(0, T, [n_k, \infty))) \text{ is contained in } S_k \text{ for } 0 \leqq t \leqq T\},$$

where n_k is the largest integer less than or equal to $k\delta T/5$ for $-1 \leqq k \leqq N$. Let

$$E'(T) = E(T) \cap \left(\bigcap_{k=-1}^{N} E_k(T) \right).$$

We will prove that

$$\{S \text{ is open}\} \supset E'(T). \tag{44}$$

Since the slope of the tube R is ρ and the slopes of the tubes S_k are all equal to λ, Theorem 3 implies that $P(E'(T)) \to 1$ as $T \to \infty$, so Lemma 11 follows from (44).

Assume that $E'(T)$ occurs. To show that S is open, we start by defining

$$f_r(s) = R(\xi(0, s, (-\infty, 0])), \quad 0 \leqq s \leqq T;$$

$$f_l(s) = L(\xi(0, s, [n_k, \infty))) \quad \text{for } s \in [0, t] \text{ such that}$$

$$k + 1 \leqq 5(\rho - \lambda)s/\delta T < k + 2.$$

Since $E'(T)$ occurs, the graph of f_r is contained in S for $0 \leqq s \leqq T$, and the graph of f_l is contained in S_k if $k + 1 \leqq 5(\rho - \lambda)s/\delta T < k + 2$, $0 \leqq s \leqq T$, and $-1 \leqq k \leqq N$. By computing where the edges of the tubes R and S cross the edges of the tubes S_k, the reader can check that it follows from this last statement that

$$\begin{gathered} f_l(s) < f_r(s) \quad \text{and} \\ \{x\colon f_l(s) \leqq x \leqq f_r(s)\} \times \{s\} \subset S \quad \text{for } s \in [0, T]. \end{gathered} \tag{45}$$

(The strict inequality in (45) is easy to show when $k + 1 \neq 5(\rho - \lambda)s/\delta T$. It then follows for $k + 1 = 5(\rho - \lambda)s/\delta T$ by the right continuity of f_l and f_r.) By (45), we can prove that S is open if we can find a nontrivial $[0, T]$-path π such that $f_l(s) \leqq L(\pi(s))$ and $R(\pi(s)) \leqq f_r(s)$ for all $s \in [0, T]$. By Lemma 5(a) it is enough to show that the following function satisfies (19)–(21):

$$\tilde{\pi}(s) = \left[\xi(0, s, (-\infty, 0]) \cup \left(\bigcup_{k=-1}^{N} \xi(0, s, [n_k, \infty)) \right) \right] \cap [f_l(s), f_r(s)].$$

The proof of this fact is similar to the proof that the function $\tilde{\pi}$ defined in the proof of Lemma 13 satisfies (19)–(21). The important points are that each of the functions $\xi(0, s, (-\infty, 0])$ and $\xi(0, s, [n_k, \infty))$ satisfies (19)–(21) and that the interval $[f_l(s), f_r(s)]$ satisfies properties analogous to (40)–(42). (Special care should be taken when s satisfies $k + 1 = 5(\rho - \lambda)s/\delta T$, but there is no difficulty here since in this case, $f_l(s^-)$ is always less than $f_l(s)$, so that the appropriate analogues to (40)–(42) will hold for such s as well.) The details are left to the reader. □

5. The inequality $\rho > \lambda$ and other results. As in the previous section, we assume throughout that β_x and δ_x^ε are as in (37) or (38). For each $\varepsilon \geqq 0$, Theorem 3 gives us asymptotic speeds $\rho(\varepsilon)$ and $\lambda(\varepsilon)$ for $R(\xi(0, t, (-\infty, 0]))$ and $L(\xi(0, t, [0, \infty)))$, respectively. We will prove

(46) if $\varepsilon < \tilde{\varepsilon}$ and $\rho(\tilde{\varepsilon})$ is finite, then $\rho(\varepsilon) \geqq \rho(\tilde{\varepsilon}) + (\tilde{\varepsilon} - \varepsilon)$.

By Theorem 3, $\rho(\varepsilon_c) \geqq \lambda(\varepsilon_c)$. Furthermore, it is known that $\lambda(\varepsilon)$ is nondecreasing in ε (see Durrett (1980) or carry out the analogue of our proof of (46) to show that $\lambda(\varepsilon)$ is strictly increasing in ε). Thus, (46) implies that $\rho(\varepsilon) > \rho(\varepsilon_c) \geqq \lambda(\varepsilon_c) \geqq \lambda(\varepsilon)$, completing the proof of Theorem 2.

The key to proving (46) is Theorem 4 below, which concerns the distribution of $\xi(0, t, (-\infty, 0])$ as viewed from the site $R(\xi(0, t, (-\infty, 0]))$. To state it, we need some terminology. Let $\mathscr{A}$ and $\mathscr{B}$ be Ξ-valued random variables. We will say that $\mathscr{A}$ *contains* $\mathscr{B}$ *in distribution* if there exists a probability measure π on the measurable subsets of $\Xi \times \Xi$ with first and second marginals equal to the distributions of $\mathscr{A}$ and $\mathscr{B}$, respectively, such that $\mu((\xi^1, \xi^2)\colon \xi^1 \supset \xi^2) = 1$.

THEOREM 4. *If* $t \geqq 0$ *and* $0 \leqq m \leqq n$, *then*

$$(\xi(0, t, (-\infty, 0]) + m) \cap (-\infty, R(\xi(0, t, (-\infty, 0]))] \quad \textit{contains}$$

$$(\xi(0, t, (-\infty, 0]) + n) \cap (-\infty, R(\xi(0, t, (-\infty, 0]))] \quad \textit{in distribution.}$$

We could paraphrase Theorem 4 by saying that for $0 \leqq m < n$, the part of $\xi(0, t, (-\infty, 0])$ that lies to the left of $R(\xi(0, t, (-\infty, 0])) - m$ is bigger in a certain sense than the part that lies to the left of $R(\xi(0, t, (-\infty, 0])) - n$. As a special case ($m = 0$, $n = 1$) we have

(47) $\xi(0, t, (-\infty, 0])$ contains $(\xi(0, t, (-\infty, 0]) \setminus \{R(\xi(0, t, (-\infty, 0]))\}) + 1$ in distribution.

Before proving Theorem 4, we will show how (47) can be used to obtain (46):

PROOF OF (46) BASED ON (47). Fix ε such that $0 \leqq \varepsilon < \varepsilon_c$. Let $(\xi(s, t, A);\ s \leqq t, A \in \Xi)$ be the system with rates β_x and δ_x^ε, constructed graphically as in Section 2. Let the underlying probability space $(\Omega, \mathscr{F}, P)$ be enlarged so that we can define a sequence $\tau_1, \tau_2, \tau_3, \ldots$ of iid exponentially distributed random variables with mean $1/(\varepsilon_c - \varepsilon)$ which are independent of the grahpical representation. Let $\sigma_n = \tau_1 + \tau_2 + \cdots + \tau_n$ for $n \geqq 1$ and let $\sigma_0 = 0$. We will define a process $(\xi'(t);\ t \geqq 0)$ inductively on the intervals $[\sigma_n, \sigma_{n+1})$ for $n \geqq 0$. Let $\xi'(t) = \xi(0, t, (-\infty, 0])$ for $t \in [0, \sigma_1)$. Having defined $\xi'(t)$ on $[0, \sigma_n)$, let

$$\xi'(t) = \xi(\sigma_n, t, \xi'(\sigma_n^-) \setminus \{R(\xi'(\sigma_n^-))\}), \qquad t \in [\sigma_n, \sigma_{n+1}).$$

The process $(\xi'(t);\ t \geqq 0)$ behaves just like a process with rates β_x and δ_x^ε and initial state $\{x \leqq 0\}$, except that at each of the times σ_n, an additional death occurs at the rightmost site contained in $\xi(\sigma_n^-)$. These extra deaths occur at rate $\varepsilon_c - \varepsilon$. Thus, we could say that the process $(\xi'(t);\ t \leqq 0)$ has birth rates β_x and death rates δ_x^ε except at $R(\xi'(t))$, where the death rate is $\delta_x^{\varepsilon_c}$, with $x = R(\xi'(t))$.

If we let $(\xi'(s,t,A);\, s \leqq t,\, A \in \Xi)$ be the process with rates β_x and $\delta_x^{\varepsilon_c}$, then a standard argument using the basic coupling shows that $\xi'(t)$ contains $\xi'(0,t,(-\infty,0])$ in distribution for all $t \geqq 0$, since $\delta_x^{\varepsilon_c} > \delta_x^{\varepsilon}$. It follows from this and Theorem 3 that

$$\liminf_{t\to\infty} R(\xi'(t))/t \geqq \rho(\varepsilon_c). \tag{48}$$

We will prove inductively using (47) that for all $n \geqq 0$,

$$\xi(0,\sigma_n,(-\infty,0]) \text{ contains } \xi'(\sigma_n) + n \text{ in distribution.} \tag{49}$$

Once (49) is proved, we can complete the proof of (46) as follows. By the strong law of large numbers, $n/\sigma_n \to \varepsilon_c - \varepsilon$ a.s. as $n \to \infty$. By (48),

$$\liminf_{t\to\infty} R(\xi'(\sigma_n) + n)/\sigma_n \geqq \rho(\varepsilon_c) + (\varepsilon_c - \varepsilon) \quad \text{a.s.}$$

By Theorem 3, $R(\xi(0,\sigma_n,(-\infty,0]))/\sigma_n \to \rho(\varepsilon)$ a.s. as $n \to \infty$. It follows from (49) that $\rho(\varepsilon) \geqq \rho(\varepsilon_c) + (\varepsilon_c - \varepsilon)$, which implies (46).

We now prove (49). To save space, let $\xi(t) = \xi(0,t,(-\infty,0])$. Since $\xi'(0) = \xi(0)$, the case $n = 0$ is trivial. Assume that (49) holds for some $n \geqq 0$. By the inductive hypothesis, there exists a measure μ with marginals equal to the distributions of $\xi(\sigma_n)$ and $\xi'(\sigma_n) + n$ such that $\mu((\xi^1,\xi^2)\colon\, \xi^1 \supset \xi^2) = 1$. Let $\mathscr{A}$ and $\mathscr{B}$ be Ξ-valued random variables which are jointly distributed according to μ, so that $\mathscr{A}$ is distributed like $\xi(\sigma_n)$ and $\mathscr{B}$ like $\xi'(\sigma_n) + n$. By enlarging the probability space once more, we can assume that $\mathscr{A}$ and $\mathscr{B}$ are independent of the original graphical representation and of the random variables $\tau_1, \tau_2, \tau_3, \dots$. Let

$$\eta_{n+1} = \xi(\sigma_n, \sigma_{n+1}, \mathscr{A}) \quad \text{and} \quad \eta'_{n+1} = \xi(\sigma_n, \sigma_{n+1}, \mathscr{B}).$$

Since $\mathscr{A}$ is distributed like $\xi(\sigma_n)$, it follows from (5) that η_{n+1} is distributed like $\xi(\sigma_{n+1}) = \xi(0,\sigma_{n+1},(-\infty,0])$. Since σ_{n+1} is independent of the graphical representation, it follows from (47) that $\xi(\sigma_{n+1})$ contains $(\xi(\sigma_{n+1}) \setminus \{R(\xi(\sigma_{n+1}))\}) + 1$ in distribution. Therefore, η_{n+1} contains $(\eta_{n+1} \setminus \{R(\eta_{n+1})\}) + 1$ in distribution. Since $\mathscr{A} \supset \mathscr{B}$ a.s., it follows from (6) that $\eta_{n+1} \supset \eta'_{n+1}$ a.s., so η_{n+1} contains $(\eta'_{n+1} \setminus \{R(\eta'_{n+1})\}) + 1$ in distribution. Since η_{n+1} is distributed like $\xi(\sigma_{n+1})$, in order to prove (49), it is enough to show that $(\eta'_{n+1} \setminus \{R(\eta'_{n+1})\}) + 1$ is distributed like $\xi'(\sigma_{n+1}) + n + 1$. By the Markov property and the fact that σ_n and σ_{n+1} are independent of the graphical representation, η'_{n+1} is distributed like $\xi(\sigma_n, \sigma_{n+1}, \xi'(\sigma_n) + n)$, which is distributed like $\xi(\sigma_n, \sigma_{n+1}, \xi'(\sigma_n)) + n$ by translation invariance. By definition, $\xi(\sigma_n, \sigma_{n+1}, \xi'(\sigma_n)) \setminus \{R(\xi(\sigma_n, \sigma_{n+1}, \xi'(\sigma_n)))\} = \xi'(\sigma_{n+1})$. It follows that $(\eta'_{n+1} \setminus \{R(\eta'_{n+1})\}) + 1$ is distributed like $\xi'(\sigma_{n+1}) + n + 1$. □

It remains to prove Theorem 4. The argument will rely on the existence of certain special dominating $[s,t]$-paths:

LEMMA 14. *Let π_i be an $[s,t]$-path to B_i out of A for $i = 1,2$. Assume that $L(B_1) \leqq L(B_2)$. Then there exists an $[s,t]$-path π to B_2 out of A such that $\pi(u) \subset \pi_1(u) \cup \pi_2(u)$ and $L(\pi(u)) \geqq \max\{L(\pi_1(u)), L(\pi_2(u))\}$ for all $u \in [s,t]$.*

PROOF. This is proved in the same way as Lemma 13, namely, one shows that

$$\tilde{\pi}(u) = (\pi_1(u) \cup \pi_2(u)) \cap [L(\pi_1(u)), \infty) \cap [L(\pi_2(u)), \infty)$$

satisfies (19)–(21). The details are omitted. □

We have a particular case of Lemma 14 in mind. Take $s = 0$, $A = \{x \leqq 0\}$, and $B = \{R(\xi(0, t, A)\}$. Since $B \subset \xi(0, t, A)$, Theorem 1 implies the existence of at least one $[0, t]$-path to B out of A. By Lemma 8, there are at most finitely many. If we apply Lemma 14 to all possible pairs of such paths, with $B_1 = B_2 = B$, then it follows that there exists a $[0, t]$-path π to B out of A such that if π' is any other $[0, t]$-path to B out of A, then $L(\pi(u)) \geqq L(\pi'(u))$ for all $u \in [0, t]$. While the path π is not necessarily unique, the function $L(\pi)$ clearly is uniquely determined by t and ω, so we can define

$$L^t(u) = L(\pi(u)) \quad \text{for } u \in [0, t].$$

The importance of the random function L^t is that it behaves like a kind of a stopping time in space–time. In order to explain this, we need some notation. If S is a Borel subset of $\mathbb{Z} \times \mathbb{R}$, define

$$\mathscr{F}_S = \bigvee_{x, k, l} (\sigma\{B(x, k) \cap S\} \vee \sigma\{D(x, l) \cap S\}).$$

This is the σ-algebra generated by the birth and death points that lie in S. One of the implications of (19)–(21) in Lemma 9 is that one can check whether a function π is an $[s, t]$-path by looking at the birth and death points that lie in the graph of π. Therefore, for any $s \leqq t$,

$$\{S \text{ contains the graph of a nontrivial } [s, t]\text{-path}\} \in \mathscr{F}_S. \tag{50}$$

We have already used a version of this fact implicitly when we stated in Section 4 that the events $\{c(m, n) \text{ is open}\}$ and $\{c(m', n') \text{ is open}\}$ are independent if $c(m, n)$ and $c(m', n')$ are not neighbors. We also used the fact that

$$\mathscr{F}_S \text{ and } \mathscr{F}_{S^c} \text{ are independent,} \tag{51}$$

which follows from the independence properties of the Poisson point locations that make up the graphical representation.

Let $f\colon [0, t] \to \mathbb{Z}$ be a right continuous function with left limits and define

$$\mathscr{F}_f = \mathscr{F}_{\{(x, s): s \in [0, t] \text{ and } x \geqq f(s)\}}.$$

It follows from (50) and the definition of L^t that

$$\{L^t \geqq f\} \in \mathscr{F}_f. \tag{52}$$

This is the property that makes L^t analogous to a stopping time. (It, together with (51) allows us to use L^t in much the same way that the rightmost path is used in percolation theory. See Kesten (1982), Lemma 6.1, Step (b).) If we define the "stopped" σ-algebra

$$\mathscr{F}^t = \Big\{\mathscr{A} \in \mathscr{F} \colon \mathscr{A} \cap \{L^t \geqq f\} \in \mathscr{F}_f \text{ for all right continuous } f\colon [0, t] \to \mathbb{Z} \text{ with left limits}\Big\}$$

then it can be shown, just as with stopping times, that (52) implies

(53) L^t is $\mathscr{F}^t$-measurable.

The analogy between L^t and stopping times goes further, in that there is a property similar to the strong Markov property. Heuristically speaking, given L^t, the part of the graphical representation which lies to the left of the graph of L^t is independent of the part that lies to the right. To state this property more precisely, let $\Gamma = (B(x,k), D(x,l))_{x,k,l}$ be the original graphical representation and assume that $\Gamma' = (B'(x,k), D'(x,l))_{x,k,l}$ is a second graphical representation which is independent of and distributed identically to Γ. Use Γ to determine the function L^t as usual, and then define a "hybrid" graphical representation which we call $\Gamma \bmod \Gamma'$ (Γ modified by Γ') as follows: Use Γ' to determine the locations of all transition points at points (x, s) such that $s \in [0, t]$ and $x < L^t(s)$, and use Γ to determine the transition points in the rest of space–time. In other words, start with Γ, remove those transition points that lie to the left of the graph of L^t, and then fill the vacant area with transition points from the corresponding part of Γ'. Note that in this new graphical representation, the function L^t remains unchanged since we have only changed transition points that do not affect L^t. We claim that

(54) Γ and $\Gamma \bmod \Gamma'$ have the same distribution.

Unfortunately, the proof of (54) is tedious and highly technical. It relies on (51) and (53), but matters are made difficult by the fact that the event $\{L^t = f\}$ has probability 0 for all f. We will merely give a very brief sketch of one approach. The idea is to use a discrete time approximation to the graphical representation. The space–time graph is $\mathbb{Z} \times \{k/n\colon k \in \mathbb{Z}\}$, where we eventually let n go to ∞. Independent choices are made at each point $(x, k/n)$ to determine the locations of transition points, with the probabilities appropriately scaled according to the value of n. The process is defined using analogues to (11) and (12). A problem arises because simultaneous births and deaths occur. This is a nuisance, since Lemma 14 fails in general if we allow births or deaths to occur simultaneously at the neighboring sites. One way around this is to first determine births and deaths at even sites, then use the resulting state to determine births and deaths at odd sites during each time unit. Once all this is carried out, the discrete time analogue to (54) can be proved and limits can be taken as $n \to \infty$.

We are now ready for:

Proof of Theorem 4. Fix $t \geqq 0$ and let Γ, Γ', and $\Gamma \bmod \Gamma'$ be as above. Fix m and n with $0 \leqq m \leqq n$. We define yet another graphical representation $\Gamma'' = (B''(x,k), D''(x,l))$ by shifting Γ' $n - m$ units to the right. Thus, for example, $(x, u) \in B''(x, k))$ iff $(x - n + m, u) \in B'(x - n + m, k)$. By translation invariance and (54), the two graphical representations $\Gamma \bmod \Gamma'$ and $\Gamma \bmod \Gamma''$ are identically distributed. Let $\xi(s,t,A)$, $\xi'(s,t,A)$, and $\xi''(s,t,A)$ be defined in terms of the graphical representations Γ, $\Gamma \bmod \Gamma'$, and $\Gamma \bmod \Gamma''$ as in Section 2 for $s \leqq t$ and $A \in \Xi$. Now fix $t \geqq 0$ and let

$$\xi' = \xi'(0, t, [0, \infty)) \quad \text{and} \quad \xi'' = \xi''(0, t, [0, \infty)).$$

By (54), ξ' and ξ'' are distributed like $\xi(0, t, [0, \infty))$. Therefore, to prove Theorem 4, it suffices to show that

$$(\xi'' + m) \cap (-\infty, R(\xi'')] \text{ contains } (\xi' + n) \cap (-\infty, R(\xi')] \text{ in distribution.}$$

We have set things up so that we can actually prove the stronger statement

$$(\xi'' + m) \cap (-\infty, R(\xi'')] \supset (\xi' + n) \cap (-\infty, R(\xi')]. \tag{55}$$

Choose $x \in (\xi' + n) \cap (-\infty, R(\xi')]$. By Theorem 1, there is a $[0, t]$-path π' to $x - n$ out of $(-\infty, 0]$ in the graphical representation $\Gamma \bmod \Gamma'$. Let

$$\tilde{\pi}(u) = \pi'(u) + n - m \quad \text{for } u \in [0, t].$$

By the definition of L^t (which by construction is the same for Γ, $\Gamma \bmod \Gamma'$, and $\Gamma \bmod \Gamma''$), there is a $[0, t]$-path π to $\{L^t(t)\}$ out of $(-\infty, 0]$ such that $L^t(u) = L(\pi(u))$ for all $u \in [0, t]$. (Note that π is a $[0, t]$-path in all three graphical representations.) Let

$$\pi''(u) = \big(\tilde{\pi}(u) \cap (-\infty, L^t(u)]\big) \cup \pi(u) \quad \text{for } u \in [0, t].$$

We claim that π'' satisfies (19)–(21) in the graphical representation $\Gamma \bmod \Gamma''$. Note that since $x \leqq R(\xi') = L^t(t)$ and since $\tilde{\pi}(t) = \pi'(t) + n - m = \{x - m\}$, it must be that $x - m \in \pi''(t)$. Thus (55) follows from the claim and Lemma 5(a).

The proof of the claim is similar to the proofs of Lemmas 13 and 14. It relies on the fact that π satisfies (19)–(21) in all three graphical representations and that π' satisfies (19)–(21) in $\Gamma \bmod \Gamma'$ (from which it follows that $\tilde{\pi}$ and π'' satisfy (19)–(21) in $\Gamma \bmod \Gamma''$, at least at space–time points that lie to the left of the graph of L^t—some special checking is needed at points of the form $(L^t(u) - 1, u)$). We omit the details. □

We conclude the paper with one last result which follows relatively easily from the work done up to this point. It gives two alternative ways of characterizing the critical value ε_c. A similar result was obtained for the contact process by Durrett and Griffeath.

Theorem 5. *Let β_x and δ_x^ε be as in* (38). *Then*

$$\begin{aligned} \varepsilon_c &= \sup\{\varepsilon\colon \lambda(\varepsilon) < \rho(\varepsilon)\} \\ &= \sup\{\varepsilon\colon P(\xi(0, t, A) \text{ is nonempty for all } t \geqq 0) > 0 \text{ for some finite } A\}. \end{aligned} \tag{56}$$

Proof. If $\varepsilon < \varepsilon_c$, then $\lambda(\varepsilon) < \rho(\varepsilon)$ by (46) and the argument that follows (46). On the other hand, if $\lambda(\varepsilon) < \rho(\varepsilon)$, then Lemmas 10–12 apply, from which it is not hard to show that a $(-\infty, 0]$-path to $\{0\}$ exists with positive probability. It follows from (31) that $\mu_1 \neq \mu_0$, so $\varepsilon \leqq \varepsilon_c$, proving the first equality in (56).

To prove the second equality in (56), assume first that $\lambda(\varepsilon) < \rho(\varepsilon)$. Then as before, Lemmas 10–12 apply, from which it can be easily shown that for some N, there exists with positive probability a $[0, \infty)$-path π such that $\pi(0) \subset [-N, N]$ and $\pi(t)$ is nonempty for all $t \geq 0$. It follows from (15) that

$$P(\xi(0, t, [-N, N]) \text{ is nonempty for all } t \geq 0) > 0. \tag{57}$$

On the other hand, if ε is chosen so that

$$P(\xi(0, t, A) \text{ is nonempty for all } t \geq 0) > 0$$

for some finite A, then (57) holds for some N by (6). For all ω such that the event in (57) occurs,

$$\begin{aligned} \lambda(\varepsilon) &= \lim_{t \to \infty} L(\xi(0, t, [-N, \infty)))/t \leqq \lim L(\xi(0, t, [-N, N]))/t \\ &\leqq \lim R(\xi(0, t, [-N, N]))/t \leqq \lim R(\xi(0, t, (-\infty, N]))/t \\ &= \rho(\varepsilon). \end{aligned}$$

Since $-\infty < \lambda(\varepsilon)$ and $\rho(\varepsilon) < \infty$ for all ε, it follows that $\lambda(\varepsilon)$ and $\rho(\varepsilon)$ are finite, with $\lambda(\varepsilon) \leqq \rho(\varepsilon)$. Now apply (46) and the fact that $\lambda(\varepsilon)$ is nondecreasing in ε to obtain the second equality in (56). □

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