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A NOTE ON FELLER'S STRONG LAW OF LARGE NUMBERS

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Let X_n, $n \geq 1$, be i.i.d. random variables with common distribution function $F(x)$ and γ_n, $n \geq 1$, be a sequence of constants such that γ_n/n is nondecreasing in n. Set $S_n = X_1 + \cdots + X_n$. The main theorem of this paper gives an integral test which determines the infinite limit points of $\{S_n/\gamma_n\}$. This result extends and combines Feller's (1946) strong law of large numbers (SLLN) and the results of Kesten (1970) and Erickson (1973).

1. Introduction. Let X, X_n, $n \geq 1$, be independent identically distributed random variables with common distribution function $F(x)$. And let γ_n, $n \geq 1$, be a sequence of positive constants such that γ_n/n is nondecreasing in n. Then, the sequence $S_n = X_1 + \cdots + X_n$, $n \geq 1$, is called a random walk and the normalized random walk which we shall study is $\{S_n/\gamma_n\}$.

Define

$$\begin{aligned} \gamma(x) &= \gamma_n, && x = n, \quad n \geq 0, \quad \gamma_0 = 0 \\ &= \gamma_n + (\gamma_{n+1} - \gamma_n)(x - n), && n \leq x < n + 1, \quad n \geq 0. \end{aligned}$$

And let $\gamma^{-1}(\cdot)$ denote the inverse function of $\gamma(\cdot)$. Set $c = EX/(\lim \gamma_n/n)$ if $E|X| < \infty$ and $c = 0$ otherwise. Feller (1946) obtained the following remarkable result

$$\begin{aligned} P\{\lim S_n/\gamma_n = c\} = 1 \quad &\text{iff } E\gamma^{-1}(|X|) < \infty \\ &\text{iff } P\{\limsup |S_n/\gamma_n| < \infty\} = 1, \end{aligned} \tag{1.1}$$

where iff stands for if and only if.

However, as far as $\limsup S_n/\gamma_n$ and $\liminf S_n/\gamma_n$ are concerned, Feller's strong law of large numbers does not cover the case where $E\gamma^{-1}(|X|) = \infty$. Define

$$m_+(x) = \int_0^x P\{X^+ \geq t\}\,dt, \qquad x^+ = \max(x, 0),$$

$$m_-(x) = \int_0^x P\{X^- \geq t\}\,dt, \qquad x^- = \max(-x, 0),$$

$$J_+(\gamma) = \int_0^\infty \min\big(\gamma^{-1}(x), x/m_-(x)\big)\,dF(x),$$

and

$$J_-(\gamma) = \int_0^\infty \min\big(\gamma^{-1}(x), x/m_+(x)\big)\,d(1 - F(-x)).$$

The following theorem gives an integral test which extends (1.1).

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THEOREM 1. [*No assumption on* $F(x)$]

(i) $J_+(\gamma) = \infty$ *iff* $P\{\limsup S_n/\gamma_n = \infty\} = 1$.

(ii) $J_-(\gamma) < J_+(\gamma) = \infty$ *iff*

$$P\{\liminf S_n/\gamma_n = \liminf(|X_1| + \cdots + |X_n|)/\gamma_n\} = P\{\limsup S_n/\gamma_n = \infty\} = 1.$$

(iii) $J_+(\gamma) + J_-(\gamma) < \infty$ *iff* $E\gamma^{-1}(|X|) < \infty$.

REMARK. Similar statements may be made, by symmetry, about the case where $J_-(\gamma) = \infty$ or the case where $J_+(\gamma) < J_-(\gamma) = \infty$. It follows from (1.1) that (iii) is implied by (i). However, (iii) is purely an analytic fact and a proof of (iii) is given in the next section by a simple nonprobabilistic argument.

Kesten (1970) and Erickson (1973) studied the case where $\gamma_n = n$ and obtained similar results. The integral test in Theorem 1 is the same as that in Theorem 2 of Erickson (1973) for $\gamma_n = n$. Like Kesten (1970), Theorem 1 is obtained by investigating the positive and negative contributions to the random walk $\{S_n\}$ with respect to the sequence γ_n. In fact, we have the following stronger

THEOREM 2. *Suppose that either* $E|X| = \infty$ *or* $\lim \gamma_n/n = \infty$. *Then, one of the following alternatives must prevail*:

(i) $J_+(\gamma) = \infty$ *and* $P\{\limsup X_n^+/(\gamma_n + X_1^- + \cdots + X_n^-) = \infty\} = 1$;

(ii) $J_+(\gamma) < \infty$ *and*

$$P\{\lim(X_1^+ + \cdots + X_n^+)/(\gamma_n + X_1^- + \cdots + X_n^-) = 0\} = 1.$$

Theorem 1 and Theorem 2 will be proved in Section 2. We have two corollaries below which follow easily from Theorem 2.

COROLLARY 1. *Suppose that* $E|X| + \lim \gamma_n/n = \infty$. *Then*, $J_+(\gamma) < \infty$ *iff*

$$S_n/\gamma_n = \left(-\sum_{i=1}^{n} X_i^-/\gamma_n\right)(1 + o(1)) + o(1) \quad a.s.$$

COROLLARY 2. *Suppose that* $\lim \gamma_n/n = \infty$. *Then, it is impossible for any random walk to have*

$$-\infty < \liminf S_n/\gamma_n < 0 \quad a.s. \tag{1.2}$$

Erickson (1976, page 818) pointed out that it is impossible to have

$$0 < \liminf n^{-\alpha} S_n < \infty \quad \text{a.s. for } \alpha < 1.$$

Actually, it follows from his argument that it is impossible for any random walk to have

$$0 < \liminf S_n/b_n < \infty \quad \text{a.s. for } b_n > 0 \text{ and } \lim b_n/n = 0. \tag{1.3}$$

Corollary 2 is the analogue of (1.3) for $\lim \gamma_n/n = \infty$.

2. Proofs. Lemma 1 gives an inequality for truncated expectations of partial sums of i.i.d. nonnegative random variables. The inequality has its own interest and may be used for other purposes.

LEMMA 1. *Let $Y, Y_1, \dots, Y_n$ be i.i.d. nonnegative random variables. Set $S_n = Y_1 + \cdots + Y_n$ and $m(x) = \int_0^x P\{Y \geq y\}\, dy$. Let $C > 0$ be a constant. Then*

$$E \min(S_n, C) \leq \min(C, nm(C)) \leq 16E \min(S_n, C).$$

PROOF. Let $S_n' = \sum_{i=1}^n \min(Y_i, C)$. We shall discuss two cases, $nm(C) > 3C$ and $nm(C) \leq 3C$. For the first case, $nm(C) = uC > 3C$,

$$\begin{aligned} P\{S_n \leq C\} &= P\{ES_n' - S_n' \geq (u-1)C\} \\ &\leq (u-1)^{-2} C^{-2} nE(\min(Y, C))^2 \leq u/(u-1)^2 \leq 3/4, \\ E \min(S_n, C) &\geq CP\{S_n \geq C\} \geq C/4, \\ C &\leq 16E \min(S_n, C). \end{aligned}$$

For $nm(C) = uC \leq 3C$,

$$\begin{aligned} nm(C) &= ES_n' \\ &\leq 8E \min(S_n, C) + ES_n' I\{S_n' > 8C\}, \\ ES_n' I\{S_n' > 8C\} &\leq \left(nE(\min(Y, C))^2 + (nm(C))^2\right)\big/(8C) \\ &\leq (Cnm(C) + 3Cnm(C))\big/(8C) = nm(C)/2, \\ nm(C) &\leq 16E \min(S_n, C). \end{aligned}$$

The inequality in the other direction is obvious. □

We shall study the ratio of two independent nonnegative random walks instead of the ratio of the positive and negative contributions of the random walk $\{S_n, n \geq 1\}$.

THEOREM 3. *Let $\{W_n\}$ and $\{V_n\}$ be two independent sequences of i.i.d. nonnegative random variables. Suppose that $EW_1 + EV_1 + \lim \gamma_n/n = \infty$. Then the following statements are equivalent:*

(i) $\lim(W_1 + \cdots + W_n)/(\gamma_n + V_1 + \cdots + V_n) = 0$ *a.s.*;
(ii) $\limsup(W_1 + \cdots + W_n)/(\gamma_n + V_1 + \cdots + V_n) < \infty$ *a.s.*;
(iii) $\limsup W_n/(\gamma_n + V_1 + \cdots + V_n) < \infty$ *a.s.*;
(iv) $\sum_{n=1}^\infty P\{\delta W_n > \gamma_n + V_1 + \cdots + V_n\} < \infty$ *for some* $\delta > 0$;
(v) $\sum_{n=1}^\infty P\{\delta W_n > \gamma_n + V_1 + \cdots + V_n\} < \infty$ *for all* $\delta > 0$;
(vi) $\int_0^\infty \min(\gamma^{-1}(x), x/\int_0^x P\{V_1 \geq t\}\, dt)\, dP\{W_1 \leq x\} < \infty$.

PROOF. It suffices to prove (iii) ⇒ (iv) ⇒ (v) ⇒ (vi) ⇒ (iv) ⇒ (i) since it is obvious that (i) ⇒ (ii) ⇒ (iii).

(iii) ⇒ (iv). There are constants $\delta > 0$ and $k < \infty$ such that

$$P\left\{\bigcup_{n=k}^{\infty} [\delta W_n > \gamma_n + V_1 + \cdots + V_n]\right\} \leq 1 - \delta < 1.$$

Set

$$A_n = [\delta W_n > \gamma_n + V_1 + \cdots + V_n],$$

$$A_n A_{n+m} \subset A_n[\delta W_{n+m} > \gamma_m + V_{n+1} + \cdots + V_{n+m}],$$

$$P\left\{A_n \bigcup_{j=1}^{\infty} A_{n+jk}\right\} \le P\{A_n\} P\left\{\bigcup_{j=1}^{\infty} A_{jk}\right\} \le P\{A_n\}(1-\delta),$$

$$\delta \sum_{n=1}^{N} P\{A_{nk+i}\} \le \sum_{n=1}^{N} P\left\{A_{nk+i}\left[\bigcup_{j=1}^{\infty} A_{(n+j)k+i}\right]^c\right\} \le 1, \qquad i = 1,\dots,k.$$

(iv) ⇒ (v). Since $\gamma(2n) \ge 2\gamma(n)$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\{2\delta W_n > \gamma_n + V_1 + \cdots + V_n\} \\
&\qquad \le 1 + 2\sum_{n=1}^{\infty} P\{2\delta W_{2n} > \gamma(2n) + V_1 + \cdots + V_{2n}\} \\
&\qquad \le 1 + 4\sum_{n=1}^{\infty} P\{\delta W_n > \gamma_n + V_1 + \cdots + V_n\}.
\end{aligned}$$

(v) ⇒ (vi) ⇒ (iv). Let $T = \inf\{k\colon\ S_k > C\}$, $S_n = \sum_{i=1}^{n} V_i$, and $S'_k = \sum_{i=1}^{k} \min(V_i, C)$. Then, it follows from Wald's lemma that

$$\int_0^C P\{V_1 \ge t\}\,dt\,E\min(T,n) = ES'_{\min(T,n)},$$

$$\min(S_n, C) \le S'_{\min(T,n)} \le 2\min(S_n, C),$$

$$E\min(T,n) = 1 + \sum_{k=1}^{n-1} P\{V_1 + \cdots + V_k \le C\}.$$

Therefore, by Lemma 1,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\{W_n > \gamma_n + V_1 + \cdots + V_n\} \\
&\qquad \le \int_0^{\infty} \sum_{\gamma(1)\le\gamma(n)<x} P\{V_1 + \cdots + V_n < x\}\,dP\{W_1 \le x\} \\
&\qquad \le 2\int_0^{\infty} \min\left(\gamma^{-1}(x),\, x \Big/ \int_0^x P\{V_1 \ge t\}\,dt\right) dP\{W_1 \le x\} \\
&\qquad \le 32 + 32\int_0^{\infty} \sum_{\gamma(1)\le\gamma(n)<x} P\{V_1 + \cdots + V_n \le x\}\,dP\{W_1 \le x\} \\
&\qquad \le 32 + 32\int_0^{\infty} \sum_{n=1}^{\infty} P\{V_1 + \cdots + V_n + \gamma_n < 2x\}\,dP\{W_1 \le x\} \\
&\qquad = 32 + 32\sum_{n=1}^{\infty} P\{2W_1 > \gamma_n + V_1 + \cdots + V_n\}.
\end{aligned}$$

(iv) ⇒ (i). We shall prove (v) ⇒ (i) instead.

$$\begin{aligned}\sum_{n=1}^{\infty} P\{\delta(W_1 + W_2) > \gamma_n + V_1 + \cdots + V_n\} &\\ \le 1 + 2\sum_{n=1}^{\infty} P\{\delta(W_1 + W_2) > \gamma(2n) + V_1 + \cdots + V_{2n}\} &\\ \le 1 + 2\sum_{n=1}^{\infty} P\{\delta W_1 > \gamma_n + V_1 + \cdots + V_n\} &\\ +2\sum_{n=1}^{\infty} P\{\delta W_2 > \gamma_n + V_1 + \cdots + V_n\} &\\ < \infty. &\end{aligned}$$

Let $T(M) = \inf\{n: \gamma_n + V_1 + \cdots + V_n + M \ge \delta(W_1 + W_2)\}$. Then, by (v), $ET(M) \le ET(0) < \infty$ and we can choose large M such that $ET(M) < 2$. Set

$$T_1 = T^{(1)} = T(M),$$

$$T^{(n)} = \inf\{j: \gamma_j + V_{k+1} + \cdots + V_{k+j} + M \ge \delta(W_{2n-1} + W_{2n})\}$$

on $\{T_{n-1} = k\}$, and

$$T_n = T_{n-1} + T^{(n)}, \qquad n = 2, 3, \dots .$$

Then, by the strong law of large numbers, there exists an integer-valued random variable N such that

$$T_n = T^{(1)} + \cdots + T^{(n)} < 2n - 1 \quad \text{for any } n \ge N.$$

Since γ_n/n is nondecreasing,

$$\gamma(T^{(1)}) + \cdots + \gamma(T^{(n)}) \le \gamma(T_n) < \gamma(2n-1) \quad \text{for any } n \ge N,$$

$$\delta(W_1 + \cdots + W_{2n}) \le nM + \gamma(2n-1) + V_1 + \cdots + V_{2n-1}, \qquad n \ge N.$$

By the condition that $EW_1 + EV_1 + \lim \gamma_n/n = \infty$,

$$nM = (\gamma_n + V_1 + \cdots + V_n + W_1 + \cdots + W_n)o(1) \quad \text{a.s.}$$

Hence,

$$\limsup (W_1 + \cdots + W_n)/(\gamma_n + V_1 + \cdots + V_n) \le 1/\delta \quad \text{a.s.}$$

And (i) follows because that δ is arbitrary. □

PROOF OF THEOREM 2. We may assume that $P\{X > 0\} \ne 0$ and $P\{X < 0\} \ne 0$. Let

$$T = T_1 = T^{(1)} = \inf\{k: X_k > 0\}, \qquad T_0 = S_0 = 0,$$

$$T^{(n)} = \inf\{k > 0: X_{k+T_{n-1}} > 0\}, \qquad T_n = T_{n-1} + T^{(n)}, \qquad n \ge 2,$$

and

$$W_n = X_{T_n}, \qquad V_n = -(S_{T_n} - S_{T_{n-1}} - W_n), \qquad n \ge 1.$$

Since $T^{(n)}$ are copies of T, (W_n, V_n), $n \ge 1$, are i.i.d. random vectors [see Chow and Teicher (1978, page 136, Lemma 3)]. It is not difficult to find that W_1 is independent of V_1 [see also Kesten (1970, middle of page 1185)].

$$P\{W_1 > t\} = P\{X > t\}/P\{X > 0\}, \qquad EW_1 \ge EX^+,$$
$$V_1 = \sum_{n=1}^{T} X_n^-, \qquad EV_1 \ge EX^-, \tag{2.1}$$

$$\int_0^x P\{V_1 \ge t\}\,dt = E\min(V_1, x) \le E\sum_{n=1}^{T}\min(X_n^-, x)$$
$$= ETE\min(X^-, x), \tag{2.2}$$
$$P\{T > n\} = (P\{X \le 0\})^n,$$
$$ET < \infty \quad \text{and} \quad X_1^- \le V_1.$$

Therefore, by (2.1) and (2.2),

$$J_+(\gamma) < \infty \tag{2.3}$$

iff

$$\int_0^\infty \min\left(\gamma^{-1}(x), x\Big/\int_0^x P\{V_1 \ge t\}\,dt\right) dP\{W_1 \le x\} < \infty.$$

Let k be an integer with $ET < k$. By the strong law of large numbers.

$$P\{T_n \ge kn, \text{i.o.}\} = 0,$$
$$\limsup X_n^+/(\gamma_n + X_1^- + \cdots + X_n^-)$$
$$\ge \limsup W_n/(\gamma(T_n) + V_1 + \cdots + V_n)$$
$$\ge \limsup W_n/(\gamma(kn) + V_1 + \cdots + V_{kn})$$
$$\ge \limsup W_{kn+i}/(\gamma(kn+i) + V_1 + \cdots + V_{kn+i}) \quad \text{for any } i$$
$$= \limsup W_n/(\gamma_n + V_1 + \cdots + V_n) \quad \text{a.s.}$$

It follows from Theorem 3(iii), (vi), and (2.3) that

$$P\{\limsup X_n^+/(\gamma_n + X_1^- + \cdots + X_n^-) = \infty\} = 1 \quad \text{if } J_+(\gamma) = \infty. \tag{2.4}$$

Similarly,

$$\limsup(X_1^+ + \cdots + X_n^+)/(\gamma_n + X_1^- + \cdots + X_n^-)$$
$$\le \limsup(W_1 + \cdots + W_n)/(\gamma(T_{n-1}) + V_1 + \cdots + V_{n-1})$$
$$\le \limsup(W_2 + \cdots + W_n)/(\gamma_{n-1} + V_2 + \cdots + V_n)$$
$$= \limsup(W_1 + \cdots + W_n)/(\gamma_n + V_1 + \cdots + V_n) \quad \text{a.s.}$$

And it follows from Theorem 3(i), (vi), and (2.3) that

$$P\{\limsup(X_1^+ + \cdots + X_n^+)/(\gamma_n + X_1^- + \cdots + X_n^-) = 0\} = 1,$$

if $J_+(\gamma) < \infty$. This completes the proof.

PROOF OF THEOREM 1. (i) and (ii) follow easily from Theorem 2 and Corollary 1. Since $J_+(\gamma) + J_-(\gamma) \le E\gamma^{-1}(|X|)$, we only need to prove the only if part of (iii). Suppose that

$$J_+(\gamma) + J_-(\gamma) < \infty. \tag{2.5}$$

Since $E\gamma^{-1}(|X|) \le \gamma^{-1}(1)[E|X| + 1]$, we can further assume that $E|X| = \infty$. Set

$$h(x) = \min\left(\gamma^{-1}(x), x\Big/\int_0^x P\{|X| \ge t\}\, dt\right). \tag{2.6}$$

Then,

$$\begin{aligned}\int_0^\infty h(x)\, dP\{|X| \le x\} &= \int_0^\infty \min\big(\gamma^{-1}(x), x/(m_+(x) + m_-(x))\big)\, dP\{|X| \le x\} \\ &\le J_+(\gamma) + J_-(\gamma) < \infty \quad \text{by (2.5).}\end{aligned} \tag{2.7}$$

Since $\gamma(x)/x$ is nondecreasing,

$$h(x) \text{ is nondecreasing and } h(x)/x \text{ is nonincreasing.} \tag{2.8}$$

Choose x_0 so that $\int_{x_0}^\infty h(x)\, dP\{|X| \le x\} \le 1/2$. Then by (2.8)

$$\begin{aligned}\int_0^y P\{|X| \ge t\}\, dt &= \int_0^\infty \min(t, y)\, dP\{|X| \le t\} \\ &\le x_0 + y(h(y))^{-1}\int_{x_0}^\infty h(t)\, dP\{|X| \le t\}.\end{aligned}$$

Therefore, since $E|X| = \infty$,

$$h(y)y^{-1}\int_0^y P\{|X| \ge t\}\, dt \le x_0 h(y)/y + 1/2 \to 1/2$$

and

$$h(y) = \gamma^{-1}(y) \quad \text{for all large } y.$$

This and (2.7) finish the proof.

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