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# $C^\infty$ DENSITIES FOR WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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Let $\{X_n\}$ be a sequence of independent random variables and $\{a_n\}$ a square summable, positive nonincreasing sequence of real numbers such that $\sum a_n X_n$ is a random variable. We show that the condition $\lim_{n\to\infty} a_n^2 \log(a_n)/\sum_{k=n+1}^{\infty} a_k^2 = 0$ implies that the distribution measure $F(dx) = P(\sum a_n X_n \in dx)$ has an infinitely differentiable density for every range-splitting sequence $\{X_n\}$. The class of range-splitting sequences includes all nontrivial i.i.d. sequences with mean 0 and finite second moments. Consequences and examples are discussed.

**1. Introduction.** Let $\{X_n\}$ be a sequence of independent random variables. We say that $\{X_n\}$ is *range splitting* if there exist $\lambda > 0$ and a sequence of numbers $\{x_n\}$ such that

$$\inf_n P(X_n - x_n \geq \lambda) > 0, \tag{1.1}$$
$$\inf_n P(X_n - x_n \leq -\lambda) > 0$$

and

$$\sup_n E|X_n| < \infty.$$

Note that every nontrivial i.i.d. sequence with $E|X_1| < \infty$ is range splitting.

Let $\{a_n\}$ be a square summable sequence of real numbers such that $\{|a_n|\}$ is nonincreasing and assume the series

$$X = \sum a_n X_n \tag{1.2}$$

converges in $L_1$.

In this paper we study sufficient conditions on the sequence $\{a_n\}$ such that the distribution measure of $X$

$$F(dx) = P\Big(\sum a_n X_n \in dx\Big) \tag{1.3}$$

has a density for *every* range-splitting sequence $\{X_n\}$.

When we say for every range-splitting sequence, we mean for every range-splitting sequence for which (1.2) is well defined, and since $\{a_n\}$ is square summable,

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this class contains

(a) all i.i.d. sequences with $E(X_n) = 0$ and $E(X_n^2) < \infty$,
(b) if $\Sigma|a_n| < \infty$, all range-splitting sequences [since in (1.1) we assumed $\sup_n E|X_n| < \infty$],
(c) if $\Sigma|a_n| = \infty$, all range-splitting sequences for which $\sup_n E(X_n^2) < \infty$ and $\Sigma a_n E(X_n)$ is well defined.

Some results and examples on (1.3) appeared in [2] and [3]. In particular, we showed in [3], Theorem 1 that a necessary condition for (1.3) to hold is

$$\lim_{n\to\infty} \frac{a_n^2}{\sum_{k=n+1}^{\infty} a_k^2} = 0.$$

In this paper we shall prove that the slightly stronger condition

$$\lim_{n\to\infty} \frac{a_n^2 \log(|a_n|)}{\sum_{k=n+1}^{\infty} a_k^2} = 0$$

not only is a sufficient condition for (1.3), but in fact proves the following stronger result.

THEOREM. *If $\{a_n\}_{n=1}^{\infty}$ is a square summable sequence such that $\{|a_n|\}_{n=1}^{\infty}$ is nonincreasing and*

$$\lim_{n\to\infty} \frac{a_n^2 \log(|a_n|)}{\sum_{k=n+1}^{\infty} a_k^2} = 0,$$

*then the distribution measure*

$$F(dx) = P\left(\sum_{n=1}^{\infty} a_n X_n \in dx\right)$$

*has an infinitely differentiable density for every range-splitting sequence $\{X_n\}_{n=1}^{\infty}$.*

REMARK. The condition that $\{|a_n|\}$ is nonincreasing is no real restriction if $EX_n = 0$ and $\sup EX_n^2 < \infty$, since any rearrangement of $\{a_n X_n\}$ alters $\Sigma a_n X_n$ on a set of probability 0. For general information see Garsia [1] and Reich [2], [3].

**2. Proof of the theorem.** To prove the theorem we will need Lemma 2 from Reich [3], which states:

LEMMA R. *Let $\{X_n\}$ be a range-splitting sequence. Then there exist $\delta$, $\lambda > 0$ such that*

$$|E(e^{iuX_n})| \le e^{-\lambda u^2}$$

*for all $0 \le |u| \le \delta$ and all $n$.*

We also need

LEMMA 1. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers such that*

$$\lim_{n\to\infty} \frac{1}{x_n} \sum_{j=1}^{n} \frac{1}{x_j} = 0.$$

*Then*

$$\sum_{n=1}^{\infty} e^{-\lambda x_n} < \infty$$

*for all* $\lambda > 0$.

PROOF. Clearly, it follows from the assumption that $\lim_{n\to\infty} x_n = \infty$. Thus we can rearrange the sequence $\{x_n\}$ in nondecreasing order, $\{x_{k_n}\}$.

Since $\{x_{k_n}\}$ is simply a rearrangement of $\{x_n\}$,

$$\sum_{n=1}^{\infty} e^{-\lambda x_n} = \sum_{n=1}^{\infty} e^{-\lambda x_{k_n}}.$$

With $l_n = \max_{1\le j\le n} k_j$, we have

$$\sum_{j=1}^{n} \frac{1}{x_{k_j}} \le \sum_{j=1}^{l_n} \frac{1}{x_j},$$

and since $x_{l_n} \le x_{k_n}$,

$$\frac{1}{x_{k_n}} \sum_{j=1}^{n} \frac{1}{x_{k_j}} \le \frac{1}{x_{l_n}} \sum_{j=1}^{l_n} \frac{1}{x_j}.$$

But the right-hand side goes to 0 by the hypothesis; therefore we get that

$$o_n(1) = \frac{1}{x_{k_n}} \sum_{j=1}^{n} \frac{1}{x_{k_j}} \ge \frac{n}{x_{k_n}^2},$$

the last inequality since $\{x_{k_n}\}$ is nondecreasing. This implies $x_{k_n} \ge \sqrt{n}$ for large $n$, and the lemma is proved. □

In proving our theorem we may assume $a_n > 0$ for all $n$, since any negative sign can be transferred to the corresponding $X_n$ without changing its range-splitting property. So from now on $\{a_n\}$ will be a nonincreasing sequence of positive numbers.

DEFINITION 1. To a given square summable sequence $\{a_n\}$, we associate a positive, nonincreasing continuous function defined on $[0, \infty)$ with $f(n) = a_n$ for $n \ge 1$. We now define

$$H(x) = -\log \int_x^{\infty} f^2(t)\, dt$$

and

$$R(n) = a_n^{-2} \sum_{k=n+1}^{\infty} a_k^2.$$

By multiplying $a_n$ and $f$ by a fixed constant, we may without loss of generality assume that $\int_0^\infty f^2(t)\,dt = 1$, or, equivalently, that $H(0) = 0$. Thus $H(x)$ is nonnegative and increasing.

LEMMA 2. *Let $\{a_n\}$ be a square summable sequence. Then*

(a) $$a_n^2 = \sum_{j=2}^{\infty} a_j^2 \left[ R(n) \prod_{j=2}^{n} [1 + (1/R(j))] \right]^{-1} \quad \textit{for } n \geq 2,$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{R(n)} = \infty.$$

PROOF. From the definition of $R(n)$ we get $R(n-1)a_{n-1}^2 - R(n)a_n^2 = a_n^2$, which implies

$$\frac{a_n^2}{a_{n-1}^2} = \frac{R(n-1)}{R(n)+1} \quad \text{for } n \geq 2.$$

Therefore

$$\frac{a_n^2}{a_1^2} = \frac{R(1)R(2)\cdots R(n-1)}{(R(2)+1)(R(3)+1)\cdots(R(n)+1)}$$
$$= R(1)\left[ R(n) \prod_{j=2}^{n} [1 + (1/R(j))] \right]^{-1}$$

and from here it follows that

$$a_n^2 = \sum_{j=2}^{\infty} a_j^2 \left[ R(n) \prod_{j=2}^{n} [1 + (1/R(j))] \right]^{-1},$$

since $a_1^2 R(1) = \sum_{j=2}^{\infty} a_j^2$.

To prove (b) we simply observe that

$$\sum_{k=n+1}^{\infty} a_k^2 = a_n^2 R(n) = \sum_{j=2}^{\infty} a_j^2 \left[ \prod_{j=2}^{n} [1 + 1/R(j)] \right]^{-1}.$$

If $\sum_{j=1}^{\infty} 1/R(j)$ were finite, the product $\prod_{j=2}^{n}(1 + 1/R(j))$ would converge to a finite limit and so $\sum_{k=n+1}^{\infty} a_k^2$ would not tend to zero, which is a contradiction. □

PROPOSITION 1. *Let $\{a_n\}$ be a square summable sequence such that $\{|a_n|\}$ is nonincreasing and*

$$\lim_{n\to\infty} \frac{1}{R(n)} \sum_{j=1}^{n} \frac{1}{R(j)} = 0.$$

*Then the distribution measure*

$$F(dx) = P\left( \sum_{n=1}^{\infty} a_n X_n \in dx \right)$$

*has an infinitely differentiable density for every range-splitting sequence* $\{X_n\}$.

PROOF. We fix a range-splitting sequence $\{X_n\}$. The Fourier transform of $F(dx)$ is given by

$$\hat{F}(u) = \prod_{n=1}^{\infty} E(e^{iua_nX_n})$$

and therefore

$$|\hat{F}(u)| \leq \prod_{n=N}^{\infty} |E(e^{iua_nX_n})| \tag{2.1}$$

for every $N \geq 1$. By Lemma R,

$$|E(e^{iua_nX_n})| \leq e^{-\lambda u^2 a_n^2}$$

for $|ua_n| \leq \delta$. And since the sequence $\{a_n\}$ is nonincreasing, this implies

$$|E(e^{iua_jX_j})| \leq e^{-\lambda u^2 a_j^2} \tag{2.2}$$

for $|ua_n| \leq \delta$ and $j \geq n$.

Now (2.1) and (2.2) imply that

$$|\hat{F}(u)| \leq \exp\left( -\lambda u^2 \sum_{j=n+1}^{\infty} a_j^2 \right) \tag{2.3}$$

for $\delta/a_n \leq |u| \leq \delta/a_{n+1}$.

We will prove that

$$\int_{-\infty}^{\infty} |u|^k |\hat{F}(u)|\, du < \infty \tag{2.4}$$

for all positive integers $k$, which proves the proposition.

By (2.3)

$$\int_{\infty}^{\infty} |u|^k |\hat{F}(u)|\, du \leq 2(\delta/a_2) + 2 \sum_{N=2}^{\infty} \int_{\delta/a_N}^{\delta/a_{N+1}} u^k \exp\left( -\lambda u^2 \sum_{j=N+1}^{\infty} a_j^2 \right) du$$
$$= 2(\delta/a_2) + 2 \sum_{N=2}^{\infty} I_N.$$

Now

$$I_N \leq \int_{\delta/a_N}^{\infty} u^k \exp(-\lambda a_N^2 R(N) u^2)\, du$$
$$= \frac{1}{(\lambda a_N^2 R(N))^{1/2(k+1)}} \int_{\delta(\lambda R(N))^{1/2}}^{\infty} t^k e^{-t^2}\, dt$$
$$\leq C_k \frac{1}{(a_N^2 R(N))^{1/2(k+1)}} \exp(-\tfrac{1}{2}\lambda\delta^2(R(N)),$$

since $\int_\alpha^\infty t^k e^{-t^2}\, dt \leq C_k e^{-\alpha^2/2}$ if $\alpha > 0$.

By Lemma 2, with $\Sigma_{k=2}^{\infty} a_k^2 = 1$, and the assumption $[1/R(N)]\Sigma_{j=1}^{N} 1/R(j) \to 0$, hence also $R(N) \to \infty$, it follows that for $N$ large enough

$$\begin{aligned} I_N &\le C\exp\{\tfrac{1}{2}(k+1)\log a_N^{-2} - \bar{\lambda}R(N)\} \\ &= C\exp\left\{\tfrac{1}{2}(k+1)\log R(N) + \tfrac{1}{2}(k+1)\sum_{j=2}^{N}\log(1+1/R(j)) - \bar{\lambda}R(N)\right\} \\ &\le C\exp\left(-R(N)\left[\bar{\lambda} - \frac{\tfrac{1}{2}(k+1)\log R(N)}{R(N)} - \frac{\tfrac{1}{2}(k+1)}{R(N)}\sum_{j=2}^{N}\frac{1}{R(j)}\right]\right) \\ &\le \exp\left(-\frac{\bar{\lambda}}{2}R(N)\right). \end{aligned}$$

Finally, Lemma 1 implies (2.4). □

LEMMA 3. *Let $\{a_n\}$ be a square summable sequence such that $\{|a_n|\}$ is nonincreasing. Then the following statements are equivalent*:

(a) $$\lim_{n\to\infty}\frac{1}{R(n)}\sum_{j=1}^{n}\frac{1}{R(j)} = 0,$$

(b) $$\lim_{x\to\infty} H(x)H'(x) = 0,$$

(c) $$\lim_{n\to\infty}\frac{a_n^2\log(|a_n|)}{\Sigma_{k=n+1}^{\infty} a_k^2} = 0.$$

PROOF. (a) $\Rightarrow$ (b). Let $n$ be a positive integer and $n \le x \le n+1$. Since $f(x)$ is nonincreasing, $f^2(x) \le f^2(n)$, and

$$\int_x^{\infty} f^2(t)\,dt = \int_n^{\infty} f^2(t)\,dt - \int_n^{x} f^2(t)\,dt \ge \sum_{k=n+1}^{\infty} f^2(k) - f^2(n).$$

Therefore

$$H'(x) = \frac{f^2(x)}{\int_x^{\infty} f^2(t)\,dt} \le \frac{f^2(n)}{\Sigma_{k=n+1}^{\infty} f^2(k) - f^2(n)} \le \frac{1}{R(n)-1} \le \frac{2}{R(n)}. \tag{2.5}$$

The last inequality follows since by (a) $\lim_{n\to\infty} R(n) = \infty$ and therefore is true for $n$ sufficiently large. We may assume this for all $n$ without loss of generality. Now write

$$\begin{aligned} H(x) &= H(1) + H(x) - H(n) + \sum_{j=1}^{n-1} H(j+1) - H(j) \\ &= H(1) + \sum_{j=1}^{n} H'(x_j). \end{aligned} \tag{2.6}$$

This equality uses the mean value theorem and $x_j \in [j, j+1]$ for $j = 1, 2, \ldots, n-1$ and $x_n \in [n, x]$.

Now we use (2.5) and (2.6) to obtain

$$H'(x)H(x) \le \frac{2}{R(n)}\left[H(1) + \sum_{j=1}^{n} \frac{2}{R(j)}\right],$$

hence $\lim_{x\to\infty} H'(x)H(x) = 0$ by (a).

(b) $\Rightarrow$ (a). Let $g(x) = 2H'(x)H(x)$. Since we assume $H(0) = 0$, $H^2(x) = \int_0^x g(t)\,dt$ and

$$\int_x^\infty f^2(t)\,dt = \exp(-H(x)) = \exp\left(-\left(\int_0^x g(t)\,dt\right)^{1/2}\right). \tag{2.7}$$

Since by (b) $g(x) \to 0$, $x \to \infty$, (2.7) implies

$$\lim_{x\to\infty} \sup_{|y|\le r} \frac{\int_x^\infty f^2(t)\,dt}{\int_{x+y}^\infty f^2(t)\,dt} = 1$$

for all $r > 0$.

Hence, if $n \le x \le n+1$, and $n$ is sufficiently large,

$$\int_x^\infty f^2(t)\,dt \le 2\int_{n+1}^\infty f^2(t)\,dt \le 2\sum_{k=n+2}^{\infty} f^2(k). \tag{2.8}$$

We will assume (2.8) for all $n$. Hence

$$H'(x) = \frac{f^2(x)}{\int_x^\infty f^2(t)\,dt} \ge \frac{f^2(n+1)}{2\sum_{k=n+2}^\infty f^2(k)} = \frac{1}{2R(n+1)}. \tag{2.9}$$

Together with (2.6) this implies

$$H'(x)H(x) \ge \frac{1}{2R(n+1)}\left[H(1) + \sum_{j=2}^{n+1} \frac{1}{2R(j)}\right],$$

which proves (a).

(b) $\Rightarrow$ (c). By (2.8)

$$\int_x^\infty f^2(t)\,dt \le 2f^2(n+1)R(n+1).$$

Together with (2.9) this implies

$$H(x)H'(x) \ge \frac{\left|\log[2f^2(n+1)R(n+1)]\right|}{2R(n+1)} = \frac{|\log(2) + \log R(n+1) + 2\log f(n+1)|}{2R(n+1)},$$

and since $\log(2)/2\,R(n+1)$ and $\log R(n+1)/R(n+1)$ converge to 0 as $n\to\infty$, it follows that

$$\lim_{n\to\infty} \frac{\log f(n+1)}{R(n+1)} = 0$$

which proves (c).

(c) $\Rightarrow$ (b). Note that

$$\frac{a_n^2 \log(a_n)}{\sum_{k=n+1}^{\infty} a_k^2} = \frac{\log(a_n)}{R(n)}.$$

Hence, assuming (c) implies $\lim_{n \to \infty} R(n) = \infty$, so (2.5) holds.

Now

$$\begin{aligned} H(x) &= -\log\left(\int_x^{\infty} f^2(t)\,dt\right) \le -\log\left(\int_{n+1}^{\infty} f^2(t)\,dt\right) \\ &\le -\log\left[\sum_{k=n+1}^{\infty} f^2(k) - f^2(n+1)\right] \\ &= -\log\{f^2(n)[R(n) - (f^2(n+1)/f^2(n))]\} \\ &\le -\log[f^2(n)(R(n) - 1)] \\ &\le -\log[f^2(n)2^{-1}R(n)], \end{aligned}$$

the last inequality for $n$ sufficiently large. Together with (2.5) this implies

$$H(x)H'(x) \le \frac{4|\log f(n) + 2\log 2 + 2\log R(n)|}{R(n)},$$

from which (b) follows. □

The theorem follows now from Proposition 1 and Lemma 3.

## 3. Examples.

EXAMPLE 1. Let $\beta \ge \alpha > \frac{1}{2}$ and $\beta - \alpha < \frac{1}{2}$. Then if $\{a_n\}$ is a sequence such that $\{|a_n|\}$ is nonincreasing and

$$\underline{C}n^{-\beta} \le |a_n| \le \bar{C}n^{-\alpha}$$

for some $\underline{C}, \bar{C} > 0$, then

$$\lim_{n \to \infty} \frac{a_n^2 \log(|a_n|)}{\sum_{k=n+1}^{\infty} a_k^2} = 0.$$

Hence, the conclusion of the theorem follows.

The proof is obvious. The condition $\beta - \alpha < \frac{1}{2}$ is sharp as was shown in [3].

In the following example we show how part (b) of Lemma 3 can be used to construct sequences $\{a_n\}$ which satisfy the hypothesis of our theorem.

EXAMPLE 2. Let $p(x)$, $x \ge 0$ be a positive nonincreasing function such that $\lim_{x \to \infty} p(x) = 0$ and $\int_0^{\infty} p(x)\,dx = \infty$.

For any number $C > 0$, let

$$a_n^2 = \frac{Cp(n)}{2\left(\int_0^n p(t)\,dt\right)^{1/2}} \exp\left(-\left(\int_0^n p(t)\,dt\right)^{1/2}\right). \tag{3.1}$$

Then the sequence $\{a_n\}$ satisfies the assumption, and hence also the conclusion of the theorem.

To prove this define $f$ by

$$\int_x^{\infty} f^2(t)\,dt = C\exp\left(-\left(\int_0^x p(t)\,dt\right)^{1/2}\right).$$

Then $f(n) = a_n$, so $f$ is associated to $\{a_n\}$ in the sense of Definition 1. Also since $\int_0^{\infty} p(x)\,dx = \infty$, $f \in L^2(0, \infty)$, and so $a_n$ is square summable.

Now $H(x) = -\log \int_x^{\infty} f^2(t)\,dt$, and thus $H'(x)H(x) = Cp(x) \to 0$, $x \to \infty$, and the result follows by Lemma 3.

By taking $p(n) \geq o_n^2(1)$, where $o_n(1)$ is an arbitrary sequence of positive numbers tending to zero, the sequence $\{a_n\}$ given by (3.1) satisfies $a_n \leq e^{-o_n(1)\sqrt{n}}$. Thus a sequence can decay much faster than polynomially and still satisfy the hypothesis of the theorem.

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