The Annals of Probability
1986, Vol. 14, No. 3, 984–995

REGULAR VARIATION AND THE STABILITY OF MAXIMA[1]

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The notion of regular variation of functions is generalized by defining $l(t) = \liminf_{x \to \infty} L(tx)/L(x)$, $t > 0$, for any positive nondecreasing function L. It is shown that l must obey one of: (i) $l(t) = +\infty$ for every $t > 1$; (ii) $l(t) > 1$ for every $t > 1$ and $l(t) \downarrow 1$ as $t \downarrow 1$; or (iii) $l(t) = 1$ for some $t > 1$. Each of these classes is characterized in terms of the convergence or divergence of the integral $I(r, \delta) = \int_1^\infty \exp\{rL(\delta x) - L(x)\}\, dL(x)$ for $r \geq 1$, $\delta < 1$.

Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function F. Define $\mu_n = F^{-1}(1 - n^{-1})$, $M_n = \max(X_1, \ldots, X_n)$, and $L(x) = -\log(1 - F(x))$. $\{M_n\}$ is almost surely stable iff $M_n/\mu_n \to 1$ a.s., and this is known to be equivalent to the convergence of $I(1, \delta)$ for every $\delta < 1$. Necessary and sufficient conditions for $\sum_{n=1}^\infty n^\alpha P[|(M_n/\mu_n) - 1| > \varepsilon] < \infty$ are presented, where $\alpha \geq -1$. In particular, that series converges iff $I(\alpha + 2, (1 + \varepsilon)^{-1}) < \infty$. Moreover, the series $\sum_{n=1}^\infty n^\alpha P[|(M_n/\mu_n) - 1| > \varepsilon]$ converges for all $\varepsilon > 0$ and some $\alpha > -1$ iff it converges for every $\alpha > -1$ and every $\varepsilon > 0$.

1. Introduction. Consider a sequence $X_1, X_2, \ldots$ of independent, identically distributed (i.i.d.) random variables (r.v.'s) with common distribution function (d.f.) $F(x) = P[X_1 \leq x]$. For $n \geq 1$, let $M_n = \max(X_1, \ldots, X_n)$.

If $F(x) < 1$ for all x, Gnedenko (1943) called the sequence $\{M_n\}$ *relatively stable* if $M_n/a_n \to 1$ in probability for some real sequence $\{a_n\}$, and proved that relative stability is equivalent to

$$\lim_{x \to \infty} (1 - F(tx))/(1 - F(x)) = 0 \quad \text{for every } t > 1. \tag{1.1}$$

More generally, $\{M_n\}$ is called *almost surely* (*relatively*) *stable* if $M_n/a_n \to 1$ almost surely (a.s.). Barndorff-Nielsen (1963) showed that a.s. stability occurs if and only if (iff)

$$\int_1^\infty [1 - F(\delta x)]^{-1}\, dF(x) < \infty \quad \text{for every } \delta < 1. \tag{1.2}$$

This result was refined by Resnick and Tomkins (1973), who showed that a.s. stability is guaranteed if $\limsup_{n \to \infty} M_n/\mu_n = 1$ a.s., where $\mu_n = F^{-1}(1 - n^{-1})$ and F^{-1} is defined by

$$F^{-1}(y) = \inf\{x\colon F(x) \geq y\}, \qquad 0 \leq y \leq 1.$$

Received July 1984; revised July 1985.

[1]This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

AMS 1980 *subject classifications.* Primary 60F15; secondary 26A12, 26A48.

Key words and phrases. Sample maxima, i.i.d. random variables, relative stability, a.s. stability of maxima, complete convergence, regularly varying functions, generalized regular variation.

With these results in mind, it is natural to seek necessary and sufficient conditions for

$$\sum_{n=1}^{\infty} P\left[\left|\frac{M_n}{a_n} - 1\right| > \varepsilon\right] < \infty \quad \text{for every } \varepsilon > 0 \tag{1.3}$$

and some sequence $a_n \uparrow$. Notice that $M_n/a_n \to 1$ completely [in the sense of Hsu and Robbins (1947)] if (1.3) holds, and that (1.3) implies a.s. stability by virtue of the Borel–Cantelli lemma.

Section 3 will present some necessary and sufficient conditions for the more general result

$$\sum_{n=1}^{\infty} n^{\alpha} P\left[\left|\frac{M_n}{a_n} - 1\right| > \varepsilon\right] < \infty \quad \text{for every } \varepsilon > 0, \tag{1.4}$$

where $\alpha \geq -1$. In fact, if (1.4) holds for some $\alpha > -1$ [in particular, if (1.3) holds], then (1.4) must hold for *every* $\alpha > -1$. Moreover, it may be assumed that $a_n \sim \mu_n$ (i.e., $a_n/\mu_n \to 1$) in this case.

Section 3 will also establish a connection between the convergence or divergence of the series in (1.4) and the behaviour of the function

$$l(t) = \liminf_{x \to \infty} L(tx)/L(x), \qquad t > 0,$$

where $L(x) = -\log(1 - F(x))$. The definition of $l(t)$ is reminiscent of regular variation: A positive, monotone function L is called *regularly varying* if

$$\lambda(t) = \lim_{x \to \infty} L(tx)/L(x)$$

exists for all $t > 0$; see, for example, Feller (1971), pages 275–279, or Seneta (1976), page 1. Regular variation has been studied extensively and generalized; two papers by Bingham and Goldie (1982a, 1982b) are especially noteworthy in this regard.

Connections between regular variation and limiting behaviour of M_n have been discovered (e.g., Feller (1971), page 278, or de Haan (1970)) and some results of Section 3 extend this connection.

Section 2 investigates the properties of $l(t)$ for general, monotone positive functions L. It will be shown (cf. Theorem 2.1) that any such L must satisfy exactly one of: (i) $l(t) = +\infty$ for every $t > 1$; (ii) $l(t) > 1$ for $t > 1$ but $\inf_{t>1} l(t) = 1$; or (iii) $l(t) = 1$ for some $t > 1$. Each of these three categories is then characterized in terms of the integral $\int_1^{\infty} \exp\{rL(\delta x) - L(x)\}\, dL(x)$ and the difference $L(x) - rL(\delta x)$, where $0 < \delta < 1$ and $r \geq 1$.

Finally, Section 4 will indicate how the results of Section 3 may be adapted to produce convergence results in the case where $F(x) = 1$ for some x.

2. A generalization of regular variation.

A positive monotone function L defined on $(0, \infty)$ is called *regularly varying* if

$$\lim_{x \to \infty} L(tx)/L(x) = \lambda(t) \tag{2.1}$$

exists for every $t > 0$. In fact, if L is regularly varying, then $\lambda(t) = t^{\rho}$ for all $t > 0$ and some ρ, $-\infty \le \rho \le +\infty$ (where $t^{\infty} = 0$ if $t < 1$, $t^{\infty} = \infty$ if $t > 1$), and L is said to have exponent ρ. It is common to say that L is *slowly varying* if $\rho = 0$ [i.e., $\lambda(t) = 1$ for all $t > 0$], and that L is *rapidly varying* if $\rho = \pm\infty$. These and other properties of regularly varying functions have been extensively expounded; see, for example, de Haan (1970), Feller (1971), or Seneta (1976). Several generalizations of regular variation have been studied by Bingham and Goldie (1982a, 1982b). Because the limit in (2.1) need not exist for all $t > 0$, functions may exist which are not regularly varying (in this connection, see Remark 2 following Theorem 2.2). This section will extend the notion of regular variation so as to include every monotone function in exactly one of three classes analogous to the cases $\rho = \pm\infty$, $0 < |\rho| < \infty$ and $\rho = 0$ of regular variation.

Suppose L is a positive nondecreasing function defined on $(0, \infty)$. For any $t > 0$, define

$$l(t) = \liminf_{x \to +\infty} L(tx)/L(x). \tag{2.2}$$

Certain functions of this form have arisen in the study of stochastic compactness [Maller (1981)] and domains of partial attraction [Maller (1980) and Goldie and Seneta (1982)]. The function l has been studied in its own right by Bingham and Goldie (1982a, 1982b) and Matuszewska (1962). The function l will be used in Section 3 for $L(x) = -\log(1 - F(x))$ for certain distribution functions F.

Clearly $l(t)$ is well defined for all $t > 0$ and, trivially, $l(1) = 1$. Moreover, l is nondecreasing since L is nondecreasing.

The following theorem shows that every function l defined by (2.2) falls into exactly one of three categories not unlike the rapid, regular and slow variation classifications. This result could be derived from work of Bingham and Goldie (1982b) on Karamata indices, but a simple proof is presented here for completeness.

THEOREM 2.1. *For a nondecreasing positive function L on* $(0, \infty)$, *define l by* (2.2). *Then exactly one of the following properties holds*:

(i) $l(t) = +\infty$ *for every* $t > 1$;
(ii) $l(t) > 1$ *for every* $t > 1$ *but* $\inf_{t>1} l(t) = 1$;
(iii) $l(t) = 1$ *for some* $t > 1$.

PROOF. Clearly $\inf_{t>1} l(t) \ge 1$. If $\inf_{t>1} l(t) = 1$, then (ii) or (iii) must hold. Now suppose $\beta \equiv \inf_{t>1} l(t) > 1$. It is easy to see [cf. Matuszewska (1962)] that $l(t_1 t_2) \ge l(t_1) l(t_2)$ for $t_1, t_2 > 0$ and, by induction, $l(t^n) \ge l^n(t)$. Therefore, if $t_0 > 1$, $l(t_0) \ge l^n(t_0^{1/n}) \ge \beta^n \to \infty$, so $l(t) = \infty$ for all $t > 1$ whenever $\beta > 1$. □

REMARKS.

1. Notice that (i) holds iff L and $1/L$ are rapidly varying iff l is *not* right-continuous at $t = 1$.

2. It is easy to create a function which is not regularly varying for any ρ, but which satisfies (ii) or (iii). If L_1 is regularly varying with exponent ρ, $0 < \rho < \infty$,

then let $L(x) = 2^n L_1(x)$ if $2^n \le x < 2^{n+1}$, $n \ge 0$. Then it is not hard to show that $l(t) = 2^m t^p$ and $\limsup_{x \to 0} L(tx)/L(x) = 2^{m+1} t^p$ whenever $2^m < t \le 2^{m+1}$. So L obeys (ii) in this case. By defining $L(x) = L_1(2^n)$ for $2^n \le x < 2^{n+1}$, it follows that $L(t2^n)/L(2^n) = 1$ if $1 \le t < 2$, so $l(t) = 1$ if $1 \le t < 2$; thus L obeys (iii). But L is not slowly varying because $L(2^{n+1})/L(2^n) \to 2^p > 1$.

Each of the three categories in Theorem 2.1 can be characterized in terms of functions of the form

$$(2.3) \qquad D(x; r, \delta) = L(x) - rL(\delta x), \qquad 0 < \delta < 1, \quad r \ge 1, \quad x > 0,$$

and integrals

$$(2.4) \qquad I(r, \delta) = \int_1^\infty \exp\{rL(\delta x) - L(x)\}\, dL(x), \qquad 0 < \delta < 1, \quad r \ge 1.$$

Since $I(r, \delta) < \infty$ and $\lim_{x \to \infty} D(x; r, \delta) = +\infty$ for every $\delta < 1$ if $r < 1$, and $I(r, \delta) = +\infty$ and $\liminf_{x \to \infty} D(x; r, \delta) = -\infty$ for every $r > 1$ if $\delta \ge 1$, it will be assumed that $0 < \delta < 1$ and $r \ge 1$.

Consider, first, the following key lemma.

LEMMA 2.2. *Fix $r \ge 1$ and $\delta < 1$. Define $I(r, \delta)$ by* (2.4).

(i) *If $I(r, \delta) < \infty$ then*

$$(2.5) \qquad \lim_{x \to \infty} \{L(x) - rL(\delta x)\} = +\infty.$$

(ii) *If* (2.5) *holds, then $l(\delta^{-1}) \ge r$.*
(iii) *If $l(\delta^{-1}) > r$ then $I(r, \delta) < \infty$.*

PROOF. Define $D(x; r, \delta)$ by (2.3). Since $d(e^{-L(x)}) = -e^{-L(x)}\, dL(x)$,

$$\exp\{-D(x; r, \delta)\} = \int_{y=x}^\infty \exp\{rL(\delta x) - L(y)\}\, dL(y)$$
$$\le \int_{y=x}^\infty \exp(-D(y; r, \delta))\, dL(y) \to 0$$

as $x \to \infty$ if $I(r, \delta) < \infty$, so (i) holds.

Note that (2.5) is equivalent to $L(\delta x)\{L(x)/L(\delta x) - r\} \to \infty$ as $x \to \infty$. Since $L(\delta x) \ge 0$ for all x, $L(x)/L(\delta x) > r$ for all large x. Therefore $l(\delta^{-1}) \ge r$, so (ii) holds.

Now suppose $l(\delta^{-1}) > r$. Choose s such that $r < s < l(\delta^{-1})$. Then a number c exists such that $L(x) > sL(\delta x)$ for all $x \ge c$. Letting $V = 1 - r/s > 0$,

$$\int_c^\infty \exp\{rL(\delta x) - L(x)\}\, dL(x) \le \int_c^\infty \exp(-VL(x))\, dL(x)$$
$$= V^{-1} \exp(-VL(c)) < \infty,$$

so (iii) holds. □

With the aid of this lemma, a characterization of rapid variation can be established.

THEOREM 2.3. *The following statements are equivalent*:

(i) $l(t) = +\infty$ *for every* $t > 1$;
(ii) $I(r, \delta) < \infty$ *for every* $\delta < 1$ *and every* $r > 1$;
(iii) $I(r, \delta) < \infty$ *for every* $\delta < 1$ *and some* $r > 1$;
(iv) (2.5) *holds for every* $\delta < 1$ *and some* $r > 1$;
(v) (2.5) *holds for every* $\delta < 1$ *and every* $r > 1$.

PROOF. Lemma 2.2(iii) ensures (ii) when (i) holds. Obviously, (ii) implies (iii), which in turn implies (iv) by Lemma 2.2(i). But (ii) implies (v) by Lemma 2.2(i), and (v) implies (iv), trivially. It remains only to observe that (iv) implies (i) by Theorem 2.1 and Lemma 2.2(ii). □

The following theorem characterizes the functions L obeying $l(t) = 1$ for some $t > 1$.

THEOREM 2.4. *The following assertions are equivalent*:

(i) $l(t) = 1$ *for some* $t > 1$;
(ii) *for some* $\delta < 1$, $I(r, \delta) = \infty$ *for every* $r > 1$;
(iii) *for some* $\delta < 1$, (2.5) *fails to hold for any* $r > 1$; *i.e.*,

$$\liminf_{x \to \infty} D(x; r, \delta) < \infty \tag{2.6}$$

for every $r > 1$.

PROOF. Suppose $l(\gamma) = 1$ for some $\gamma > 1$. Then $l(\gamma) < r$ for every $r > 1$, so (2.6) holds with $\delta = \gamma^{-1}$ for every $r > 1$ by Lemma 2.2(ii). Thus, (i) implies (iii). By Lemma 2.2(i), (iii) implies (ii). Finally, if $I(r, \delta) = \infty$ for some $\delta < 1$ and every $r > 1$, then $l(\delta^{-1}) \leq r$ for all $r > 1$ by Lemma 2.2(iii). Hence $l(\delta^{-1}) = 1$, so (ii) implies (i). □

Theorems 2.1, 2.3, and 2.4 combine to yield the following characterizations of the case where $l(t) > 1$ for all $t > 1$ and $\inf_{t>1} l(t) = 1$.

THEOREM 2.5. *The following statements are equivalent*:

(i) $l(t) > 1$ *for every* $t > 1$ *and* $\inf_{t>1} l(t) = 1$;
(ii) *for every* $\delta < 1$, *there exists* $r > 1$ *such that* (2.5) *holds, and for each* $r > 1$, *there exists* $\delta < 1$ *such that* (2.6) *holds*;
(iii) *for every* $\delta < 1$, *there exists* $r > 1$ *such that* $I(r, \delta) < \infty$, *and for each* $r > 1$, *there exists* $\delta < 1$ *such that* $I(r, \delta) = \infty$.

Finally, a line of reasoning similar to that used to prove Theorem 2.4 leads to a characterization of those functions L for which $l(t) = 1$ for all $t \geq 1$.

THEOREM 2.6. *The following are equivalent*:

(i) $l(t) = 1$ *for every* $t > 1$;
(ii) *for every* $r > 1$, (2.6) *holds for every* $\delta < 1$;
(iii) *for some* $r > 1$, (2.6) *holds for every* $\delta < 1$;
(iv) *for some* $r > 1$, $I(r, \delta) = \infty$ *for every* $\delta < 1$;
(v) *for every* $r > 1$, $I(r, \delta) = \infty$ *for every* $\delta < 1$.

PROOF. Suppose $l(t) = 1$ for all $t > 1$. If $\delta < 1$ and $r > 1$, $l(\delta^{-1}) = 1 < r$, so (2.6) holds for all $r > 1$, $\delta < 1$ by Lemma 2.2(ii); thus, (i) implies (ii). Now, (ii) implies (iii) trivially, while (iii) implies (iv) by Lemma 2.2(i). If (iv) holds, then $I(r, \delta) = \infty$ for some $r > 1$ and all $\delta < 1$, so $l(\delta^{-1}) \le r$ for some $r > 1$ and all $\delta < 1$ by Lemma 2.2(iii); thus $\sup_{t>1} l(t) \le r < \infty$. But then, for any $t > 1$, $1 \le l(t) \le l^{1/n}(t^n) \le r^{1/n}$ for every $n \ge 1$ (cf. the proof of Theorem 2.1). Since $r^{1/n} \to 1$, $l(t) = 1$, i.e., (i) holds.

If (i) holds, then $l(\delta^{-1}) < r$ for all $r > 1$, $\delta < 1$, so (v) holds by Lemma 2.2. Since (v) implies (iv), trivially, the theorem is proved. □

REMARK. It follows from Theorems 2.3 and 2.5 that $I(1, \delta) < \infty$ and $\lim_{x\to\infty} D(x; 1, \delta) = \infty$ for every $\delta < 1$ whenever $l(t) > 1$ for all $t > 1$. But if $l(t) = 1$ for some $t > 1$, even if L is slowly varying, it is possible that both of these assertions hold, or that neither holds, or that only $\lim_{x\to\infty} D(x; 1, \delta) = \infty$. [Note that $I(1, \delta) < \infty$ guarantees $\lim_{x\to\infty} D(x; 1, \delta) = \infty$ by Lemma 2.2(i).] For instance, $L(x) = \log x$ is slowly varying, but $L(x) - L(\delta x) = -\log \delta$ and $I(1, \delta) = \delta^{-1} \int_1^\infty dL(x) = \infty$ for all $\delta < 1$. But the slowly varying function $L(x) = (\log x)^2$ obeys $I(1, \delta) < \infty$ and, hence, $\lim_{x\to\infty} D(x; 1, \delta) = \infty$ for every $\delta < 1$. Finally, if $L(x) = c \log x \log\log x$ for $x > e^2$ and some $c > 0$, then $\lim_{x\to\infty} D(x; 1, \delta) = \infty$ for all $\delta < 1$, but $I(1, \delta) = \infty$ if $\delta > e^{-c}$ [cf. Resnick and Tomkins (1973)].

3. A generalization of stability. Throughout this section $M_n = \max(X_1, \dots, X_n)$, where $X_1, X_2, \dots$ are i.i.d. r.v.'s with common d.f. F with $F(x) < 1$ for all x. Let $\mu_n = F^{-1}(1 - n^{-1})$. This section focusses on establishing necessary and sufficient conditions for

$$\sum_{n=1}^{\infty} n^{r-2} P\left[\left|\frac{M_n}{\mu_n} - 1\right| > \varepsilon\right] < \infty \quad \text{for every } \varepsilon > 0. \tag{3.1}$$

The following lemma is due to Tomkins (1984); its proof is given here for completeness.

LEMMA 3.1. *Let* $\{c_n\}$ *be a real sequence such that* $c_n \uparrow +\infty$. *Suppose* $-\infty < p \le 1$.

(i) *If* $\sum_{n=1}^{\infty} n^{-p}[1 - F^n(c_n)] < \infty$ *then* $F^n(c_n) \to 1$.
(ii) *If* $\sum_{n=1}^{\infty} n^{-p} F^n(c_n) < \infty$ *then* $F^n(c_n) \to 0$.

PROOF. Because the summands of a convergent series necessarily converge to zero, both parts are easy to prove if $p \le 0$, so assume $0 < p \le 1$. Then

$$1 - F^n(c_{2n}) = 2 \sum_{j=n+1}^{2n} (2n)^{-1}(1 - F^n(c_{2n})) \le 2 \sum_{j=n+1}^{2n} j^{-p}\left[1 - F^j(c_j)\right].$$

Under the hypotheses of (i), clearly $F^n(c_{2n}) \to 1$, so that $F^{2n}(c_{2n}) \to 1$. But, since $c_n \uparrow$, it follows that

$$1 \ge F^{2n+1}(c_{2n+1}) \ge F(c_{2n+1})F^{2n}(c_{2n}) \to 1,$$

so (i) holds.

Similarly,

$$F^{2n}(c_n) = 2 \sum_{j=n+1}^{2n} (2n)^{-1}F^{2n}(c_n) \le 2 \sum_{j=n+1}^{2n} j^{-p}F^j(c_j),$$

so the hypothesis of (ii) guarantees $F^{2n}(c_n) \to 0$ and hence, $F^n(c_n) \to 0$. □

Necessary and sufficient conditions for (3.1) will follow from the next theorem, in conjunction with results from Section 2.

THEOREM 3.2. *Let $X_1, X_2, \dots$ be i.i.d. r.v.'s with common d.f. F with $F(x) < 1$ for all x. Define $M_n = \max(X_1, \dots, X_n)$ and $\mu_n = F^{-1}(1 - n^{-1})$. Let $r \ge 1$ and $\varepsilon > 0$. Then the following assertions are equivalent:*

(i) $$\sum_{n=1}^{\infty} n^{r-2}P\left[\left|\frac{M_n}{\mu_n} - 1\right| > \varepsilon\right] < \infty;$$

(ii) $$\sum_{n=1}^{\infty} n^{r-2}P\left[M_n > (1 + \varepsilon)\mu_n\right] < \infty;$$

(iii) $$\sum_{n=1}^{\infty} n^{r-1}(1 - F((1 + \varepsilon)\mu_n)) < \infty;$$

(iv) $$\int_{-\infty}^{\infty} [1 - F(y/(1 + \varepsilon))]^{-r}\, dF(y) < \infty.$$

PROOF. Clearly (i) implies (ii). Now, by definition of μ_n, $F^n((1 + \varepsilon)\mu_n) \ge (1 - n^{-1})^n \to e^{-1}$. If $e\alpha < 1$, then there exists $N > 0$ such that $F^n((1 + \varepsilon)\mu_n) \ge \alpha$ for all $n \ge N$. It is well known [e.g., Feller (1968), page 66] that

$$(3.2) \qquad 1 - t \le -\log t \le (1 - t)/t \quad \text{for } 0 < t \le 1.$$

Letting $t = F^n((1 + \varepsilon)\mu_n)$, it follows that

$$1 - F^n((1 + \varepsilon)\mu_n) \le -n \log F((1 + \varepsilon)\mu_n) \le \alpha^{-1}(1 - F^n((1 + \varepsilon)\mu_n)$$

for all $n \ge N$. But $(y - 1)/\log y \to 1$, so $-n \log F((1 + \varepsilon)\mu_n) \sim n[1 - F((1 + \varepsilon)\mu_n)]$; thus the equivalence of (ii) and (iii) follows from the limit comparison test for series.

Now define $\delta = (1+\varepsilon)^{-1}$, $x_0 = F^{-1}(0)$ and $\mu_x = F^{-1}(1 - x^{-1})$. Then, for $r \geq 1$,

$$\begin{aligned}(n+1)^{r-1}\left[1 - F((1+\varepsilon)\mu_{n+1})\right] &\leq 2^{r-1}n^{r-1}\left[1 - F((1+\varepsilon)\mu_{n+1})\right]\\ &\leq 2^{r-1}\int_n^{n+1} x^{r-1}\left[1 - F((1+\varepsilon)\mu_x)\right] dx\\ &\leq 2^{r-1}(n+1)^{r-1}\left[1 - F((1+\varepsilon)\mu_n)\right]\\ &\leq 4^{r-1}n^{r-1}\left[1 - F((1+\varepsilon)\mu_n)\right].\end{aligned}$$

Summation over n yields the equivalence of (iii) and (iv), since

$$\begin{aligned}r\int_1^\infty x^{r-1}\left[1 - F((1+\varepsilon)\mu_x)\right] dx &= r\int_{x=1}^\infty x^{r-1}\int_{y=(1+\varepsilon)\mu_x}^\infty dF(y)\, dx\\ &= r\int_{y=x_0}^\infty \int_{x=1}^{[1-F(\delta y)]^{-1}} x^{r-1}\, dx\, dF(y)\\ &= \int_{y=x_0}^\infty \left[1 - F(\delta y)\right]^{-r} dF(y) - 1.\end{aligned}$$

Finally, it will be shown that (iv) implies (i). In view of the established equivalence of (ii) and (iv), it will suffice to show that (iv) implies

$$\sum_{n=1}^\infty n^{r-2}P\left[M_n \leq (1-\varepsilon)\mu_n\right] = \sum_{n=1}^\infty n^{r-2}F^n((1-\varepsilon)\mu_n) < \infty.$$

But $1 - \varepsilon = (1-\varepsilon^2)/(1+\varepsilon) < \delta$, so it will be enough to show that (iv) implies

$$\sum_{n=1}^\infty n^{r-2}F^n(\delta\mu_x) < \infty. \tag{3.3}$$

From the definition of μ_x, $x[1 - F(\delta\mu_x)] > 1$. Moreover, let $A > 0$ be a constant such that $y^r e^{-y} \leq A$ for all $y \geq 1$. Then a modification of a technique of Resnick and Tomkins (1973) yields

$$\begin{aligned}n^{r-2}F^n(\delta\mu_n) &\sim \frac{n^{r-1}}{n+1}F^{n+1}(\delta\mu_n)\\ &\leq \int_n^{n+1}\frac{x^{r-1}}{x}F^x(\delta\mu_x)\, dx\\ &= \int_n^{n+1} x^{r-2}\exp\{x\log F(\delta\mu_x)\}\, dx\\ &\leq \int_n^{n+1} x^{r-2}\exp\{-x[1 - F(\delta\mu_x)]\}\, dx \quad \text{by (3.2)}\\ &\leq A\int_n^{n+1} x^{-2}\left[1 - F(\delta\mu_x)\right]^{-r} dx,\end{aligned}$$

so (3.3) holds if $I \equiv \int_1^\infty x^{-2}[1 - F(\delta\mu_x)]^{-r}\, dx < \infty$. But the change of variable $y = \mu_x$ yields $I = \int_{F^{-1}(0)}^\infty [1 - F(\delta y)]^{-r}\, dF(y)$, so (iv) implies (3.3). □

REMARK. Resnick and Tomkins (1973) showed that $\{M_n\}$ is a.s. stable iff $\limsup_{n\to\infty} M_n/\mu_n = 1$ a.s. The equivalence of (i) and (ii) in Theorem 3.2 is analogous.

COROLLARY 3.3. *Let $X_1, X_2, \ldots$ be i.i.d. r.v. with common d.f. F with $F(x) < 1$ for all x. Define $M_n = \max(X_1, \ldots, X_n)$ and $\mu_n = F^{-1}(1 - n^{-1})$. Let $r \geq 1$. Then a real sequence $\{a_n\}$ exists such that $a_n \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} n^{r-2} P\left[\left|\frac{M_n}{a_n} - 1\right| > \varepsilon\right] < \infty \quad \text{for every } \varepsilon > 0 \tag{3.4}$$

iff (3.1) *holds.*

PROOF. If (3.4) holds, then $F^n((1+\varepsilon)a_n) \to 1$ and $F^n((1-\varepsilon)a_n) \to 0$ for every ε, $0 < \varepsilon < 1$, by Lemma 3.1. It follows readily that $M_n/a_n \to 1$ in probability. But then $a_n \sim \mu_n$ by Lemma 2 of Resnick (1972), and (3.1) follows easily. Since (3.1) obviously implies (3.4), the corollary is proved. □

In the special case $r = 1$, Theorem 3.2 leads to a new criterion for a.s. stability.

COROLLARY 3.4 [Tomkins (1984)]. *Define M_n and μ_n as in Theorem 3.2. Then $\{M_n\}$ is a.s. stable iff*

$$\sum_{n=1}^{\infty} n^{-1} P\left[\left|\frac{M_n}{a_n} - 1\right| > \varepsilon\right] < \infty \quad \text{for every } \varepsilon > 0 \tag{3.5}$$

and some real sequence $\{a_n\}$. Moreover, when (3.5) *holds, it may be assumed without loss of generality that $a_n = \mu_n$.*

PROOF. By Theorem 3.2 and Corollary 3.3, (3.5) is equivalent to the convergence of the integral (1.2) for every $\delta < 1$, which is, in turn, tantamount to a.s. stability [cf. Barndorff-Nielsen (1963) and Resnick and Tomkins (1973)]. That one may take $a_n = \mu_n$ in (3.5) is clear from Corollary 3.3. □

Turning now to the case where $r > 1$, the next theorem shows that the complete convergence of M_n/a_n is directly linked to the generalization of regular variation discussed in Section 2.

THEOREM 3.5. *Let $X_1, X_2 \ldots$ be i.i.d. r.v.'s with common d.f. F, where $F(x) < 1$ for all x. Define $M_n = \max(X_1, \ldots, X_n)$ and*

$$L(x) = -\log(1 - F(x)) \tag{3.6}$$

for all real x. Let $r > 1$. Then

$$\sum_{n=1}^{\infty} n^{r-2} P\left[\left|\frac{M_n}{a_n} - 1\right| > \varepsilon\right] < \infty \quad \text{for every } \varepsilon > 0$$

for some sequence $\{a_n\}$, $a_n \uparrow \infty$, *if and only if the function L is rapidly varying, i.e.,*

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = +\infty \quad \textit{for every } t > 1.$$

Moreover, it may be assumed without loss of generality that $a_n = \mu_n$.

PROOF. Theorem 3.2 and Corollary 3.3 show that (3.4) holds iff the integral

$$\int_{-\infty}^{\infty} [1 - F(\delta y)]^{-r}\, dF(y) \tag{3.7}$$

converges for every $\delta < 1$. But $1 - F(y) = \exp(-L(y))$ and $dF(y) = \exp(-L(y))\, dL(y)$, from which it is easy to see that (3.7) holds iff $I(r, \delta) < \infty$, where $I(r, \delta)$ is defined by (2.4). Thus (3.4) holds iff $I(r, \delta) < \infty$ for every $\delta < 1$ which, in turn, happens iff L is rapidly varying, by Theorem 2.3. Finally, Corollary 3.3 indicates that one may assume $a_n = \mu_n$. □

Theorems 2.3, 3.2, and 3.5 combine to produce the following important and, perhaps, surprising result.

THEOREM 3.6. *Define* M_n *and* μ_n *as in Theorem* 3.5. *Let* $\{a_n\}$ *be a real sequence such that* $a_n \uparrow \infty$. *If* (3.4) *holds for some* $r > 1$, *then* (3.4) *holds for every* $r > 1$.

REMARK. Theorems 3.5 and 3.6 answer the question raised in Section 1 about the complete convergence [in the sense of Hsu and Robbins (1947)] of M_n/a_n to 1: $\sum_{n=1}^{\infty} P[|(M_n/a_n) - 1| > \varepsilon] < \infty$ for every $\varepsilon > 0$ and some sequence $a_n \to \infty$ iff (3.4) holds for all $r > 1$ iff $-\log(1 - F(x))$ is rapidly varying.

The final result completes the connection between the convergence or divergence of the series in (3.4) and the generalization of regular variation in Section 2.

THEOREM 3.7. *Let* $X_1, X_2, \ldots$ *be i.i.d. with d.f. F such that* $F(x) < 1$ *for all* x. *For* $n \geq 1$, *let* $M_n = \max(X_1, \ldots, X_n)$ *and* $\mu_n = F^{-1}(1 - n^{-1})$; *define L by* (3.6) *and l by* (2.2).

(i) *For every* $0 < \varepsilon < 1$, *a number* $r_1 > 1$ *exists such that the series*

$$\sum_{n=1}^{\infty} n^{r-2} P\left[\left|\frac{M_n}{\mu_n} - 1\right| > \varepsilon\right] \tag{3.8}$$

converges when $r = r_1$, *and for every* $r > 1$, *there exists* $\varepsilon > 0$ *such that* (3.8) *diverges iff* $l(t) > 1$ *for every* $t > 1$ *and* $\inf_{t>1} l(t) = 1$.

(ii) *There exists* $\varepsilon > 0$ *such that the series* (3.8) *diverges for all* $r > 1$ *iff* $l(t_0) = 1$ *for some* $t_0 > 1$.

(iii) *The series* (3.8) *diverges for all* $r > 1$ *and all* $\varepsilon > 0$ *iff* $l(t) = 1$ *for all* $t > 1$.

PROOF. Define $I(r, \delta)$ by (2.2). By Theorem 3.2, the convergence of the series (3.8) is equivalent to the convergence of $I(r, (1 + \varepsilon)^{-1})$. Then parts (i), (ii), and (iii) follow from Theorems 2.8, 2.7, and 2.9, respectively. □

REMARKS.

1. Mindful of the last four theorems of Section 2, it is clear from Theorems 3.5 and 3.7 that the convergence properties of the series (3.8) can be derived from examining $\liminf_{x\to\infty} L(tx)/L(x)$, or the integrals $I(r, \delta) = \int_1^\infty [1 - F(\delta y)]^{-r}\, dF(y)$ or $\liminf_{x\to\infty}(L(x) - rL(\delta x))$, where L is defined by (3.6). Notice, in particular, that $\lim_{x\to\infty}(L(x) - rL(\delta x)) = \infty$ iff $\lim_{x\to\infty}(1 - F(x)/[1 - F(\delta x)]^r) = 0$; the special case $r = 1$ is condition (1.1), tantamount to relative stability of M_n [cf. Gnedenko (1943)].

2. In view of Remark 1 at the end of Section 2, if $\{M_n\}$ is not a.s. stable then $l(t_0) = 1$ for some $t_0 > 1$. Furthermore, if $L(x) = -\log(1 - F(x))$ is slowly varying, then M_n may be a.s. stable [e.g., $L(x) = (\log x)^2$], only relatively stable [e.g., $L(x) = c \log x \log\log x$, $c > 0$] or not stable in either sense [e.g., $L(x) = \log x$].

4. The bounded case. Again, let $X_1, X_2, \dots$ be i.i.d. r.v.'s with d.f. F, but this time suppose $F(x) = 1$ for some x. Define $M_n = \max(X_1, \dots, X_n)$ and $x_1 = F^{-1}(1)$. Then $M_n \le x_1$ a.s. Moreover, it is easily shown that $M_n \to x_1$ completely and, consequently, a.s. and in probability. One way of studying the rate of this convergence is to ask: Under what circumstances does a real sequence $\{a_n\}$ exist such that $a_n(x_1 - M_n) \to 1$ a.s.? If one sets $X_n^* = (x_1 - X_n)^{-1}$ and $M_n^* = \max(X_1^*, \dots, X_n^*)$, then $a_n(x_1 - M_n) = a_n/M_n^*$, so the question becomes equivalent to seeking criteria for $M_n^*/a_n \to 1$ a.s. In view of Barndorff-Nielsen (1963) and Resnick and Tomkins (1973), it follows that $a_n(x_1 - M_n) \to 1$ a.s. iff $\int_1^\infty [1 - F^*(\delta y)]^{-1}\, dF^*(y) < \infty$ for all $\delta < 1$, where F^* is the d.f. of X_1^*, i.e., $F^*(x) = F(x_1 - x^{-1})$, $0 < x < \infty$.

Similarly, criteria for $a_n(x_1 - M_n) \to 1$ in probability can be derived from Theorem 2 of Gnedenko (1943), and criteria for $\sum_{n=1}^\infty n^{r-2} P[|a_n(x_1 - M_n) - 1| > \varepsilon]$ to converge may be derived from the results of Section 3.

Acknowledgments. The author is grateful to Professors C. Goldie and E. Seneta for their helpful comments on parts of this work, and for drawing certain references to his attention; and to the referee and the editor for having pointed out some errors in the original manuscript.

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