The Annals of Probability
1986, Vol. 14, No. 3, 974–983

THE ASYMPTOTIC DISTRIBUTION OF SUMS OF EXTREME VALUES FROM A REGULARLY VARYING DISTRIBUTION

BY SÁNDOR CSÖRGŐ[1] AND DAVID M. MASON[2]

University of California, San Diego and University of Munich

Let $X_{1,n} \leqq \cdots \leqq X_{n,n}$ be the order statistics of n independent and identically distributed positive random variables with common distribution function F satisfying $1 - F(x) = x^{-\alpha}L^*(x)$, $x > 0$, where α is any positive number and L^* is any function slowly varying at infinity. We give a complete description of the asymptotic distribution of the sum of the top k_n extreme values $X_{n+1-k_n,n}, X_{n+2-k_n,n}, \dots, X_{n,n}$ for any sequence of positive integers k_n such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$.

1. The result. Let $X_1, \dots, X_n$ be independent and identically distributed positive random variables (rv's) with common distribution function $F(x) = \Pr\{X \leqq x\}$, $-\infty < x < \infty$, and quantile function

$$Q(u) = \inf\{x: F(x) \geqq u\}, \qquad 0 < u \leqq 1, \qquad Q(0) = Q(0+).$$

We assume that the upper tail of F is regularly varying with exponent $-1/a$, i.e.,

$$1 - F(x) = x^{-1/a}L^*(x), \qquad x > 0, \tag{1.1}$$

where a is an arbitrary positive number and L^* is a function slowly varying at infinity. This is equivalent to the condition that

$$Q(1-s) = s^{-a}L(s), \qquad 0 < s < 1, \tag{1.2}$$

where L is some function slowly varying at zero.

Let $X_{1,n} \leqq \cdots \leqq X_{n,n}$ be the order statistics of $X_1, \dots, X_n$. In this paper, we are interested in the asymptotic distribution of sums of extreme values

$$T_n(k) = \sum_{j=1}^{k} X_{n+1-j,n}.$$

It is easy to show that if $k \geqq 1$ is any fixed integer then

$$\frac{1}{n^a L(1/n)} \sum_{j=1}^{k} X_{n+1-j,n} \to_{\mathscr{D}} \sum_{j=1}^{k} (S_j)^{-a},$$

where $\to_{\mathscr{D}}$ denotes convergence in distribution as $n \to \infty$, and $S_j = E_1 +$

Received August 1984; revised January 1985.

[1] On leave from Szeged University; supported by a NSERC Canada grant to D. A. Dawson.

[2] On leave from University of Delaware; supported by a NSERC Canada grant to M. Csörgő and by the Alexander von Humboldt foundation.

AMS 1980 *subject classifications*. Primary 60F05; secondary 62G30.

Key words and phrases. Regular variation, sums of extreme values, asymptotic distribution.

$\cdots + E_j$, $j = 1, \ldots, k$, where $E_1, \ldots, E_k$ are independent exponentially distributed rv's with mean one.

Many statistical procedures concerning the tail behaviour of the underlying distribution are based on an increasing number k_n of upper order statistics such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. In fact, sums of the form $T_n(k_n)$ play an important role in these procedures. The prime examples are the estimation of the tail index of a distribution [Hall (1982a), Mason (1982), and S. Csörgő, Deheuvels, and Mason (1985)] and the estimation of an endpoint of a distribution [Hall (1982b) and S. Csörgő and Mason (1984)]. Hence it is only natural to consider the probabilistic problem concerning when such partial sums of an increasing number of extreme values can be normalized and centralized such that they converge in distribution to a nondegenerate rv. In S. Csörgő and Mason (1985) we proved that if (1.1) holds, then for any $a > 0$ and for any sequence k_n of positive integers such that $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, we have

$$\frac{1}{k_n^{1/2}}\left\{\sum_{j=1}^{k_n} \log X_{n+1-j,n} - n\int_{1-k_n/n}^{1} \log Q(s)\,ds\right\} \to_{\mathscr{D}} N(0, 2a^2),$$

where $N(0, v)$ stands for a normal rv with mean 0 and variance $v > 0$. Assuming the condition of regular variation in (1.1), the aim of this paper is to give a complete solution to the problem posed above for the sums $T_n(k_n)$ of the extreme values themselves rather than for those of their logarithms.

The case $a > \frac{1}{2}$, when F is in the domain of attraction of a nonnormal stable law, has been treated in S. Csörgő, Horváth, and Mason (1986). For the sake of completeness we include this result in the theorem below.

For any $a > \frac{1}{2}$ let $\Delta_{1/a}$ denote a completely asymmetric stable rv with characteristic function

$$\phi_{1/a,\beta,\gamma,\theta}(t) = \exp\{i\theta t - \gamma|t|^{1/a}[1 - i\beta\operatorname{sgn}(t)\omega(t, 1/a)]\},$$

where

$$\omega(t, 1/a) = \begin{cases} \tan\dfrac{\pi}{2a} & \text{if } a \neq 1, \\ -\dfrac{2}{\pi}\log|t| & \text{if } a = 1, \end{cases}$$

given by the skewness parameter $\beta = 1$, scale parameter

$$\gamma = \gamma(1/a) = \begin{cases} \Gamma\left(1 - \dfrac{1}{a}\right)\cos\dfrac{\pi}{2a} & \text{if } a > 1, \\ \dfrac{\pi}{2} & \text{if } a = 1, \\ \dfrac{1}{a}\left(\dfrac{1}{a} - 1\right)^{-1}\Gamma\left(2 - \dfrac{1}{a}\right)\left|\cos\dfrac{\pi}{2a}\right| & \text{if } \frac{1}{2} < a < 1, \end{cases}$$

and location parameter

$$\theta = \theta(1/a) = \begin{cases} 0 & \text{if } a \neq 1, \\ \displaystyle\int_0^\infty \left(\frac{\sin x}{x^2} - \frac{1}{x(1+x)} \right) dx & \text{if } a = 1. \end{cases}$$

The complete description of the asymptotic distribution of sums of extreme values under regular variation is given by the following results.

THEOREM. *Let k_n be any sequence of positive integers such that $n + 1 - k_n \leqq n$, $k_n \to \infty$, and $k_n/n \to 0$.*
(i) *If $a > \frac{1}{2}$, then*

$$\frac{1}{n^a L(1/n)} \left\{ \sum_{j=1}^{k_n} X_{n+1-j,n} - C_n(k_n) \right\} \to_{\mathscr{D}} \Delta_{1/a},$$

where

$$C_n(k_n) = \begin{cases} 0 & \textit{if } a > 1, \\ n \displaystyle\int_{1-k_n/n}^{1-1/n} Q(s)\, ds & \textit{if } a = 1, \\ n \displaystyle\int_{1-k_n/n}^{1} Q(s)\, ds & \textit{if } \frac{1}{2} < a < 1. \end{cases}$$

(ii) *If $0 < a \leqq \frac{1}{2}$, then*

$$\frac{1}{A_n(a, k_n)} \left\{ \sum_{j=1}^{k_n} X_{n+1-j,n} - n \int_{1-k_n/n}^{1} Q(s)\, ds \right\} \to_{\mathscr{D}} N(0,1),$$

where

$$A_n(a, k_n) = \begin{cases} n^{1/2} \left(\dfrac{k_n}{n} \right)^{1/2-a} L\left(\dfrac{k_n}{n} \right) \left(\dfrac{2a^2}{(1-2a)(1-a)} \right)^{1/2} & \textit{if } 0 < a < \frac{1}{2}, \\ n^{1/2} \left(\displaystyle\int_{1/n}^{k_n/n} s^{-1} L^2(s)\, ds \right)^{1/2} & \textit{if } a = \frac{1}{2}. \end{cases}$$

The case (i) is contained as a special case in Theorem 3 of S. Csörgő, Horváth, and Mason (1986), and for a more detailed discussion of this case the reader is referred to this paper. Hence the content of the present paper is the proof of case (ii) of the theorem. Just as in this former paper [and also in S. Csörgő and Mason (1984, 1985) and in S. Csörgő, Deheuvels, and Mason (1985)], the proof is based on a new Brownian bridge approximation to the uniform empirical process in weighted supremum norms. This is described at the beginning of the proof in Section 3, while Section 2 contains some technical lemmas concerning regularly varying functions.

REMARK. Very slight changes in the proof of the theorem show that the asymptotic normality statement given in (ii) remains true when $Q(1-s) = A - L(s)s^{-a}$ for all $s > 0$ sufficiently close to zero, where $-\infty < A < \infty$ and $-\infty < a < 0$. $A_n(a, k_n)$ and $C_n(k_n)$ in this case are defined in the same way as above for the case $0 < a < \frac{1}{2}$, but now inserting $-\infty < a < 0$ into $A_n(a, k_n)$.

2. Technical lemmas. In the following first four lemmas, $L(s)$, $0 < s < 1$, denotes any function slowly varying at zero. The proofs of the first two lemmas are found in de Haan (1970), while Lemma 3 is proven in S. Csörgő, Horváth, and Mason (1986) (cf. their Lemma 2).

LEMMA 1. *If* $\beta > -1$, *then*

$$\lim_{s \downarrow 0} \int_0^s u^\beta L(u)\, du / (s^{\beta+1} L(s)) = \frac{1}{1+\beta},$$

while if $\beta < -1$ *then there exists a* $0 < \delta < 1$ *such that*

$$\lim_{s \downarrow 0} \int_s^\delta u^\beta L(u)\, du / (s^{\beta+1} L(s)) = \frac{1}{|1+\beta|}.$$

LEMMA 2. *For all* $0 < \tau \leqq \gamma < \infty$,

$$\lim_{s \downarrow 0} \sup_{\tau s \leqq t \leqq \gamma s} L(t)/L(s) = 1$$

and

$$\lim_{s \downarrow 0} \inf_{\tau s \leqq t \leqq \gamma s} L(t)/L(s) = 1.$$

LEMMA 3. *Let* k_n *be any sequence of positive integers such that* $k_n \to \infty$ *and* $k_n/n \to 0$. *Then for any* $0 < \beta < \infty$, *we have*

$$\lim_{n \to \infty} \left\{ (k_n/n)^\beta L(k_n/n) \right\} \Big/ \left\{ (1/n)^\beta L(1/n) \right\} = \infty.$$

LEMMA 4. *Let* k_n *be any sequence of positive integers such that* $k_n \to \infty$ *and* $k_n/n \to 0$. *Then*

$$\lim_{n \to \infty} L(1/n) \Big/ \int_{1/n}^{k_n/n} s^{-1} L(s)\, ds = 0 \tag{2.1}$$

and

$$\lim_{n \to \infty} L(k_n/n) \Big/ \int_{1/n}^{k_n/n} s^{-1} L(s)\, ds = 0. \tag{2.2}$$

PROOF. Choose any positive integer m. Notice that for all n large enough,

$$L(1/n) \Big/ \int_{1/n}^{k_n/n} s^{-1} L(s)\, ds \leqq L(1/n) \Big/ \int_{1/n}^{m/n} s^{-1} L(s)\, ds$$
$$\leqq \sup_{1/n \leqq s \leqq m/n} \{ L(1/n)/L(s) \} / \log m.$$

Applying Lemma 2 we see that

$$\limsup_{n\to\infty} L(1/n)\Big/\int_{1/n}^{k_n/n} s^{-1}L(s)\,ds \leqq 1/\log m,$$

which, since m can be made arbitrarily large, implies (2.1).

Next, observe that for all n sufficiently large,

$$L(k_n/n)\Big/\int_{1/n}^{k_n/n} s^{-1}L(s)\,ds \leqq L(k_n/n)\Big/\int_{k_n/mn}^{k_n/n} s^{-1}L(s)\,ds.$$

Thus in exactly the same way as above we obtain (2.2). □

In the following two lemmas Q denotes a nonnegative quantile function. Here and in what follows we use the convention

$$\int_a^b f(u)\,dQ(u) = \int_{[a,\,b)} f(u)\,dQ(u)$$

for any right-continuous f and for any $0 < a < b < 1$. Since Q is left continuous, the usual formula for integration by parts is valid whenever f is right continuous. If $a = 0$ and/or $b = 1$, then the corresponding integrals are interpreted as improper Riemann–Stieltjes integrals.

LEMMA 5. *Assume that for some* $0 < a < \frac{1}{2}$, $Q(1-s) = s^{-a}L(s)$ *for* $0 < s < 1$, *and let*

$$\sigma^2(s) = \int_{1-s}^{1}\int_{1-s}^{1} (u\wedge v - uv)\,dQ(u)\,dQ(v).$$

Then

$$\lim_{s\downarrow 0} \sigma^2(s)/(s^{1-2a}L^2(s)) = 2a^2/((1-a)(1-2a)).$$

PROOF. It is easy to show that

$$\sigma^2(s) = \int_0^s u^{-2a}L^2(u)\,du + (1-s)s^{1-2a}L^2(s)$$
$$-\left(\int_0^s u^{-a}L(u)\,du\right)^2 - 2(1-s)s^{-a}L(s)\int_0^s u^{-a}L(u)\,du.$$

The lemma now follows from Lemma 1. □

LEMMA 6. *Assume that for some* $0 < a \leqq \frac{1}{2}$, $Q(1-s) = s^{-a}L(s)$ *for* $0 < s < 1$, *and let*

$$\sigma_n^2(k_n) = \int_{1-k_n/n}^{1-1/n}\int_{1-k_n/n}^{1-1/n} (u\wedge v - uv)\,dQ(u)\,dQ(v),$$

where k_n *is a sequence of positive integers such that* $k_n \to \infty$ *and* $k_n/n \to 0$. *Then whenever* $0 < a < \frac{1}{2}$,

$$\lim_{n\to\infty} \sigma_n^2(k_n)/\big((k_n/n)^{1-2a}L^2(k_n/n)\big) = 2a^2/((1-a)(1-2a)), \tag{2.3}$$

and when $a = \frac{1}{2}$

$$\lim_{n\to\infty} \sigma_n^2(k_n) \Big/ \int_{1/n}^{k_n/n} s^{-1}L^2(s)\,ds = 1. \tag{2.4}$$

PROOF. First assume that $0 < a < \frac{1}{2}$. Notice that

$$\big(\sigma(k_n/n) - \sigma(1/n)\big)^2 \leqq \sigma_n^2(k_n) \leqq \sigma^2(k_n/n).$$

Applying Lemmas 3 and 5, we obtain

$$\lim_{n\to\infty} \sigma(1/n)/\big((k_n/n)^{1/2-a}L(k_n/n)\big) = 0.$$

Therefore, by the above inequality and Lemma 5 we have (2.3).

Now we assume that $a = \frac{1}{2}$. We can write

$$\begin{aligned}\sigma_n^2(k_n) &= \int_{1/n}^{k_n/n} s^{-1}L^2(s)\,ds + \left(1 - \frac{k_n}{n}\right)L^2(k_n/n) + L^2(1/n)\\ &\quad -\left[n^{-1/2}L(1/n) + \int_{1/n}^{k_n/n} s^{-1/2}L(s)\,ds\right]^2\\ &\quad -2\left(1 - \frac{k_n}{n}\right)(k_n/n)^{-1/2}L(k_n/n)\left[n^{-1/2}L(1/n) + \int_{1/n}^{k_n/n} s^{-1/2}L(s)\,ds\right]\\ &=: \int_{1/n}^{k_n/n} s^{-1}L^2(s)\,ds + R_n^{(1)} + R_n^{(2)} - R_n^{(3)} - R_n^{(4)}.\end{aligned}$$

Hence to complete the proof of (2.4) it is enough to show that

$$\lim_{n\to\infty} R_n^{(j)} \Big/ \int_{1/n}^{k_n/n} s^{-1}L^2(s)\,ds = 0 \quad \text{for } j = 1,\dots,4. \tag{2.5}$$

Since L^2 is slowly varying at zero, we see by Lemma 4 that (2.5) holds for $j = 1, 2$. Also, applying first the c_r-inequality and Schwarz's inequality, we see by Lemma 4 again that (2.5) holds for $j = 3$. Similarly, Lemma 4 implies (2.5) for $j = 4$ after applying the Schwarz inequality and the fact that $Q(1 - s) = s^{-1/2}L(s)$ is a nonincreasing function of s. □

3. Proof of the theorem (ii). M. Csörgő, S. Csörgő, Horváth, and Mason (1986) have constructed a probability space $(\Omega, \mathscr{A}, P)$ carrying an infinite sequence $U_1, U_2, \dots$ of independent rv's uniformly distributed on $(0,1)$ and a sequence of Brownian bridges $B_n(u)$, $0 \leqq u \leqq 1$, $n = 1, 2, \dots$, such that for the empirical process

$$\alpha_n(u) = n^{1/2}\{G_n(u) - u\}, \qquad 0 \leqq u \leqq 1,$$

where

$$G_n(u) = n^{-1}\#\{k\colon 1 \leqq k \leqq n, U_k \leqq u\}$$

we have

$$\sup_{1/n \leq u \leq 1-1/n} n^{\nu} \frac{|\alpha_n(u) - B_n(u)|}{(u(1-u))^{1/2-\nu}} = O_p(1) \tag{3.1}$$

as $n \to \infty$, where ν is any fixed number such that $0 \leq \nu < \frac{1}{4}$. This is Corollary 2.2 in the above paper.

If for any n, $U_{1,n} \leq \cdots \leq U_{n,n}$ denote the order statistics corresponding to $U_1, \ldots, U_n$, then

$$\sum_{j=1}^{k_n} X_{n+1-j,n} =_{\mathscr{D}} \sum_{j=1}^{k_n} Q(U_{n+1-j,n})$$

for each n, where $=_{\mathscr{D}}$ denotes equality in distribution. Therefore, from now on we work without loss of generality with this representation of our sums of extreme values, on the above space $(\Omega, \mathscr{A}, P)$.

Applying integration by parts we see that

$$\begin{aligned}
&\frac{1}{A_n(a,k_n)}\left\{\sum_{j=1}^{k_n} Q(U_{n+1-j,n}) - n\int_{1-k_n/n}^{1} Q(s)\,ds\right\} \\
&= -\frac{1}{D_n(a,k_n)}\int_{1-k_n/n}^{1-1/n} \alpha_n(s)\,dQ(s) - \frac{1}{D_n(a,k_n)}\int_{1-1/n}^{1} \alpha_n(s)\,dQ(s) \\
&\quad + \frac{n}{A_n(a,k_n)}\int_{U_{n-k_n,n}}^{1-k_n/n}\left(1 - G_n(s) - \frac{k_n}{n}\right) dQ(s) \\
&=: \Delta_n^{(1)} + \Delta_n^{(2)} + \Delta_n^{(3)},
\end{aligned}$$

where $D_n(a, k_n) = A_n(a, k_n)/n^{1/2}$. We will show that for any $0 < a \leq \frac{1}{2}$,

$$\begin{aligned}
\Delta_n^{(1)} &= -\frac{1}{D_n(a,k_n)}\int_{1-k_n/n}^{1-1/n} B_n(s)\,dQ(s) + o_p(1) \\
&=: Z_n + o_p(1)
\end{aligned} \tag{3.2}$$

and

$$\Delta_n^{(j)} \to_P 0, \qquad j = 2,3. \tag{3.3}$$

Then we will verify that for $0 < a \leq \frac{1}{2}$,

$$\operatorname{Var} Z_n \to 1 \quad \text{as } n \to \infty. \tag{3.4}$$

Since for each $n = 1, 2, \ldots$, Z_n is a normal rv with mean zero, (3.2), (3.3), and (3.4) will imply case (ii) of the theorem.

First consider (3.2). Choose any $0 < \nu < \frac{1}{4}$. Applying (3.1) we have

$$\frac{1}{D_n(a,k_n)}|\Delta_n^{(1)} - Z_n| = O_p(n^{-\nu})\frac{1}{D_n(a,k_n)}\int_{1-k_n/n}^{1-1/n}(1-s)^{1/2-\nu}\,dQ(s). \tag{3.5}$$

Integrating by parts, we see that

$$\int_{1-k_n/n}^{1-1/n}(1-s)^{1/2-\nu}\,dQ(s)$$

$$(3.6)\qquad = \left\{(1/n)^{1/2-\nu-a}L(1/n)-(k_n/n)^{1/2-\nu-a}L(k_n/n)\right\} + \left(\tfrac{1}{2}-\nu\right)\int_{1/n}^{k_n/n} s^{-1/2-\nu-a}L(s)\,ds.$$

When $0 < a < \frac{1}{2}$ we choose $\nu > 0$ such that $\frac{1}{2} + \nu + a < 1$. Then we see by Lemma 3 that the difference is eventually negative and an application of the first statement of Lemma 1 to the integral shows that (3.2) follows from (3.5) and (3.6) in this case. Applying the second statement of Lemma 1 and Lemma 4 and using also that Q and hence L is nonnegative, it can be seen that (3.2) follows from (3.5) and (3.6) also in the case when $a = \frac{1}{2}$.

We turn to (3.3) in the case $j = 2$. Notice that

$$|\Delta_n^{(2)}| \leqq \frac{n^{1/2}}{D_n(a,k_n)}\int_{1-1/n}^{1}(1-s)\,dQ(s) + \frac{n^{1/2}}{D_n(a,k_n)}\int_{1-1/n}^{1}(1-G_n(s))\,dQ(s)$$

$$=: \Delta_{n,1}^{(2)} + \Delta_{n,2}^{(2)}.$$

By integrating by parts and using (1.2),

$$\Delta_{n,1}^{(2)} = \frac{-(1/n)^{1/2-a}L(1/n)}{D_n(a,k_n)} + \frac{n^{1/2}}{D_n(a,k_n)}\int_0^{1/n} s^{-a}L(s)\,ds,$$

which by Lemma 1 is for all n sufficiently large

$$\leqq \frac{1}{1-a}\,\frac{L(1/n)(1/n)^{1/2-a}}{D_n(a,k_n)}.$$

When $0 < a < \frac{1}{2}$, the latter bound goes to zero by Lemma 3, and when $a = \frac{1}{2}$, it goes to zero by Lemma 4. Therefore, for all $0 < a \leqq \frac{1}{2}$, $\Delta_{n,1}^{(2)} = o_p(1)$. Since $E\Delta_{n,2}^{(2)} = \Delta_{n,1}^{(2)}$, we see that $\Delta_{n,2}^{(2)} = o_p(1)$. Hence we have (3.3) for the case $j = 2$.

To prove (3.3) for the case $j = 3$, we need the following fact: Let k_n be any sequence of positive integers such that $k_n \to \infty$ and $k_n/n \to 0$. Then

$$(3.7)\qquad \frac{n}{k_n^{1/2}}\left(U_{n-k_n,n} - \left(1-\frac{k_n}{n}\right)\right) \to_{\mathscr{D}} N(0,1).$$

[See Balkema and de Haan (1975).]

Choose any $0 < d < \infty$, and set

$$T_n(d) = \frac{n}{A(a,k_n)}\left|1 - G_n\left(1-\frac{k_n}{n}\right) - \frac{k_n}{n}\right|\left\{Q(r_n^+(d)) - Q(r_n^-(d))\right\},$$

where

$$r_n^-(d) = 1 - \frac{k_n}{n} - d\frac{k_n^{1/2}}{n} \quad\text{and}\quad r_n^+(d) = 1 - \frac{k_n}{n} + d\frac{k_n^{1/2}}{n}.$$

Notice that since for all s in the closed interval formed by $U_{n-k_n,n}$ and $1-(k_n/n)$, $|1-G_n(s)-(k_n/n)| \leqq |1-G_n(1-(k_n/n))-(k_n/n)|$, we have

$$\lim_{d\to\infty}\liminf_{n\to\infty} P\{|\Delta_n^{(3)}| \leqq T_n(d)\} \geqq \lim_{d\to\infty}\liminf_{n\to\infty} P\left\{\frac{n}{k_n^{1/2}}\left|U_{n-k_n,n}-\left(1-\frac{k_n}{n}\right)\right| \leqq d\right\},$$

where the lower bound equals one on account of (3.7). Hence to prove (3.3) for $j=3$, it is sufficient to show that for each $0<d<\infty$,

$$T_n(d) \to_P 0. \tag{3.8}$$

Choose any $1<\lambda<\infty$. We see that for all n sufficiently large

$$\begin{aligned} ET_n(d) &\leq \frac{(k_n/n)^{1/2}}{D_n(a,k_n)}\left\{Q\left(1-\frac{1}{\lambda k_n}\right)-Q\left(1-\frac{\lambda k_n}{n}\right)\right\} \\ &= \frac{(k_n/n)^{1/2-a}}{D_n(a,k_n)}\left\{\lambda^a L\left(\frac{k_n}{\lambda n}\right)-\lambda^{-a}L\left(\frac{\lambda k_n}{n}\right)\right\}. \end{aligned} \tag{3.9}$$

Assume that $0<a<\frac{1}{2}$. In this case the latter bound is

$$\left\{\frac{((1-2a)(1-a))^{1/2}}{(2a^2)^{1/2}}\right\}\left\{\frac{\lambda^a L\left(\frac{k_n}{\lambda n}\right)-\lambda^{-a}L\left(\frac{\lambda k_n}{n}\right)}{L(k_n/n)}\right\},$$

the second factor of which, since L is slowly varying at zero, converges to $\lambda^a-\lambda^{-a}$. Since $\lambda>1$ can be chosen arbitrarily close to one, for $0<a<\frac{1}{2}$ we have

$$ET_n(d)\to 0 \quad \text{as } n\to\infty. \tag{3.10}$$

If $a=\frac{1}{2}$, then the upper bound in (3.9) equals

$$\left\{\lambda^{1/2}L\left(\frac{k_n}{\lambda n}\right)-\lambda^{-1/2}L\left(\frac{\lambda k_n}{n}\right)\right\}\Big/\left(\int_{1/n}^{k_n/n} s^{-1}L^2(s)\,ds\right)^{1/2}.$$

This expression converges to zero on account of Lemma 4. Hence we have (3.10) for the case $a=\frac{1}{2}$ as well. This implies (3.8).

Finally, to finish the proof of the theorem we only have to verify that (3.4) holds. But

$$\operatorname{Var} Z_n = \sigma_n^2(k_n)/D_n^2(a,k_n)$$

and hence (3.4) follows directly from Lemma 6.

Acknowledgments. We thank a referee for pointing out that the original proof of (3.3) could be shortened in the case $j=3$. This work was done while both authors were visiting the Laboratory for Research in Statistics and Probability, Carleton University, Ottawa, Canada.

REFERENCES

BALKEMA, A. and DE HAAN, L. (1975). Limit laws for order statistics. In *Colloquia Math. Soc. J. Bolyai 11. Limit Theorems of Probability* (P. Révész, ed.) 17–22. North-Holland, Amsterdam.

CSÖRGŐ, M., CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986). Weighted empirical and quantile processes. *Ann. Probab.* **14** 31–85.

CSÖRGŐ, S., DEHEUVELS, P. and MASON, D. M. (1985). Kernel estimates of the tail index of a distribution. *Ann. Statist.* **13** 1050–1077.

CSÖRGŐ, S., HORVÁTH, L. and MASON, D. M. (1986). What portion of the sample makes a partial sum asymptotically stable or normal? *Z. Wahrsch. verw. Gebiete.* To appear.

CSÖRGŐ, S. and MASON, D. M. (1984). Simple estimators of the endpoint of a distribution. Unpublished manuscript.

CSÖRGŐ, S. and MASON, D. M. (1985). Central limit theorems for sums of extreme values. *Math. Proc. Cambridge Philos. Soc.* **98** 547–558.

DE HAAN, L. (1970). *On Regular Variation and its Application to the Weak Convergence of Sample Extremes.* Tract No. 32, Mathematical Centre, Amsterdam.

HALL, P. (1982a). On some simple estimates of an exponent of regular variation. *J. Roy. Statist. Soc. Ser. B* **44** 37–42.

HALL, P. (1982b). On estimating the endpoint of a distribution. *Ann. Statist.* **10** 556–568.

MASON, D. M. (1982). Laws of large numbers for sums of extreme values. *Ann. Probab.* **10** 754–764.

BOLYAI INSTITUTE
SZEGED UNIVERSITY
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED
HUNGARY

MATHEMATICS INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTRASSE 39
D-8000 MUNICH 2
WEST GERMANY