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RANDOM MULTILINEAR FORMS

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We study convergence of multilinear forms $\sum f(n_1, \dots, n_k) X_{n_1} \cdots X_{n_k}$ in symmetric independent random variables. We show that the multilinear form converges if and only if its "tetrahedronal" part and "diagonal" parts of different orders converge simultaneously. For "tetrahedronal" forms a.s. and L_0 convergence are equivalent. Moreover, they are equivalent to L_p convergence provided (X_k) satisfies a Marcinkiewicz–Paley–Zygmund condition for $p \geq 2$.

Introduction. In this paper we study convergence of multilinear forms

$$Q_n^k = Q_n^k(f, X) = \sum_{1 \leq n_1, \dots, n_k \leq n} f(n_1, \dots, n_k) X_{n_1} \cdots X_{n_k},$$

where $X = (X_n)$ is a sequence of independent random variables and f is a real function defined on N^k.

The main applications of random multilinear forms are connected with multiple stochastic integration. In fact Q_n^k is exactly a multiple stochastic integral of a step function with respect to an independently scattered random measure. There are nice applications of stochastic multiple integrals in statistics and quantum mechanics. For further information we refer to the papers of Rosiński and Szulga (1982), Rosiński and Woyczyński (1984a, b; 1986), and Sjörgen (1982). Rademacher multilinear forms have appeared in harmonic analysis, cf. Bonami (1970); they were used by Pisier and Zinn (1977) to establish certain limit theorems in Banach spaces. A contribution of p-stable bilinear forms to operator theory is due to Cambanis et al. (1985) and Sjörgen (1982) (Gaussian case).

In Section 1 we introduce notation and present basic facts applied throughout. We introduce a Marcinkiewicz–Paley–Zygmund condition which combines and extends those of Marcinkiewicz and Zygmund (1937) and of Paley and Zygmund (1932).

In Section 2 we deal with Rademacher multilinear forms. All finite moments are comparable (Bonami (1970)), which is easily shown. It turns out that Q_n^k converges in L_0 if and only if its "tetrahedronal" part T_n^k and "diagonal" part D_n^k converge in L_0 simultaneously. The main result of this section states the equivalence of a.s. and L_0 convergence of T_n^k. Moreover, every convergent multilinear form in Rademacher random variables possesses an exponential $2/k$ moment, i.e., $E \exp(c|Q_n^k|^{2/k}) < \infty$ for a $c > 0$ (the case $k = 2$

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is due to Pisier and Zinn (1977)). We show that Rademacher multilinear forms satisfy a Marcinkiewicz–Paley–Zygmund condition with all positive exponents p (cf. also Borell (1984) if $p \geq 2$ and f are vector functions).

In Section 3 we extend some of these results to the case of independent symmetric random variables. A coincidence of a.s. and L_0 convergence still holds in such a general situation.

We characterize the space

$$C(X) = \left\{\alpha = (\alpha_n) \in R^N \colon \sum \alpha_n X_n \text{ converges a.s.}\right\}$$

as a generalized Orlicz sequence space l_ϕ. We show that for every predictable sequence $A = (A_n)$

$$\left\{\sum A_n X_n \text{ converges}\right\} = \{A \in C(X)\} \quad \text{a.s.},$$

which generalizes a result for p-stable random variables due to Cambanis et al. (1985) and answers a question posed by S. Kwapień. It allows study of a.s. convergence of bilinear forms in terms of series of independent random variables in an Orlicz space l_ϕ.

We obtain a useful contraction principle for multilinear forms in independent random variables which generalizes the analogous property for the series of random variables (cf., e.g., Hoffmann-Jørgensen (1977b)).

In Section 4 we study the behaviour of tetrahedronal multilinear forms in independent zero mean random variables which satisfy a Marcinkiewicz–Paley–Zygmund condition. We show that this condition is hereditary with respect to taking tetrahedronal forms of an arbitrary order k, provided $p > 2$, and bilinear forms if $p = 2$.

1. Notation and probability background. We denote by $X = (X_n)$ a sequence of real r.v.'s defined on a probability space $(\Omega, \mathscr{F}, P)$. The letter ε is reserved for a Rademacher sequence (ε_k), i.e., ε_k are independent identically distributed r.v.'s taking values $+1$ or -1 with probabilities $\frac{1}{2}$. The following property of a Rademacher sequence is known as the *Khinchine inequality*: for any p, $0 < p < \infty$, there are constants k_p and K_p such that for every integer n and for an arbitrary sequence of real numbers $(c_1, \dots, c_n)$

$$k_p\left(\sum_{k=1}^{n} |c_k|^2\right)^{1/2} \leq \left(E\left|\sum_{k=1}^{n} c_k\varepsilon_k\right|^p\right)^{1/p} \leq K_p\left(\sum_{k=1}^{n} |c_k|^2\right)^{1/2}. \tag{1.1}$$

Let k, n be integers such that $k \leq n$. Given a real function f on N^k, a *multilinear form in X of order k* is defined as a sequence of sums

$$Q_n^k(f, X) = \sum_{1 \leq n_1, \dots, n_k \leq n} f(n_1, \dots, n_k) X_{n_1} \cdots X_{n_k}. \tag{1.2}$$

Set $f'(n_1, \dots, n_k) = f(n_1, \dots, n_k)$ if all integers $n_1, \dots, n_k$ are different and

$f'(n_1,\dots,n_k) = 0$, otherwise. Then

$$Q_n^k = T_n^k + D_n^k, \tag{1.3}$$

where $T_{n_k}^k = T_n^k(f, X) = Q_n^k(f', X)$ is called the *tetrahedronal part* of Q_n^k, and $D_n^k = D_n^k(f, X) = Q_n^k(f - f', X)$ is called a *diagonal part* of Q_n^k. It is easy to see that

$$Q_n^k(f, X) = Q_n^k(\bar{f}, X), \tag{1.4}$$

where $\bar{f}(n_1,\dots,n_k) = 1/k!\Sigma f(n_{\pi(1)},\dots,n_{\pi(k)})$ and the summation is extended over all permutations π of the $\{1,\dots,k\}$. We call f *symmetric* if $f = \bar{f}$. Another decomposition will be useful in the sequel:

$$T_n^k(f, X) = \sum_{i=1}^{n} \tilde{T}_{i-1}^{k-1}(f, X)X_i, \tag{1.5}$$

where

$$\tilde{T}_{i-1}^{k-1}(f, X) = \sum_{1 \le n_1 < \cdots < n_{k-1} < i} k!\bar{f}(n_1,\dots,n_{k-1},i)X_{n_1} \cdots X_{n_{k-1}}.$$

If $X \subseteq L_1$ is a martingale difference sequence then $(T_n^k,\ n \ge k)$ is a martingale. If $X \subseteq L_2$ consists of zero mean independent r.v.'s then $\tilde{T}_{i-1}^{k-1}X_i$, $i = k,\dots,n$, are uncorrelated.

By $I(A)$ we denote the indicator function of a set A and for a real number a we write $a^* = \min(1, |a|)$. For $Z \in L_p$, $0 < p < \infty$, we put $\|Z\|_p = (E|Z|^p)^{1/p}$ if $p > 0$ and $\|Z\|_0 = EZ^*$.

The following inequality had its origins with Paley and Zygmund (1932) and Marcinkiewicz and Zygmund (1937) (cf. also Kahane (1968)).

LEMMA 1.1. *Let $0 < q < p$ and $0 < t < 1$. Then for $Z \in L_p$ we have*

$$P\big(|Z| > t\|Z\|_q\big) \ge \Big[(1 - t^{q^*})^{1/q^*}\|Z\|_q/\|Z\|_p\Big]^r, \tag{1.6}$$

where $1/q = 1/p + 1/r$.

PROOF. Put $Z' = ZI(|Z| > t\|Z\|_q)$. We have

$$\|Z'\|_q \le \|Z\|_p\big[P\big(|Z| > t\|Z\|_q\big)\big]^{1/r}$$

by the Hölder inequality and also

$$\|Z\|_q^{q^*} \le \|Z'\|_q^{q^*} + t^q\|Z\|_q^{q^*}.$$

Combining these inequalities we get

$$(1 - t^{q^*})^{1/q^*}\|Z\|_q \le \|Z'\|_q \le \|Z\|_p\big[P\big(|Z| > t\|Z\|_q\big)\big]^{1/r},$$

and this completes the proof. □

COROLLARY 1.2. *If $0 < t < \|Z\|_q/\|Z\|_p$ then*

$$P\big(|Z| > t\|Z\|_p\big) \ge \Big[\big(\|Z\|_q/\|Z\|_p\big)^{q^*} - t^{q^*}\Big]^{r/q^*}.$$

PROPOSITION 1.3. *Let $\mathscr{Z} \subseteq L_p$, $p > 0$. Then the following conditions are equivalent*:

$$\text{there is a } \delta_p > 0 \text{ such that for every } Z \in \mathscr{Z} \quad P(|Z| > \delta_p \|Z\|_p) > \delta_p; \tag{1.7}$$

$$\text{there is a } q, 0 < q < p \text{ and an } m_{p,q} > 0 \text{ such that for every } Z \in \mathscr{Z} \quad \|Z\|_p \le m_{p,q}\|Z\|_q; \tag{1.8}$$

$$\text{for every } q, 0 < q \le p, \text{ there is an } m_{p,q} > 0 \text{ such that for every } Z \in \mathscr{Z} \quad \|Z\|_p \le m_{p,q}\|Z\|_q. \tag{1.9}$$

PROOF. (1.7) ⇒ (1.9). Suppose we are given a $\delta_p > 0$ satisfying (1.7). Then the estimate

$$\|Z\|_q \ge \left\| ZI(|Z| > \delta_p \|Z\|_p) \right\|_q \ge \delta_p^{1+1/q} \|Z\|_p$$

gives (1.9).

(1.9) ⇒ (1.8) is evident.

(1.8) ⇒ (1.7) follows by Corollary 1.2. □

COROLLARY 1.4. *If a sequence $(X_n) \subseteq L_p$ satisfies one of the equivalent conditions of Proposition* 1.3 *and it converges in L_0 (respectively, is bounded in L_0), then X_n converges in L_q for every q, $0 < q < p$ (respectively, is bounded in L_p).*

DEFINITION 1.5. We say that a family $\mathscr{Z} \subseteq L_p$, $0 < p < \infty$, satisfies the Marcinkiewicz–Paley–Zygmund condition with the exponent p (MPZ(p), in short) if one of the assertions (1.7)–(1.9) holds.

It is worth noting the important role the condition MPZ(p) plays in probability theory. In fact for certain p's it has been used under different names quite frequently (cf. Gundy (1967), Kahane (1968), Stout (1974), Hoffmann-Jørgensen (1977a), Sjörgen (1982), to name but a sample).

The collection of multilinear forms becomes a vector space. Its metric nature is shown by means of the following results.

LEMMA 1.6. *Let $p \ge 1$ and $X \subseteq L_p$ be a sequence of zero mean independent r.v.'s. Then for every function f on N^k*

$$k!\,|\bar{f}(n_1, \dots, n_k)| \le \|T_n^k(f, X)\|_p / (\|X_{n_1}\|_p \cdots \|X_{n_k}\|_p). \tag{1.10}$$

PROOF. A straightforward computation shows that

$$E\left(T_n^k(f, X) - k!\,\bar{f}(n_1, \dots, n_k) X_{n_1} \cdots X_{n_k} | X_{n_1} \cdots X_{n_k}\right) = 0.$$

This observation explains (1.10). □

PROPOSITION 1.7. *Let $p \geq 1$ and X be a sequence of zero mean independent r.v.'s. Then $\mathscr{M}_p^k(X) = \{f: N^k \to R: f$ are symmetric and $\sup_n \|T_n^k(f, X)\|_p < \infty\}$ is a Banach space under the norm $|f|_p = \sup_n \|T_n^k(f, X)\|_p$.*

PROOF. Let (f_m) be a Cauchy sequence in $\mathscr{M}_p^k(X)$. Hence $f(n_1, \dots, n_k) = \lim_{m \to \infty} f_m(n_1, \dots, n_k)$ exists for all distinct integers $n_1, \dots, n_k$. Given $\gamma > 0$, there exists an $M > 0$ such that for $m, m' \geq M$ and for $n \geq k$

$$\left\| T_n^k(f_m - f_{m'}', X) \right\|_p < \gamma.$$

Hence, letting $m' \to \infty$, we get for $m \geq M$ and $n \geq k$

$$\left\| T_n^k(f_m - f, X) \right\|_p \leq \gamma.$$

Therefore, $f \in \mathscr{M}_p^k(X)$ and $|f - f_m|_p \to 0$. □

2. Rademacher multilinear forms. In the case of a Rademacher multilinear form $Q_n^k(f, \varepsilon)$ the diagonal form $D_n^k(f, \varepsilon)$ is a finite sum of tetrahedronal forms of lower orders. Put

$$K_n^{k,0} = \left\{(n_1, \dots, n_k): 1 \leq n_1, \dots, n_k \leq n,\ \#\{i: n_i = n_j\} \text{ is even},\ j = 1, \dots, k\right\},$$

$$g_n^{k,0} = \sum_{K_n^{k,0}} f(n_1, \dots, n_k),$$

and, for $m = 1, \dots, k$ and $1 \leq i_1 < \cdots < i_m \leq n$,

$$\begin{aligned} K_n^{k,m}(i_1, \dots, i_m) := \Big\{&(n_1, \dots, n_k): 1 \leq n_1, \dots, n_k \leq n,\ i_j \in \{n_1, \dots, n_k\}, \\ &\#\{i: n_i = n_j\} \text{ is even for } n_j \notin \{i_1, \dots, i_m\}, \\ &\#\{i: n_i = n_j\} \text{ is odd for } n_j \in \{i_1, \dots, i_m\},\ j = 1, \dots, k\Big\}, \end{aligned}$$

$$g_n^{k,m}(i_1, \dots, i_m) := \sum_{K_n^{k,m}(i_1, \dots, i_k)} f(n_1, \dots, n_k).$$

Define

$$T_n^{k,0} = g_n^{k,0},$$

$$T_n^{k,m} = T_n^m\left(g_n^{k,m}, \varepsilon\right) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} g_n^{k,m}(i_1, \dots, i_m)\varepsilon_{i_1} \cdots \varepsilon_{i_m}.$$

Note that $T_n^{k,m} = 0$ if $k + m$ is an odd number. We check easily that

$$D_n^k = \sum_{m=0}^{k-1} T_n^{k,m} \quad \text{and} \quad T_n^k = T_n^{k,k}, \tag{2.1}$$

and that $T_n^{k,m}$, $m = 0, 1, \dots, k$, are uncorrelated. Unfortunately, for the most part, $(T_n^{k,m}, n \geq k)$ are not martingales for $m < k$.

LEMMA 2.1. *Let $\varepsilon^1 = (\varepsilon_i^1)$ and $\varepsilon^2 = (\varepsilon_i^2)$ be independent Rademacher sequences. Then for every function f on N^k, every $p \geq 1$ and for all choices*

$a_1, \dots, a_k \in \{1,2\}$ *we have*

$$\|T_n^k(f, \varepsilon)\|_p \le k!2^k \Big\| \sum_{1 \le i_1 < \cdots < i_k \le n} \bar{f}(i_1, \dots, i_k)\varepsilon_{i_1}^{a_1} \cdots \varepsilon_{i_k}^{a_k} \Big\|_p . \tag{2.2}$$

PROOF. Denote by $\mathscr{R}$ the σ-field spanned by all sums $\varepsilon_i^1 + \varepsilon_i^2$, $i \in N$. Observe that for every Borel subset $B_1, \dots, B_k \subseteq R$ the law

$$\mathscr{L}\Big(\varepsilon_{i_1}^{a_1} \cdots \varepsilon_{i_k}^{a_k} I\big(\varepsilon_{i_1}^1 + \varepsilon_{i_1}^2 \in B_1\big) \cdots I\big(\varepsilon_{i_k}^1 + \varepsilon_{i_k}^2\big) \in B_k\Big)$$

does not depend on a choice of $(a_1, \dots, a_k) \in \{1,2\}^k$. Therefore

$$E\big(\varepsilon_{i_1}^{a_1} \cdots \varepsilon_{i_k}^{a_k} | \mathscr{R}\big) = 2^{-k} \prod_{j=1}^{k} \big(\varepsilon_{i_j}^1 + \varepsilon_{i_j}^2\big). \tag{2.3}$$

We check that the couple $(T_n^k(f, \varepsilon^1), T_n^k(f, \varepsilon^1 + \varepsilon^2))$ forms a martingale. Using (2.3) and the Jensen inequality we proceed as follows:

$$\begin{aligned}
\|T_n^k(f, \varepsilon)\|_p &= \|T_n^k(f, \varepsilon^1)\|_p \\
&\le \|T_n^k(f, \varepsilon^1 + \varepsilon^2)\|_p \\
&= k!2^k \Big\| E\Big(\sum_{1 \le i_1 < \cdots < i_k \le n} \bar{f}(i_1, \dots, i_k)\varepsilon_{i_1}^{a_1} \cdots \varepsilon_{i_k}^{a_k} | \mathscr{R}\Big) \Big\|_p \\
&\le k!2^k \Big\| \sum_{1 \le i_1 < \cdots < i_k \le n} \bar{f}(i_1, \dots, i_k)\varepsilon_{i_1}^{a_1} \cdots \varepsilon_{i_k}^{a_k} \Big\|_p ,
\end{aligned}$$

whatever $(a_1, \dots, a_k) \in \{1,2\}^k$ is chosen. The lemma is proved. □

The forthcoming auxiliary statement collects results used frequently in the sequel.

LEMMA 2.2. *The following conditions are valid*:

(i) $$\|Q_n^k\|_2^2 = \|T_n^k\|_2^2 + \|D_n^k\|_2^2 = \sum_{m=0}^{k} \|T_n^{k,m}\|_2^2;$$

(ii) $$\|T_n^{k,m}\|_2^2 = \sum_{i=k}^{n} \big\|\tilde{T}_{i-1}^{k,m-1}\big\|_2^2, \qquad m = 1, \dots, k;$$

(iii) *There is a constant a depending only on k such that*

$$\|T_n^{k,m}\|_p \le ap^{m/2}\|T_n^{k,m}\|_2; \; .$$

(iv) *There is a constant a′ depending only on k such that*

$$\|Q_n^k\|_p \le a'p^{k/2}\|Q_n^k\|_2.$$

PROOF. Since appropriate summands are uncorrelated random variables, (i) and (ii) are evident.

Let $a_{p,m}$ denote the optimal constant in equality (iii) (i.e., the smallest of numbers c such that $\|T_n^{k,m}(f,\varepsilon)\|_p \le c\|T_n^{k,m}(f,\varepsilon)\|_2$ for all integers n and all functions f on N^k). Taking $a_1 = \cdots = a_{k-1} = 1$ and $a_k = 2$ in (2.2) and using the Khinchine inequality we infer that

$$\|T_n^{k,m}\|_p \le 2^k K_p \left\| \left(\sum_{i=k}^{n} |\tilde{T}_{i-1}^{k,m-1}|^2 \right)^{1/2} \right\|_p,$$

where $\tilde{T}_{i-1}^{k,m-1}$ is defined by (1.5). Then a version of the Hölder inequality and (ii) of this lemma allow estimation of the latter quantity, consecutively, by

$$2^k K_p \left(\sum_{i=k}^{n} \|\tilde{T}_{i-1}^{k,m-1}\|_p^2 \right)^{1/2} \le 2^k K_p a_{p,m-1} \left(\sum_{i=1}^{n} \|\tilde{T}_{i-1}^{k,m-1}\|_2^2 \right)^{1/2}$$

$$\le 2^k K_p a_{p,m-1} \|T_n^{k,m}\|_2.$$

Since $a_{p,0} = 1$, the induction argument yields $a_{p,m} \le (2^k K_p)^m$. It is well known that the Khinchine constant K_p is of order $p^{1/2}$ when $p \to \infty$ (cf., e.g., Lindenstrauss and Tzafriri (1977), page 66); hence $a_{p,m} \le ap^{m/2}$ for a suitable number a.

Finally, combining (2.1) and (iii) we obtain (iv) as follows:

$$\|Q_n^k\|_p \le \sum_{m=0}^{k} \|T_n^{k,m}\|_p \le (k+1)^{1/2} \left(\sum_{m=0}^{k} \|T_n^{k,m}\|_p^2 \right)^{1/2}$$

$$\le (k+1)^{1/2} a \|Q_n^k\|_2 \le (k+1)^{1/2} \left(2^k K_p\right)^k \|Q_n^k\|_2 \le a' p^{k/2} \|Q_n^k\|_2$$

for some number a'. The latter completes the proof. □

COROLLARY 2.3. *For every $t > 0$*

$$P\left(|T_n^{k,m}| > t\|T_n^{k,m}\|_2\right) \le \exp\left(-t^{2/m}/M\right), \qquad m = 1, \dots, k,$$

$$P\left(|Q_n^k| > t\|Q_n^k\|_2\right) \le \exp\left(-t^{2/k}M'\right),$$

where M depends only on (k, m) and M' depends only on k.

PROOF. We apply a method invented by Pisier and Zinn (1977). By the Chebyshev inequality and Lemma 2.1(iii) we have, for $t > 0$,

$$P\left(|T_n^{k,m}| > t\|T_n^{k,m}\|_2\right) \le \|T_n^{k,m}\|_p^p / \left(t^p \|T_n^{k,m}\|_2^p\right) \le a^p p^{pm/2}/t^p,$$

where $p > 0$ (the estimate is trivial if $p < 2$). Setting $p = (t/ae)^{2/m}$ we obtain the first of the required inequalities. The second follows by a similar argument. □

REMARK 2.4. The inequality (iii) of Lemma 2.1 was originally proved by Bonami (1970) and certain generalizations thereof are due to Pisier (1977/78) and Borell (1984). The present proof became simple by virtue of (2.2), which we learned from S. Kwapień.

LEMMA 2.5. *If Q_n^k converges in L_0 then $T_n^{k,m}$ is L_2 bounded, $m = 0, 1, \dots, k$. Further, $T_n^k = T_n^{k,k}$ is L_2 bounded provided Q_n^k contains an L_0-bounded subsequence.*

PROOF. By Lemma 1.1 and Lemma 2.2(i) and (iv) we infer that there exists a $\delta_p > 0$ such that for $n \geq k$ and $m = 0, 1, \dots, k$,

$$P\left(|Q_n^k| > \delta_p \|T_n^{k,m}\|_2\right) \geq P\left(|Q_n^k| > \delta_p \|Q_n^k\|_2\right) > \delta_p.$$

Hence the statement of Lemma 2.5 follows immediately (we recall that $T_n^{k,k}$ is a martingale). □

The following assertion may be deduced from Hilbert space theory (cf., e.g., Yosida (1965)), because the set $\{\varepsilon_{i_1} \cdots \varepsilon_{i_m}\colon i_1, \dots, i_m$ are different$\}$ forms an orthonormal system in L_2. We omit a standard proof.

LEMMA 2.6. *If $T_n^{k,m} = T_n^m(g_n^m, \varepsilon)$ converges in L_0 to a random variable T^m then there exists a symmetric function g^m on N^k such that*

(i) $$g_n^m \to g^m \text{ uniformly as } n \to \infty;$$

(ii) $$\sum_{1 \leq i_1 < \cdots < i_m} |g^m(i_1, \dots, i_m)|^2 < \infty;$$

(iii) $$T^m = \sum_{1 \leq i_1 < \cdots < i_m} g^m(i_1, \dots, i_m) \varepsilon_{i_1} \cdots \varepsilon_{i_m}.$$

An immediate consequence of the above auxiliary results is the main statement of this section. Note that coincidence of a.s. and L_0 convergence was also proved by Pisier (1977/78).

THEOREM 2.7. *Let $p, q > 0$ and let $Q_n^k = Q_n^k(f, \varepsilon)$ be a Rademacher multilinear form. Then the following conditions are equivalent:*

(i) *Q_n^k converges in L_p;*

(ii) *$T_n^{k,m}$ converges in L_q to a tetrahedronal multilinear form, $m = 0, 1, \dots, k$, and $T_n^k = T_n^{k,k}$ converges a.s.*

COROLLARY 2.8. *A bilinear Rademacher form $Q_n^2(f, \varepsilon)$ converges in L_p, $0 \leq p < \infty$, if and only if $Q_n^2(f, \varepsilon)$ converges* a.s.

COROLLARY 2.9. *The family $\mathscr{X} = \{Q_n^k(f, \varepsilon)\colon f$ are functions on $N^k\}$ satisfies* MPZ(p) *for every p, $0 < p < \infty$.*

In particular, by comparing the L_p and L_2 norms of a Rademacher multilinear form we obtain a property which should be called the *generalized Khinchine inequality*: for every p, $0 < p < \infty$, there are constants $c_{p,k}$ and $C_{p,k}$ such that for every $n \geq k$ and for all functions f on N^k

$$(2.4)\quad \begin{aligned} &c_{p,k}\Big(\sum_{1 \leq i_1 < \cdots < i_k \leq n} |\bar{f}(i_1,\dots,i_k)|^2\Big)^{1/2} \\ &\leq \|T_n^k(f,\varepsilon)\|_p \leq C_{p,k}\Big(\sum_{1 \leq i_1 < \cdots < i_k \leq n} |\bar{f}(i_1,\dots,i_k)|^2\Big)^{1/2}. \end{aligned}$$

A further consequence is a *contraction principle* for Rademacher multilinear forms:

$$(2.5)\quad \|Q_n^k(\bar{f}h,\varepsilon)\|_p \leq C_{p,2}C_{2,p}\|h\|_\infty \|Q_n^k(f,\varepsilon)\|_p$$

for every $n \geq k$ and for all functions f, h on N^k, where $\|h\|_\infty = \sup_{1 \leq i_1,\dots,i_k}|h(i_1,\dots,i_k)|$, provided $f = 0$ on $K_n^{k,0}$. The proof of (2.5) is standard: We use (2.1), Lemma 2.2(i), and apply Corollary 2.9 twice.

Concluding this section, we have to note that the present meaning of the convergence of a multilinear form need not be unique. Quantities

$$f(i_1,\dots,i_k)X_{i_1}\cdots X_{i_k}$$

may also be summable according to other summability methods. At this moment we are not able to compare them; however, the following result gives some information in this direction.

PROPOSITION 2.10. *Let f be a function on N^2. For $j \geq 2$ and $i = 1,\dots, j-1$ set $Z_{(j-1)(j-2)/2+i} = f(i,j)\varepsilon_i\varepsilon_j$. Then the series $\sum_n Z_n$ converges a.s. and in L_p if and only if the bilinear form $T_n^2(f,\varepsilon)$ converges in L_0.*

PROOF. Denote $S_n = \sum_{k=1}^n Z_k$. Now we have

$$S_{(n-1)(n-2)/2+m} = T_{n-1}^2 + \sum_{i=1}^m f(i,n)\varepsilon_i\varepsilon_n,$$

where $n = 2,3,\dots, m = 1,\dots, n-1$. In particular (T_n^2) is a subsequence of (S_n) and hence the "only if" part follows. On the other hand

$$\begin{aligned} &\Big(E\max_{1 \leq m \leq n-1}\big|S_{(n-1)(n-2)/2+m} - T_{n-1}^2\big|^p\Big)^{1/p} \\ &\quad = \Big(E\max_{1 \leq m \leq n-1}\Big|\sum_{i=1}^m f(i,n)\varepsilon_i\Big|^p\Big)^{1/p} \\ &\quad \leq 2K_p\Big(\sum_{i=1}^{n-1}|f(i,n)|^2\Big)^{1/2} \end{aligned}$$

by the Lévy and the Khinchine inequalities. Let now $T_n^2(f,\varepsilon)$ converge in L_0. By Theorem 2.7 $T_n^2(f,\varepsilon)$ converges in L_2, or equivalently, $\sum_{n=2}^\infty\sum_{i=1}^{n-1}|f(i,n)|^2 < \infty$.

Therefore $\sum_{i=1}^{n-1}|f(i,n)|^2 \to 0$ and thus S_n converges a.s. and in L_p, which concludes the proof. □

3. Multilinear forms in independent symmetric random variables. Throughout this section $X = (X_n)$ denotes a sequence of independent nondegenerate symmetric random variables. Let us denote

$$C(X) = \left\{(\alpha_n) \in R^N\colon \sum_n \alpha_n X_n \text{ converges a.s.}\right\}.$$

It is well known that $C(X) = l_2$ if X_n are Gaussian or Rademacher random variables (see Hoffmann-Jørgensen (1977a) and Section 4 for other examples); $C(X) = l_p$ if X_n have p-stable distribution, $0 < p \leq 2$. We introduce functions

$$\phi_n(T) = E\left(t^2|X_n|^2\right)^*, \qquad t > 0.$$

It can be immediately checked that they have the following properties:

(a) $\phi_n(t) = 0$ if and only if $t = 0$;
(b) ϕ_n are continuous and nondecreasing;
(c) $\phi_n(ct) \leq c^2\phi_n(t)$ for $c \geq 1$;
(d) $\phi_n(t) = t^2E|X_n|^2I(|X_n| \leq t^{-1}) + P(|X_n| > t^{-1})$.

Therefore a modular

$$\rho(\alpha) = \sum_n \varphi_n(|\alpha_n|), \qquad \alpha = (\alpha_n) \in R^N,$$

determines a generalized Orlicz sequence space

$$l_\phi = \{\alpha \in R^N\colon \rho(\alpha) < \infty\}.$$

Let $e_n = (0,\ldots,0,1,0,\ldots)$, with 1 in the nth place.

Proposition 3.1. *l_ϕ is a complete metric linear space under the F-norm $\|\alpha\| = \inf\{c > 0\colon \rho(\alpha/c) \leq c\}$, (e_n) is a Schauder basis of l_ϕ, and continuous linear functionals form a separating set. Moreover, if (ϕ_n) is equivalent to a sequence of convex functions (ϕ'_n) (in the sense of Musielak* (1983), *Definition* 8.16), *then the functional $|\alpha| = \inf\{c > 0\colon \rho'(\alpha/c) \leq 1\}$ defines a norm on l_ϕ.*

The proof can be found in Musielak (1983), which is our general reference concerning generalized Orlicz spaces.

Proposition 3.2. $C(X) = l_\phi$.

Proof. It suffices to apply the Kolmogorov three series theorem. □

Recall that a sequence $A = (A_n)$ of random variables is said to be *predictable* (with respect to X) if each A_n is $\mathscr{F}_{n-1}$ measurable, where $\mathscr{F}_0 = \{\varnothing, \Omega\}$ and $\mathscr{F}_n$ is the σ field generated by $X_1,\ldots,X_n$.

THEOREM 3.3. *For every sequence $X = (X_n)$ of symmetric independent random variables and for every predictable sequence $A = (A_n)$,*

$$\Big\{\sum_n A_n X_n \text{ converges}\Big\} = \{A \in C(X)\} \quad a.s.$$

PROOF. The truncated random variables $Y_n = A_n X_n I(|A_n X_n| \le 1)$ are martingale differences by the symmetry of X. Since

$$\phi_n(|A_n|) = E\big(|Y_n|^2 | \mathscr{F}_{n-1}\big) + P(|A_n X_n| > 1 | \mathscr{F}_{n-1})$$

then by the conditional three series theorem (cf. Stout (1974)) we obtain

$$\{A \in C(X)\} \subseteq \Big\{\sum_n A_n X_n \text{ converges}\Big\} \quad \text{a.s.}$$

On the other hand, by a conditional version of the Borel–Cantelli lemma (cf. Stout (1974)) we infer that, almost surely,

$$\Big\{\sum_n A_n X_n \text{ converges}\Big\} = \Big\{\sum_n Y_n \text{ and } \sum_n P(|A_n X_n| > 1 | \mathscr{F}_{n-1}) \text{ converge}\Big\}.$$

Since $|Y_n| \le 1$ a.s.,

$$\Big\{\sum_n Y_n \text{ converges}\Big\} \subseteq \Big\{\sum_n E\big(|Y_n|^2 | \mathscr{F}_{n-1}\big) < \infty\Big\} \qquad \text{a.s.}$$

(cf. Doob (1953), page 311; also Stout (1974)). Therefore

$$\Big\{\sum_n A_n X_n \text{ converges}\Big\} \subseteq \Big\{\sum_n \phi_n(|A_n|) < \infty\Big\} = \{A \in C(X)\} \quad \text{a.s.,}$$

which concludes the proof. □

Cambanis et al. (1985) proved the p-stable counterpart of Theorem 3.3 using the idea of "conditioning" due to Hill (1982). Our extension answers a problem posed by S. Kwapień.

By examining the proof we see that for independent random variables (not necessarily symmetric)

$$\{A \in C(X)\} \subseteq \Big\{\sum_n A_n X_n \text{ converges}\Big\} \quad \text{a.s.}$$

The above relation continues to also hold for arbitrary martingale differences X_n and A such that $|A_n X_n| \le c$ a.s. On the other hand, the converse inclusion fails in general, when avoiding the symmetry assumption, as the following example (essentially due to S. Kwapień) explains:

EXAMPLE. Let X_{2k-1} be independent standard Gaussian random variables and let X_{2k} take 1 with probability $1 - 1/k^2$ and $1 - k^2$ with the remaining probability. Then it can easily be checked that

$$C(X) = \Big\{\alpha \in R^N \colon \sum_n |\alpha_{2n-1}|^2 < \infty \text{ and } \sum_n \alpha_{2n} \text{ converges}\Big\}.$$

Let us choose nondegenerate A_n such that

1. $$\sum_n |A_{2n-1}|^2 < \infty \quad \text{a.s.},$$

2. $$A_{2n-1}X_{2n-1} \to 0 \quad \text{a.s.},$$

3. $$A_{2n} = -\sum_{k=1}^{2n-1} A_k X_k \quad \text{a.s.}$$

Then $\Sigma_n A_n X_n$ converges a.s. but still $A \notin C(X)$.

We shall apply Theorem 3.3 at the end of this section to establish equivalence between convergence of bilinear forms and certain Orlicz space valued series of independent random vectors.

THEOREM 3.4. *Let $X = (X_n)$ be a sequence of independent symmetric random variables. The following statements are equivalent*:

(i) $Q_n^k(f, X)$ *converges in* L_0;
(ii) $T_n^k(f, X)$ *converges a.s. and* $D_n^k(f, X)$ *converges in* L_0.

PROOF. It suffices to show that T_n^k converges a.s., provided Q_n^k does in L_0. To this end, assume that X is defined on a probability space $(\Omega_1, \mathscr{F}_1, P_1)$ and choose a Rademacher sequence independent of X, say, defined on $(\Omega_2, \mathscr{F}_2, P_2)$. Let E_1 and E_2 denote the expectations for P_1 and P_2, respectively. Being symmetric, X and $X\varepsilon = (X_n \varepsilon_n)$ are identically distributed. Suppose now $Q_n^k(f, X)$ converges in L_0, hence $Q_n^k(f, X\varepsilon)$ does in $L_0(P_1 \times P_2)$ to a random variable Z. Then $E_1 V_n(t) \to 0$ as $n \to \infty$ for every $t > 0$, where

$$V_n(t) = V_n(t, \omega_1) = P_2\Big(\big\{\omega_2 : \big|Z(\omega_1, \omega_2) - Q_n^k(f, X(\omega_1)\varepsilon(\omega_2)\big| > t\big\}\Big).$$

In particular, $V_n(t) \to 0$ in $L_0(P_1)$ for every $t > 0$. Hence we can find an $\Omega_{01} \subseteq \Omega_1$ with $P(\Omega_{01}) = 1$ and an increasing sequence (n_i) of integers such that $V_{n_i}(t, \omega_1) \to 0$ as $i \to \infty$ for every $\omega_1 \in \Omega_{01}$ and for every rational $t > 0$. It means that $Q_{n_i}^k(\bar{f}X(\omega_1), \varepsilon)$ converges in $L_0(P_2)$ for every $\omega_1 \in \Omega_{01}$. We deduce from Theorem 2.7 that there exists an $\Omega_{02} = \Omega_{02}(\omega_1) \subseteq \Omega_2$ with $P(\Omega_{02}) = 1$ such that for every $\omega_1 \in \Omega_{01}$ and for every $\omega_2 \in \Omega_{02}$, $T_n^k(\bar{f}X(\omega_1), \varepsilon(\omega_2)) = T_n^k(f, X(\omega_1)\varepsilon(\omega_2))$ converges. By virtue of the Fubini theorem $T_n^k(f, X)$ converges a.s. The other implications are obvious and the proof is complete. □

COROLLARY 3.5. *The following properties of tetrahedronal multilinear forms $T_n^k(f, X)$ in independent symmetric random variables are equivalent*:

(i) $T_n^k(f, X)$ *converges in* L_0;
(ii) $\sup_n |T_n^k(f, X)| < \infty$ *a.s.*;
(iii) $T_n^k(f, X)$ *converges a.s.*;
(iv) $|T_n^k(\bar{f}^2, X^2)|^{1/2}$ *converges a.s.*;
(v) $T_n^k(h\bar{f}, X)$ *converges a.s. for every bounded function h on N^k.*

More information can be obtained when dealing with bilinear forms.

THEOREM 3.6. *The following properties of bilinear forms $Q_n^2(f, X)$ in independent symmetric random variables are equivalent*:

(i) $Q_n^2(f, X)$ *converges in* L_0;
(ii) $Q_n^2(f, X)$ *converges a.s.*;
(iii) $T_n^2 = \sum_{k=2}^n(\sum_{j=1}^{k-1}f(j, k)X_j)X_k$ *and* $D_n^2 = \sum_{k=1}^n f(k, k)X_k^2$ *converge a.s.*;
(iv) *for every* $j \in N$, $u_j = (0,\dots,0, f(j, j+1), f(j, j+2),\dots)$ *are vectors in* l_Φ, *the series* $\sum_j u_j X_j$ *converges a.s. in* l_Φ *and the series* $\sum_k f(k, k)X_k^2$ *converges a.s.*

PROOF. Equivalence of (i), (ii), and (iii) is a consequence of Theorem 3.4.

(iii) $\Rightarrow$ (iv): Let $A_1 = 0$ and $A_k = \sum_{j=1}^{k-1} f(j, k)X_j$, $k > 1$. Suppose T_n^2 converges a.s. Then $\sum_k A_k X_k$ converges a.s.; hence, $\sum_k A_k e_k \in l_\Phi$ a.s. by Proposition 3.2. Setting $u_{jn} = \sum_{k=j+1}^{n+1} f(j, k)e_k$, we get by Theorem 3.3 the following equality:

$$\sum_{j=1}^{n} u_{jn}X_j = \sum_{k=1}^{n+1} A_k e_k. \tag{3.1}$$

It suffices now to multiply the l.h.s. in (3.1) by X_j and to use the symmetry of X to see that $u_{jn}X_j^2$ converges a.s. as $n \to \infty$. Therefore there exists $u_j = \lim_n u_{jn} = \sum_{k=j+1}^{\infty} f(j, k)e_k$. Finally, since

$$P\left(\left\|\sum_{j=m}^{m'} u_j X_j\right\| > t\right) = \lim_n P\left(\left\|\sum_{j=m}^{m'} u_{jn}X_j\right\| > t\right)$$
$$\le 2\lim_n P\left(\left\|\sum_{j=m}^{m'} u_{jn}X_j\right\| > t/2\right),$$

$\sum_j u_j X_j$ converges in L_0 or, equivalently, a.s.

(iv) $\Rightarrow$ (iii): Suppose that $u_j \in l_\Phi$ and $\sum_j u_j X_j$ converges a.s. in l_Φ. Let R_n denote the projection in l_Φ with the range spanned by $\{e_{n+2}, e_{n+3},\dots\}$. Since $R_n \to 0$ pointwise,

$$\sum_{k=1}^{n+1} A_k e_k = \sum_{j=1}^{\infty}\left(u_j - R_n(u_j)\right)X_j = \sum_{j=1}^{\infty} u_j X_j - R_n\left(\sum_{j=1}^{\infty} u_j X_j\right)$$

converges a.s. in l_Φ, which concludes the proof. □

The above result in the case of p-stable random variables was proved similarly by Cambanis et al. (1985).

LEMMA 3.7. *Let* $0 < p < \infty$. *There is an* $a_p > 0$ *such that for every sequence* X *of independent symmetric random variables and for every function* f *on* N^k *we have*

$$\left\|\max_{k \le j \le n} |T_j^k(f, X)|\right\|_p \le a_p \|T_n^k(f, X)\|_p.$$

PROOF. It suffices to limit ourselves to the case $p \le 1$ because for $p > 1$ the result follows from martingale theory (cf., e.g. Doob (1953)). Let ε be a Rademacher sequence independent of $X = X(\omega)$ and let E_2 denote the expectation with respect to ε. Since Rademacher multilinear forms are martingales, using the generalized Khinchine inequality we get, for $\omega \in \Omega$,

$$E_2 \max_{k \le j \le n} |T_j^k(f, X)|^p \le \left(E_2 \max_{k \le j \le n} \left| T_j^k(f, X(\omega)\varepsilon) \right|^2 \right)^{p/2}$$
$$\le 2^p \left(E_2 |T_n^k(f, X(\omega)\varepsilon)^2 \right)^{p/2} \le 2^p c_{p,k}^{-p} E_2 |T_n^k(f, X(\omega)\varepsilon)|^p.$$

Taking the expectation with respect to X and using the symmetry of X we derive the required inequality with $a_p \le 2 c_{p,k}^{-1}$. □

LEMMA 3.8. *For every p, $0 < p < \infty$, and for every multilinear form in independent symmetric random variables we have*

$$c_{p,k} \left\| T_n^k(\bar{f}^2, X^2)^{1/2} \right\|_p \le \left\| T_n^k(f, X) \right\|_p \le C_{p,k} \left\| T_n^k(\bar{f}^2, X^2)^{1/2} \right\|_p.$$

PROOF. The inequalities follow from the generalized Khinchine inequality and the Fubini theorem. □

LEMMA 3.9. *For every p, $0 < p < \infty$, for every multilinear form in independent symmetric random variables, and for every bounded function h on N^h we have*

$$\left\| T_n^k(h\bar{f}, X) \right\|_p \le \|h\|_\infty C_{p,k} c_{p,k}^{-1} \left\| T_n^k(f, X) \right\|_p.$$

PROOF. This follows from Lemma 3.8. □

THEOREM 3.10. *Let $0 < p < \infty$ and let X be a sequence of independent symmetric p-integrable random variables. The following properties of tetrahedronal multilinear forms in X are equivalent*:

(i) $T_n^k(f, X)$ *is* L_p-*bounded*;
(ii) $\sup_n |T_n^k(f, X)| \in L_p$;
(iii) $\sup_n \|T_n^k(\bar{f}^2, X^2)^{1/2}\|_p < \infty$;
(iv) $T_n^k(f, X)$ *converges a.s. to a* $T^k \in L_p$;
(v) $T_n^k(f, X)$ *converges in* L_p.

PROOF. (i) ⇒ (ii) follows by Lemma 3.7.
(ii) ⇒ (iii) applying Lemma 3.8.
(iii) ⇒ (iv) is a consequence of Corollary 3.5 and of the Fátou lemma.
(iv) ⇒ (v): By the symmetry of X there is $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$ $T_n^k(f, X(\omega)\varepsilon) \to T^k(f, X(\omega)\varepsilon)$ a.s., where T^k is an infinite multilinear form (which exists by Theorem 2.7) and ε is a Rademacher sequence independent of X. Hence $T_n^k(f, X(\omega)\varepsilon)$ converges in L_p to $T^k(f, X(\omega)\varepsilon)$ for

every $\omega \in \Omega_0$. The contraction principle (2.5) for Rademacher multilinear forms yields

$$\|T_n^k(f, X(\omega)\varepsilon)\|_p \leq C_{p,2}C_{2,p}\|T_n^k(f, X(\omega)\varepsilon)\|_p, \qquad \omega \in \Omega_0.$$

From the Fubini theorem we derive L_p-boundedness of $T_n^k(f, X)$ and hence $\sup_n|T_n^k(f, X)| \in L_p$ by already proved implication (i) $\Rightarrow$ (ii). Finally, (v) follows by the Lebesgue dominated theorem.

Since (v) $\Rightarrow$ (i) is obvious, the proof is completed. □

Now we extend the domain of those p for which the statement of Proposition 1.7 is valid.

PROPOSITION 3.11. *Let $0 < p < 1$. Let X be a sequence of independent symmetric random variables. Then $\mathscr{M}_p^k(X) = \{f: N^k \to R: T_n^k(f, X)$ converges in $L_p\}$ is a complete metric linear space under the F norm $|f|_p = \sup_n\|T_n^k(f, X)\|_p^p$.*

PROOF. We repeat the same argument as the one used in the proof of Proposition 1.7, applying Lemma 3.9 instead of Lemma 1.6.

REMARK 3.12. A sort of unconditional convergence of multilinear forms $Q_n^k(f, X)$ in symmetric independent random variables was defined by Rosiński and Woyczyński (1984b): $T_n^k(f, X)$ is said to converge unconditionally (a.s., in L_p, $p \geq 0$, respectively) if $T_n^k(\eta f, X)$ converges (a.s., in L_p, $p \geq 0$, respectively) for every ± 1-valued function η on N^k. However, this simply means that $T_n^k(|f|, X)$ converges in a suitable meaning.

4. Multilinear forms with Marcinkiewicz–Paley–Zygmund condition. In this section we study the behaviour of certain transforms of random multilinear forms similar to those investigated by Burkholder (1973).

PROPOSITION 4.1. *Let $0 < p < \infty$ and let $X \subseteq L_p$ be a sequence of independent random variables. Assume that for a function f on N^k, $\{T_n^k(f, X), n \geq k\}$ satisfies* MPZ(p). *Consider the following properties*:

1. *$T_n^k(f, X)$ admits a bounded subsequence*;
2. *$T_n^k(f, X)$ converges*;
3. *$T_n^k(f^2, X^2)^{1/2}$ is bounded*;
4. *$T_n^k(hf, X)$ converges for every bounded function h on N^k.*

Then

(i) *Properties* 1–4 *are equivalent provided X_n are symmetric, $0 < p < \infty$, or $EX_n = 0$, $p \geq 2$*;

(ii) *Properties* 1 *and* 2 *are equivalent provided $EX_n = 0$, $1 < p < \infty$, where the concepts of L_0 a.s. or L_p convergence* (*boundedness, respectively*) *may be used exchangeably.*

PROOF. The first part follows by an immediate combination of Corollary 3.5, Theorem 3.10, and Corollary 1.4. We need more details to prove that existence of an L_0-bounded subsequence implies L_p-boundedness. For this purpose suppose that $T_n^k(f, X)$ contains an L_0-bounded subsequence. From the condition MPZ(p) we derive its L_p-boundedness and next, choosing a suitable function h in Lemma 3.9, we infer that $T_n^k(f, X)$ is L_p-bounded.

For the second part it suffices to recall that $T_n^k(f, X)$ is a martingale. □

REMARK 4.2. If $p = 1$ and $EX_k = 0$ then Properties 1 and 2 are still equivalent under a slightly stronger assumption:

$$\{T_n^k(f, X) - T_m^k(f, X): m, n \geq k\} \quad \text{satisfies MPZ}(1).$$

The only nonimmediate case concerns the implication: existence of an L_0-bounded subsequence ⇒ L_1-convergence. Corollary 1.4 gives L_1-boundedness and L_0-convergence of T_n^k. Now the assumption yields

$$\|T_n^k - T_m^k\|_1 \leq C_{1,q}\|T_n^k - T_m^k\|_q$$

for every $q < 1$ (Proposition 1.3). Thus T_n^k converges in L_1.

The result below can be formulated and proved in a more general case of a Hilbert space valued function f (cf. Borell (1984)). Borell's proof utilized some harmonic analysis arguments. The claim needed for our purpose can be proved in a simple way. We include its proof for the sake of completeness (consult also Bonami (1970) for the case of symmetric X and even p).

LEMMA 4.3. *Let $X \subseteq L_p$, $p > 2$ be a sequence of independent zero mean random variables. If X satisfies* MPZ(p) *then so does $\mathscr{X} = \{T_n^k(f, X): n \geq k, f$ are functions on $N^k\}$.*

PROOF. It is enough to prove

$$\|T_n^k(f, X)\|_p \leq d_{p,k}\|T_n^k(f, X)\|_2,$$

where a constant $d_{p,k}$ depends only on p and k.

Put $d_p = d_p(X) = \sup_n\|X_n\|_p/\|X_n\|_2$. Using (1.5) and a square function inequality due to Burkholder (1973) we obtain

$$\|T_n^k(f, X)\|_p \leq b_p\left\|\left(\sum_{i=k}^{n}|\tilde{T}_{i-1}^{k-1}X_i|^2\right)^{1/2}\right\|_p$$

$$\leq b_p\left(\sum_{i=k}^{n}\|\tilde{T}_{i-1}^{k-1}\|_p^2\|X_i\|_p^2\right)^{1/2} \leq b_p d_p d_{p,k-1}\|T_n^k(f, X)\|_2,$$

where b_p is the Burkholder constant. Since $d_{p,1} \leq b_p d_p$, the induction argument concludes the proof; moreover $d_{p,k} \leq (b_p d_p)^k$. □

At this moment we are not able to extend Lemma 4.3 completely to the case $p = 2$. However, we have the following partial result:

LEMMA 4.4. *Let $X \subseteq L_2$ be a sequence of independent random variables with* MPZ(2). *If X_n are symmetric or if $EX_n = 0$ in case of bilinear forms ($k = 2$) then $\mathscr{T} = \{T_n^k(f, X)\colon n \geq k,\ f$ are functions on $N^k\}$ satisfies* MPZ(2) *as well.*

PROOF. Suppose first X_n are symmetric. Let ε be a Rademacher sequence independent of X and let E_2 denote the expectation with respect to ε. We write $EX^2 := (E|X_n|^2)$ and $E|X| = (E|X_n|)$. Set $d_2 = \sup_n \|X_n\|_2 / \|X_n\|_1$. We have

$$\begin{aligned}\|T_n^k(f, X)\|_2 &= \left(ET_n^k(\bar{f}^2, |X|^2)\right)^{1/2} = \left(T_n^k(\bar{f}^2, EX^2)\right)^{1/2}\\ &\leq d_2\left(T_n^k(\bar{f}^2, (E|X|)^2)\right)^{1/2}\\ &\leq C_{2,1} d_2 E_2 |T_n^k(f, \varepsilon E|X|)| \leq C_{2,1} d_2 E_2 E\left|T_n^k(f, \varepsilon X)\right|\\ &= C_{2,1} d_2 \|T_n^k(f, X)\|_1,\end{aligned}$$

which proves the first part of Lemma 4.4.

Now, let $k = 2$ and let us drop the symmetry assuming only that $EX_n = 0$. We may normalize X_n's in order to have $E|X_n|^2 = 1$. Let X' be an independent copy of X. If $T_n^2(f, X)$ converges in L_1 then (being a martingale) it converges a.s. Applying Gundy's result (1967) we infer that

$$\sum_{j=2}^{\infty}\left(\sum_{i=1}^{j-1} f(i, j) X_i\right)^2 < \infty \quad \text{a.s.}$$

and further,

$$\sum_{j=2}^{\infty}\left(\sum_{i=1}^{j-1} f(i, j) X_i\right) X_j'$$

converges a.s., because $C(X) = C(X') = l_2$. Therefore the symmetrized bilinear form

$$\sum_{j=2}^{\infty}\left(\sum_{i=1}^{j-1} f(i, j)(X_i - X_i')(X_j - X_j')\right)$$

converges a.s. as well. As already shown, the first part of Lemma 4.4 for symmetric random variables yields L_2 convergence of $T_n^2(f, X - X')$. Since

$$\|T_n^2(f, X)\|_2 = 2^{-1/2}\|T_n^2(f, X - X')\|_2,$$

the proof is completed. □

COROLLARY 4.5. *If $X \subseteq L_p$, $p \geq 2$, is a sequence of independent zero mean random variables and if additionally X_n's are symmetric in the case $p = 2$ and*

$k > 2$, then the properties listed in Proposition 4.1 *are equivalent, whichever concept (convergence or boundedness) is used, provided X satisfies MPZ(p).*

REMARK 4.6. All results contained in this paper continue to hold for random variables assuming values in a Hilbert space. We need only infinite dimensional counterparts of classical probability theorems, like the three series theorem, the Burkholder and Khinchine inequalities etc., which do extend to the Hilbert space case. A generalized Orlicz space $\ell_\Phi(H)$ can be derived as well in such a situation.

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