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ON THE RATE AT WHICH A HOMOGENEOUS DIFFUSION APPROACHES A LIMIT, AN APPLICATION OF LARGE DEVIATION THEORY TO CERTAIN STOCHASTIC INTEGRALS[1]

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Let $X(T)$ be the solution to a stochastic differential equation whose coefficients are homogeneous of degree 1 (e.g., a linear S.D.E.). Under mild conditions, it is shown that limits like

$$\lim_{T\to\infty}\frac{1}{T}\log P\big(|X(T)|/|X(0)|\geq R\big)$$

exist and a formula is provided for their computation. The techniques developed apply to a broad class of situations besides the one treated here.

1. Some preliminaries and statement of the results. The notation introduced below will be used throughout. $N\geq 2$ and $d\geq 1$ are fixed integers; $\{V_0,\dots,V_d\}\subseteq C^\infty(R^N\setminus\{0\};R^N)$ is a collection of vector fields each of which is homogeneous of degree 1 [i.e., $V_k(x)=|x|V_k(x/|x|)$], and $(\beta(t)=(\beta_1(t),\dots,\beta_d(t)),\mathscr{F}_t,P)$ is a d-dimensional Brownian motion. When dealing with a vector field V, it will often be useful to identify V with the directional derivative operator $\sum_{i=1}^N V^i\,\partial/\partial x_i$ which it determines. Thus, for example, $Vf\equiv\sum_{i=1}^N V^i\,\partial f/\partial x_i$ and $V^2f=V\circ Vf$. Also, for notational convenience when writing stochastic integrals, $\circ\, d\beta_0(t)$ will be used sometimes to denote dt.

Lemma 1.1. *For each $x\in R^N\setminus\{0\}$ there is a P-almost surely unique, right-continuous, $\{\mathscr{F}_t\colon t\geq 0\}$-progressively measurable function $X(\cdot,x)$ such that $P(X(t,x)\in R^N\setminus\{0\}$ for all $t\geq 0)=1$ and $X(\cdot,x)$ satisfies the Stratonovich stochastic integral equation*

$$X(T,x)=x+\sum_{k=0}^{d}\int_0^T V_k(X(t,x))\circ d\beta_k(t),\qquad T\geq 0. \tag{1.2}$$

Moreover, if $\rho(T,x)=\log(|X(T,x)|/|x|)$ and $\theta(T,x)=X(T,x)/|X(T,x)|$, then

$$\begin{aligned}\rho(T,x)&=\sum_{k=0}^{d}\int_0^T\sigma_k(\theta(t,x))\circ d\beta_k(t)\\&=\sum_{k=1}^{d}\int_0^T\sigma_k(\theta(t,x))\,d\beta_k(t)+\int_0^T Q(\theta(t,x))\,dt,\qquad T\geq 0,\end{aligned}\tag{1.3}$$

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and

$$\theta(T,x) = \frac{x}{|x|} + \sum_{k=0}^{d} \int_0^T W_k(\theta(t,x)) \circ d\beta_k(t), \qquad T \geq 0, \tag{1.4}$$

where $\sigma_k(\theta) = (\theta, V_k(\theta))_{R^N}$, $W_k(\theta) = V_k(\theta) - \sigma_k(\theta)\theta$, *and* $Q(\theta) = \sigma_0(\theta) + \frac{1}{2}\sum_1^d W_k(\sigma_k)(\theta)$ *for* $\theta \in S^{N-1}$.

PROOF. By the standard theory of stochastic integral equations, there is no problem about the existence and uniqueness of $X(\cdot, x)$ up until the first time $X(\cdot, x)$ hits 0. Moreover, up until that time, it is easy to check that $\rho(\cdot, x)$ and $\theta(\cdot, x)$ satisfy (1.3) and (1.4), respectively. Finally, from (1.3), it is clear that $\inf_{0<t<T}|X(t,x)|/|x| > 0$ a.s. P for each $T > 0$. Hence, P-almost surely, $X(\cdot, x)$ never hits 0 in a finite time. □

As a consequence of (1.4), it is clear that, for each $x \in R^N \setminus \{0\}$, $\theta(\cdot, x)$ is the diffusion on S^{N-1} starting at $x/|x|$ and generated by

$$L = \frac{1}{2} \sum_{k=1}^{d} W_k^2 + W_0. \tag{1.5}$$

Let $P(T, \theta, \cdot)$, $(T, \theta) \in (0, \infty) \times S^{N-1}$, denote the transition probability function for this diffusion. Henceforth it will be assumed that

$$\mathrm{Lie}(W_1, \dots, W_d)(\theta) = T_\theta(S^{N-1}), \qquad \theta \in S^{N-1}. \tag{1.6}$$

[$\mathrm{Lie}(W_1, \dots, W_d)$ denotes the Lie algebra of vector field on S^{N-1} generated by $\{W_1, \dots, W_d\}$.] In particular, by a renowned theorem of Hörmander [2], (1.6) guarantees that there is a smooth map $(T, \theta, \eta) \in (0,\infty) \times S^{N-1} \times S^{N-1} \to p(T, \theta, \eta)$ such that $P(T, \theta, d\eta) = p(T, \theta, \eta)\, d\eta$, where $d\eta$ denotes the normalized Lebesgue measure on S^{N-1}. Moreover, by the strong maximum principle [cf. Theorem (6.1) in [5]], one can easily see that $p(T, \theta, \eta) > 0$ for all $(T, \theta, \eta) \in (0, \infty) \times S^{N-1} \times S^{N-1}$. Hence, by Doeblin's theorem, there is a unique $m \in M_1(S^{N-1})$ (the probability measures on S^{N-1}) such that

$$\limsup_{T \uparrow \infty} \frac{1}{T} \log\Big(\sup_{\theta \in S^{N-1}} \|P(T, \theta, \cdot) - m\|_{\mathrm{var}} \Big) < 0.$$

Since $m = \int P(T, \theta, \cdot) m(d\theta)$, $T > 0$, it is obvious from the preceding discussion about $p(T, \theta, \cdot)$ that $m(d\eta) = \psi(\eta)\, d\eta$ where $\psi \in C^\infty(S^{N-1})$ is positive everywhere on S^{N-1}. In the future $\int f\, dm$ will be denoted by $\bar{f}$ for $f \in L^1(m)$.

The goal of this article is to prove several results about the behavior of $P(\rho(T,x)/T \in \Gamma)$, $x \in R^N \setminus \{0\}$ and $\Gamma \in \mathscr{B}_R$, as $T \uparrow \infty$. The first statement is a rather abstract existence assertion. Subsequent statements provide more concrete information.

THEOREM 1.7. *There is a lower semicontinuous, convex function* I: $R^1 \to [0, \infty) \cup \{\infty\}$ *such that, for each* $\Gamma \in \mathscr{B}_R$

$$\liminf_{T \uparrow \infty} \frac{1}{T} \log\Big(\inf_x P(\rho(T,x)/T \in \Gamma) \Big) \geq - \inf_{\rho \in \mathrm{int}\,\Gamma} I(\rho) \tag{1.8}$$

and

$$\limsup_{T\uparrow\infty} \frac{1}{T}\log\Big(\sup_x P(\rho(T,x)/T \in \Gamma)\Big) \le -\inf_{\rho\in\bar{\Gamma}} I(\rho), \tag{1.9}$$

where it is to be understood that x *varies over* $R^N \setminus \{0\}$.

In order to describe the function I, it will be useful to have some additional notation. Define the function a and the vector field $\tilde{W}$ on S^{N-1} by

$$a = \sum_{k=1}^{d} \sigma_k^2$$

and

$$\tilde{W} = \sum_{k=1}^{d} \sigma_k W_k,$$

respectively. Set

$$\alpha = \inf\left\{ \sum_{k=1}^{d} \int (\sigma_k - W_k\phi)^2\, dm \colon \phi \in C^\infty(S^{N-1}) \right\}, \tag{1.10}$$

and define the bilinear operation $\langle \cdot, \cdot \rangle$ by

$$\langle \phi_1, \phi_2 \rangle(\theta) = \sum_{k=1}^{d} (W_k\phi_1)(\theta)(W_k\phi_2)(\theta), \qquad \phi_1, \phi_2 \in C^\infty(S^{N-1}).$$

THEOREM 1.11. *Assume that* $\alpha > 0$. *Then*

$$I(\rho) = \sup_{\phi} \inf_{\mu} \left[\left(\rho - \int (Q - L\phi)\, d\mu \right)^2 \Big/ 2\sum_1^d \int (\sigma_k - W_k\phi)^2\, d\mu \right], \tag{1.12}$$

where ϕ *varies over* $C^\infty(S^{N-1})$, μ *varies over* $M_1(S^{N-1})$, *and it is understood that, when* $\sum_1^d \int (\sigma_k - W_k\phi)^2\, d\mu = 0$, *the ratio is* 0 *or* ∞ *according to whether* $\rho = \int (Q - L\phi)\, d\mu$ *or* $\rho \neq \int (Q - L\phi)\, d\mu$. *In particular, there is an* $A \in (0, \infty)$ *such that*:

$$A(\rho - \bar{Q})^2 \le I(\rho) \le (\rho - \bar{Q})^2/2\alpha, \qquad \rho \in R^1, \tag{1.13}$$

and so $I \in C(R^1)$, $I(\bar{Q}) = 0$, *and* I *is strictly increasing* [*decreasing*] *on* (Q, ∞) $[(-\infty, \bar{Q})]$.

THEOREM 1.14. *Assume that* $\alpha = 0$. *Then there is a unique* $f \in C^\infty(S^{N-1})$ *such that* $\bar{f} = 0$ *and* $W_k f = \sigma_k$, $1 \le k \le d$. *Moreover, if* $\hat{Q} \equiv Q - Lf$, *then*

$$I(\rho) = \inf\left\{ J_0(\mu) \colon \mu \in M_1(S^{N-1}) \text{ and } \rho = \int \hat{Q}\, d\mu \right\}, \tag{1.15}$$

where

$$J_0(\mu) = -\inf\left\{ \int \left(\tfrac{1}{2}\langle \phi, \phi \rangle + L\phi \right) d\mu \colon \phi \in C^\infty(S^{N-1}) \right\} \tag{1.16}$$

and it is to be understood that $I(\rho) = \infty$ *if there is no* $\mu \in M_1(S^{N-1})$ *satisfying* $\int \hat{Q}\, d\mu = \rho$. *In particular, if* $\hat{q}_{\pm} \equiv \pm \sup\{\pm \hat{Q}(\theta): \theta \in S^{N-1}\}$, *then* I *is continuous on* $(\hat{q}_-, \hat{q}_+)$ *and is infinite off of* $[\hat{q}_-, \hat{q}_+]$. *Finally,* $I(\overline{Q}) = 0$ *and there is an* $A > 0$ *such that* $I(\rho) \geq A(\rho - \overline{Q})^2$ *for all* $\rho \in R^1$. *In particular,* I *is strictly increasing on* $(\overline{Q}, \infty)$ *and strictly decreasing on* $(-\infty, \overline{Q})$.

REMARK 1.17. Referring to Theorem 1.14, observe that $Lf = \frac{1}{2}\sum_1^d W_k(\sigma_k) + W_0 f$. Thus, $\hat{Q} = \sigma_0 - W_0 f$.

COROLLARY 1.18. *If either* $\alpha > 0$ *or* $\alpha = 0$ *and* $0 \in (q_-, q_+)$, *then for any function* R: $(0,\infty) \to (0,\infty)$ *satisfying* $\lim_{T\to\infty} 1/T \log R(T) = 0$

$$\lim_{T\to\infty} \sup_x \left| \frac{1}{T} \log\big(P(|X(T,x)|/|x| \geq R(T))\big) - \Pi(\overline{Q}) \right| = 0, \tag{1.19}$$

where

$$\Pi(\overline{Q}) = \begin{cases} 0 & \text{if } \overline{Q} \geq 0, \\ -I(0) & \text{if } \overline{Q} < 0. \end{cases}$$

Moreover, if $\alpha = 0$ *and* $\overline{Q} > 0$, *then*

$$\lim_{T\to\infty} \inf_x \frac{1}{T} \log P\big(|X(T,x)|/|x| \geq R(T)\big) = 0. \tag{1.20}$$

Finally, if $\alpha = 0$ *and* $\hat{q}_+ < 0$, *then*

$$\lim_{T\to\infty} \sup_x \frac{1}{T} \log P\big(|X(T,x)|/|x| \geq R(T)\big) = -\infty. \tag{1.21}$$

2. Proofs. The proof of Theorem 1.7 follows the same pattern as that used in Chapter 6 of [4].

Given $x \in R^N \setminus \{0\}$, $T > 0$, and $\Gamma \in \mathscr{B}_R$, set $F(T,x,\Gamma) = P(\rho(T,x)/T \in \Gamma)$. Note that, from (1.3) and (1.4), $F(T,x,\Gamma) = F(T, x/|x|, \Gamma)$ and that for all $T_1, T_2 > 0$

$$\begin{aligned} &P\big((\rho(T_1+T_2,x) - \rho(T_1,x)) \in \Gamma | \mathscr{F}_{T_1}\big) \\ &\quad = F\big(T_2, \theta(T_1,x), \Gamma/T_2\big) \quad \text{a.s. } P. \end{aligned} \tag{2.1}$$

LEMMA 2.2. *There exist constants* $A \in (0,\infty)$ *and* $\varepsilon > 0$ *such that for all* $0 < \delta \leq 1$, $\Gamma \in \mathscr{B}_R$, $T \geq 2$, *and* $(x,y) \in (R^N \setminus \{0\})^2$

$$F(T,x,\Gamma) \leq A\big(F(T,y,\Gamma^{(\delta)}) + \exp(-\varepsilon\delta^2 T^2)\big), \tag{2.3}$$

where $\Gamma^{(\delta)} \equiv \{\rho \in R^1: \operatorname{dist}(\rho,\Gamma) < \delta\}$.

PROOF. First note that, by standard estimates and (1.3), there is a $B \in [1,\infty)$ and an $\varepsilon > 0$ such that $\sup_x P(|\rho(1,x)|/T \geq \delta/2) \leq B\exp(-\varepsilon\delta^2T^2)$ for all $0 < \delta \leq 1$ and $T > 0$. Second, define $M = \sup\{p(1,\theta,\eta)/p(1,\theta',\eta): \theta, \theta', \eta \in S^{N-1}\}$ and observe that for all $x, y \in R^N \setminus \{0\}$, and $f \in B(S^{N-1})^+$

$$E[f(\theta(1,x))] \leq ME[f(\theta(1,y))].$$

Using this in conjunction with (2.1), one now sees that

$$\begin{aligned} F(T,x,\Gamma) &\le P((\rho(T,x)-\rho(1,x))/T\in\Gamma^{(\delta/2)}) + B\exp\left(-\varepsilon(\delta T)^2\right)\\ &= E\left[F\left(T-1,\theta(1,x),\frac{T}{T-1}\Gamma^{(\delta/2)}\right)\right] + B\exp\left(-\varepsilon(\delta T)^2\right)\\ &\le ME\left[F\left(T-1,\theta(1,y),\frac{T}{T-1}\Gamma^{(\delta/2)}\right)\right] + B\exp\left(-\varepsilon(\delta T)^2\right)\\ &= MP((\rho(T,y)-\rho(1,y))/T\in\Gamma^{(\delta/2)}) + B\exp\left(-\varepsilon(\delta T)^2\right)\\ &\le MF(T,y,\Gamma^{(\delta)}) + B(M+1)\exp\left(-\varepsilon(\delta T)^2\right) \end{aligned}$$

for all $T \ge 2$, $0 < \delta \le 1$, and $x, y \in R^N \setminus \{0\}$. Thus (2.3) holds with $A = B(M+1)$. □

For $T > 0$ and $\Gamma \in \mathscr{B}_R$, set

$$\mathscr{P}(T,\Gamma) = \inf_x F(T,x,\Gamma);$$

for $\rho \in R^1$ and $\delta > 0$, define

$$\ell(\rho,\delta) = \inf\left\{-\frac{1}{T}\log\mathscr{P}(T,B(\rho,\delta)) \colon T > 0\right\},$$

where $B(\rho,\delta) = (\rho-\delta,\rho+\delta)$; define

$$G = \{\rho\in R^1\colon (\exists\,\delta>0)\ell(\rho,\delta)=\infty\};$$

and for $\rho \in R^1$, define

$$I(\rho) = \sup\{\ell(\rho,\delta)\colon \delta > 0\}.$$

LEMMA 2.4. *If $\rho \notin G$, then for all $\delta > 0$*

$$\lim_{T\to\infty} -\frac{1}{T}\log\mathscr{P}(T,B(\rho,\delta)) = \ell(\rho,\delta). \tag{2.5}$$

In particular, $I\colon R^1 \to [0,\infty)\cup\{\infty\}$ is lower semicontinuous and convex. Finally, if $\|Q\| = \max_\theta|Q(\theta)|$ and $\|a\| = \max_\theta|a(\theta)|$, then

$$\sup_x P(|\rho(T,x)|/T \ge R) \le 2\exp\left(-T(R-\|Q\|)^2/2\|a\|\right) \tag{2.6}$$

for all $T > 0$ and $R > \|Q\|$.

PROOF. First note that by (2.1), for any $\rho \in R^1$, $r > 0$, $x \in R^N\setminus\{0\}$, and $T_1, T_2 > 0$

$$\begin{aligned} &F(T_1+T_2,x,B(\rho,r))\\ &\quad\ge P\left(\rho(T_1,x)/T_1\in B(\rho,r),\ \frac{\rho(T_1+T_2,x)-\rho(T_1,x)}{T_2}\in B(\rho,r)\right)\\ &\quad= E\left[F(T_2,\theta(T_1,x),B(\rho,r)),\ \rho(T_1,x)/T_1\in B(\rho,r)\right]\\ &\quad\ge \mathscr{P}(T_2,B(\rho,r))F(T_1,x,B(\rho,r)). \end{aligned}$$

Hence,

(2.7) $$\mathscr{P}(T_1 + T_2, B(\rho, r)) \geq \mathscr{P}(T_1, B(\rho, r))\mathscr{P}(T_2, B(\rho, r))$$

for all $\rho \in R^1$, $r > 0$, and $T_1, T_2 > 0$.

Now let $\rho \notin G$ and $\delta > 0$ be given and set $S(T) = -\log \mathscr{P}(T, B(\rho, \delta))$ for $T > 0$. By (2.7) with $r = \delta$, S is subadditive. Thus, the equality $\lim_{T\to\infty} 1/TS(T) = \inf_{T>0} 1/TS(T)$ will follow once it is shown that there exist $0 < T_1 < T_2 < \infty$ such that $\sup\{S(T): T \in [T_1, T_2]\} < \infty$. To this end, note that since $\rho \notin G$, there is a $T_0 > 0$ and a $\beta \in (0,1]$ such that $\mathscr{P}(T_0, B(\rho, \delta/2)) = \beta$. Hence by (2.7) with $r = \delta/2$, $\mathscr{P}(nT_0, B(\rho, \delta/2)) \geq \beta^n$ for all $n \geq 1$. Choose $n_0 \geq 1$ so that $T_1 \equiv n_0 T_0 \geq 2$ and $\gamma \equiv \beta^{n_0} \geq 4A\exp(-\varepsilon(\delta T_1/2)^2)$ [cf. (2.3) for the definition of A and ε], and let θ_0 be a fixed element of S^{N-1}. Then, since $T \to F(T, \theta_0, B(\rho, \delta/2))$ is lower semicontinuous, there is a $T_2 > T_1$ such that $F(T, \theta_0, B(\rho, \delta/2)) \geq \gamma/2$ for $T \in [T_1, T_2]$. Hence, by (2.3) with $\Gamma = B(\rho, \delta/2)$ and $\delta/2$ in place of δ,

$$\begin{aligned}\gamma/2 &\leq F(T, \theta_0, B(\rho, \delta/2)) \\ &\leq A\mathscr{P}(T, B(\rho, \delta)) + A\exp\left(-\varepsilon(\delta T/2)^2\right) \\ &\leq A\mathscr{P}(T, B(\rho, \delta)) + \gamma/4\end{aligned}$$

for all $T \in [T_1, T_2]$. Clearly this proves that $\sup\{S(T): T \in [T_1, T_2]\} < \infty$.

The lower semicontinuity of I is obvious. To prove that I is convex, it suffices to consider $\rho_1, \rho_2 \notin G$. Given $\xi \in (0,1)$ and $\delta > 0$, set $\rho = \xi\rho_1 + (1-\xi)\rho_2$ and choose $\delta' > 0$ so that $\xi B(\rho_1, \delta') + (1-\xi)B(\rho_2, \delta') \subseteq B(\rho, \delta)$. Then, just as in the derivation of (2.7), one can show that

$$\mathscr{P}(T, B(\rho, \delta)) \geq \mathscr{P}(\xi T, B(\rho_1, \delta'))\mathscr{P}((1-\xi)T, B(\rho_2, \delta'))$$

for all $T > 0$. In particular, since $\rho_1, \rho_2 \notin G$ and therefore

$$\lim_{T\to\infty} -\frac{1}{T}\log \mathscr{P}(\xi T, B(\rho_1, \delta')) = \xi\ell(\rho_1, \delta') < \infty$$

and

$$\lim_{T\to\infty} -\frac{1}{T}\log \mathscr{P}((1-\xi)T, B(\rho_1, \delta')) = (1-\xi)\ell(\rho_2, \delta') < \infty,$$

it follows that $\rho \notin G$ and that

$$\begin{aligned}\ell(\rho, \delta) &\leq \xi\ell(\rho_1, \delta') + (1-\xi)\ell(\rho_2, \delta') \\ &\leq \xi I(\rho_1) + (1-\xi)I(\rho_2).\end{aligned}$$

Clearly, this completes the proof that I is convex.

Finally, from (1.3) the derivation of the estimate in (2.6) is standard. □

PROOF OF THEOREM 1.7. In view of Lemma 2.4, we need only prove (1.8) and (1.9). To prove (1.8), let Γ be an open subset of R^1 and suppose that $\rho \in \Gamma$. If $I(\rho) = \infty$, then it is clear that $\liminf_{T\to\infty} 1/T \log \mathscr{P}(T, \Gamma) \geq -I(\rho)$. If $I(\rho) < \infty$ choose $\delta_0 > 0$ so that $B(\rho, \delta_0) \subseteq \Gamma$ and let $0 < \delta \leq \delta_0$ be given. Then, since

$p \notin G$ and therefore (2.5) holds,

$$\liminf_{T\to\infty} \frac{1}{T}\log \mathscr{P}(T, \Gamma) \geq \liminf_{T\to\infty} \frac{1}{T}\log \mathscr{P}(T, B(\rho, \delta)) = -\ell(\rho, \delta).$$

Since $\ell(\rho, \delta) \uparrow I(\rho)$ as $\delta \downarrow 0$, this completes the proof of (1.8).

Next suppose that Γ is a compact subset of R^1 and set $\gamma = \inf\{I(\rho)\colon \rho \in \Gamma\}$. Given $\beta > 0$ and $\rho \in \Gamma \cap G^c$, choose $\delta(\rho) > 0$ so that $\ell(\rho, 2\delta(\rho)) \geq \gamma - \beta$ if $\gamma < \infty$ and $\ell(\rho, 2\delta(\rho)) \geq 1/\beta$ if $\gamma = \infty$. If $\rho \in \Gamma \cap G$, choose $\delta(\rho) > 0$ so that $\ell(\rho, 2\delta(\rho)) = \infty$. Since Γ is compact, there exists an $n \geq 1$ and $\rho_1, \dots, \rho_n \in \Gamma$ so that $\Gamma \subseteq \bigcup_{\nu=1}^n B(\rho_\nu, \delta_\nu)$ where $\delta_\nu = \delta(\rho_\nu)$. Thus, by (2.3) with $\delta = \delta_1 \wedge \cdots \wedge \delta_n$,

$$\begin{aligned} F(T, x, \Gamma) &\leq \sum_{\nu=1}^n F(T, x, B(\rho_\nu, \delta_\nu)) \\ &\leq A\left(\sum_{\nu=1}^n \mathscr{P}(T, B(\rho_\nu, 2\delta_\nu)) + \exp\left(-\varepsilon(\delta T)^2\right)\right) \\ &\leq 2nA \max\left\{\mathscr{P}(T, B(\rho_\nu, 2\delta_\nu)) \vee \exp\left(-\varepsilon(\delta T)^2\right)\colon 1 \leq \nu \leq n\right\} \end{aligned}$$

for all $T \geq 2$ and $x \in R^N \setminus \{0\}$. Note that

$$\lim_{T\to\infty} \frac{1}{T}\log\left(\mathscr{P}(T, B(\rho_\nu, 2\delta_\nu)) \vee \exp\left(-\varepsilon(\delta T)^2\right)\right) = \begin{cases} -\infty & \text{if } \rho_\nu \in G, \\ -\ell(\rho_\nu, 2\delta_\nu) & \text{if } \rho_\nu \notin G. \end{cases}$$

Hence,

$$\limsup_{T\to\infty} \frac{1}{T}\log\left(\sup_x F(T, x, \Gamma)\right) \leq \begin{cases} -1/\beta & \text{if } \gamma = \infty, \\ -\gamma + \beta & \text{if } \gamma < \infty. \end{cases}$$

Thus (1.9) is now proved in the case when Γ is bounded.

To complete the proof of (1.9), let Γ be a given closed subset of R^1 and for $R > \|Q\|$ define $\Gamma_R = \Gamma \cap B(0, R)$. Then, by the preceding plus (2.6),

$$\begin{aligned} \limsup_{T\to\infty} \frac{1}{T}\log\left(\sup_x F(T, x, \Gamma)\right) &\leq \left(-\inf_{\rho\in\overline{\Gamma}_R} I(\rho)\right) \vee \left(-(R - \|Q\|)^2/2\|a\|\right) \\ &\leq \left(-\inf_{\rho\in\Gamma} I(\rho)\right) \vee \left(-(R - \|Q\|)^2/2\|a\|\right) \end{aligned}$$

for all $R > \|Q\|$. Clearly (1.9) follows after one lets $R \uparrow \infty$. □

LEMMA 2.8. *$I(\overline{Q}) = 0$ and $\liminf_{|\rho|\to\infty} I(\rho)/\rho^2 > 0$. Moreover, if $\Phi \in C(R^1)$ satisfies $\limsup_{|\rho|\to\infty} |\Phi(\rho)|/\rho^2 = 0$, then*

$$\lim_{T\to\infty} \sup_x \left|\frac{1}{T}\log E\left[\exp(T\Phi(\rho(T, x)/T))\right] - \Lambda(\Phi)\right| = 0, \tag{2.9}$$

where $\Lambda(\Phi) \equiv \sup_\rho(\Phi(\rho) - I(\rho)) \in R^1$.

PROOF. To prove that $I(\bar{Q}) = 0$, note that, by the ergodic theorem and standards estimates applied to (1.4),

$$\lim_{T\to\infty} \int F\Big(T, \theta, \overline{B(\bar{Q}, \delta)}\Big) m(d\theta) = 1$$

for each $\delta > 0$. Hence, by (1.9),

$$\begin{aligned} 0 &= -\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\int F\Big(T,\theta,\overline{B(\bar{Q},\delta)}\Big)m(d\theta)\right)\right) \\ &\geq -\limsup_{T\to\infty}\frac{1}{T}\log\Big(\sup F\Big(T,\theta,\overline{B(\bar{Q},\delta)}\Big)\Big) \\ &\geq \inf_{\rho\in\overline{B(\bar{Q},\delta)}} I(\rho). \end{aligned}$$

Since I is lower semicontinuous, it follows that $I(\bar{Q}) = 0$. To see that $\liminf_{|\rho|\to\infty} I(\rho)/\rho^2 > 0$, let $R > \|Q\|$ be given. Then, by (1.8) and (2.6),

$$\begin{aligned} (R - \|Q\|)^2/2\|a\| &\leq -\liminf_{T\to\infty}\frac{1}{T}\log\inf_x F\Big(T, x, \overline{B(0,R)}^c\Big) \\ &\leq \inf_{|\rho|>R} I(\rho) \leq I(2R). \end{aligned}$$

Thus, for $|\rho| > 2\|Q\|$, $I(\rho) \geq (\rho - 2\|Q\|)^2/8\|a\|$.

Equation (2.9) is a variation on a lemma first proved by Varadhan in [6]. First note that, from the preceding, $\rho \to \Phi(\rho) - I(\rho)$ is an upper semicontinuous function which tends to $-\infty$ as $|\rho| \to \infty$ and is finite at $\bar{Q}$. Thus there exists a $\rho_0 \notin G$ such that $\Phi(\rho_0) - I(\rho_0) = \Lambda(\Phi) \in R^1$. Given $\delta > 0$, note that

$$\begin{aligned} E\big[\exp(T\Phi(\rho(T,x)/T))\big] &\geq E\big[\exp(T\Phi(\rho(T,x)/T)), \rho(T,x)/T \in B(\rho_0,\delta)\big] \\ &\geq \exp\Big(T\inf_{B(\rho_0,\delta)}\Phi(\rho)\Big)\mathscr{P}(T, B(\rho_0,\delta)). \end{aligned}$$

Thus, by (1.8), for every $\delta > 0$

$$\begin{aligned} \liminf_{T\to\infty}\frac{1}{T}\log\Big(\inf_x E\big[\exp(T\Phi(\rho(T,x)/T))\big]\Big) &\geq \inf_{B(\rho_0,\delta)}\Phi(\rho) - \inf_{B(\rho_0,\delta)} I(\rho) \\ &\geq \inf_{B(\rho_0,\delta)}\Phi(\rho) - I(\rho_0). \end{aligned}$$

Since Φ is continuous, this proves that

$$\liminf_{T\to\infty}\frac{1}{T}\log\Big(\inf_x E\big[\exp(T\Phi(\rho(T,x)/T))\big]\Big) \geq \Phi(\rho_0) - I(\rho_0) = \Lambda(\Phi).$$

To complete the proof, first choose $R_0 > \|Q\|$ so that $|\Phi(\rho)| \leq (\frac{1}{4}\|a\|)\rho^2$ for $|\rho| \geq R_0$. Then, for $R \geq R_0$

$$\begin{aligned} E\big[\exp(T\Phi(\rho(T,x)/T))\big] &= E\big[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T \leq R\big] \\ &\quad + E\big[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T > R\big]. \end{aligned}$$

By (2.6)

$$\begin{aligned} &E\left[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T > R\right] \\ &\quad \le 2\int_R^\infty \exp\left(-\frac{T}{2\|Q\|}((\rho - \|Q\|^2 - \rho^2/2)\right) d\rho \\ &\quad \le K\exp(-\lambda T R^2) \end{aligned}$$

for some $K \in (0, \infty)$ and $\lambda > 0$. Thus, for all $R \ge R_0$

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T}\log\Big(\sup_x E\left[\exp(T\Phi(\rho(T,x)/T))\right]\Big) \\ &\quad \le \limsup_{T\to\infty} \frac{1}{T}\log\Big(\sup_x E\left[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T \le R\right]\Big) \\ &\quad \vee (-\lambda R^2). \end{aligned}$$

Thus, it suffices to prove that for all $R \ge R_0$

$$\text{(2.10)} \quad \limsup_{T\to\infty} \frac{1}{T}\log\sup_x E\left[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T \le R\right] \le \Lambda(\Phi).$$

Let $R \ge R_0$ be fixed and set $M = \max_{|\rho| \le 2R}|\Phi(\rho)|$. Given $\beta > 0$, choose $0 < \delta < R$ so that $\sup\{\Phi(\sigma) - \Phi(\rho): |\sigma| \vee |\rho| \le R$ and $|\sigma - \rho| \le \delta\} < \beta$. Choose $\rho_1, \dots, \rho_n \in \overline{B(0,R)}$ so that $B(0,R) \subseteq \bigcup_1^n B(\rho_\nu, \delta)$. Then

$$E\left[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T \le R\right] \le \sum_{\nu=1}^n e^{\beta T} e^{T\Phi(\rho_\nu)} F\big(T, x, \overline{B(\rho_\nu, \delta)}\big).$$

Hence, by (1.9),

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T}\log\Big(\sup_x E\left[\exp(T\Phi(\rho(T,x)/T)), |\rho(T,x)|/T \le R\right]\Big) \\ &\quad \le \beta + \max_{1\le\nu\le n}\left[\Phi(\rho_\nu) - \inf_{\overline{B(\rho_\nu,\delta)}} I(\rho)\right] \\ &\quad \le 2\beta + \sup_\rho\left[\Phi(\rho) - I(\rho)\right] = 2\beta + \Lambda(\Phi). \end{aligned}$$

□

LEMMA 2.11. *For each* $\lambda \in R^1$ *set* $\Lambda(\lambda) = \sup_\rho[\lambda\rho - I(\rho)]$. *Then* Λ *is a continuous convex function on* R^1,

$$\text{(2.12)} \qquad \lim_{T\to\infty} \sup_x \left|\frac{1}{T}\log(E\left[\exp(\lambda\rho(T,x))\right]) - \Lambda(\lambda)\right| = 0,$$

and

$$\text{(2.13)} \qquad I(\rho) = \sup_\lambda\left[\lambda\rho - \Lambda(\lambda)\right].$$

PROOF. From its definition it is clear that Λ is a lower semicontinuous convex function. Moreover, by Lemma 2.8, $\Lambda(\lambda) \in R^1$ for all $\lambda \in R^1$ and satisfies

(2.12). In particular, Λ must be continuous. Finally, Λ is the Legendre transform of I; and so, since I is lower semicontinuous and convex, I is the Legendre transform of Λ. That is, (2.13) holds. □

LEMMA 2.14. *There is a* $K \in (0, \infty)$ *such that*

$$\|\phi - \bar{\phi}\|^2_{L^2(m)} \le K \int \langle \phi, \phi \rangle \, dm, \qquad \phi \in C^\infty(S^{N-1}). \tag{2.15}$$

PROOF. Define $\hat{W}_k\phi = -(1/\psi)W_k^*(\psi\phi)$, $\phi \in C^\infty(S^{N-1})$, where W_k^* denotes the adjoint of the operator W_k in $L^2(S^{N-1})$. Then $\int \phi_1 \cdot W_k\phi_2 \, dm = -\int \phi_2 \cdot \hat{W}_k\phi_1 \, dm$ for all $\phi_1, \phi_2 \in C^\infty(S^{N-1})$. Thus, if $\hat{L} = \frac{1}{2}\Sigma_1^d \hat{W}_k \circ W_k$ on $C^\infty(S^{N-1})$, then $\hat{L}$ is symmetric in $L^2(m)$ and

$$-\int \phi \hat{L} \phi \, dm = \tfrac{1}{2} \int \langle \phi, \phi \rangle \, dm, \qquad \phi \in C^\infty(S^{N-1}).$$

Thus (2.15) is equivalent to the existence of a $K \in (0, \infty)$ such that

$$\|\phi\|^2_{L^2(m)} \le -2K \int \phi \hat{L} \phi \, dm, \qquad \phi \in C^\infty(S^{N-1}) \quad \text{with } \bar{\phi} = 0. \tag{2.15'}$$

Noting that $\hat{W}_k = W_k + c_k$ where $c_k \in C^\infty(S^{N-1})$, recalling that (1.6) holds, and applying Hörmander's theorem and the strong maximum principle, one concludes that $\hat{L}$ is essentially self-adjoint in $L^2(m)$ and that its self-adjoint extension $\bar{\hat{L}}$ satisfies

$$\exp(t\bar{\hat{L}})(\phi) = \int \phi(\eta) \hat{p}(t, \cdot, \eta) m(d\eta), \qquad \phi \in C^\infty(S^{N-1}),$$

where $\hat{p}$ is a positive element of $C^\infty((0, \infty) \times S^{N-1})$ and, for each $t > 0$, $(\theta, \eta) \in S^{N-1} \times S^{N-1} \to \hat{p}(t, \theta, \eta)$ is a symmetric doubly stochastic kernel. In particular, $\exp(\bar{\hat{L}})$ is a compact self-adjoint operator, all of whose eigenfunctions are in $C^\infty(S^{N-1})$. Thus there exist $0 = \lambda_0 \le \lambda_1 \le \lambda_n \le \cdots$ and an $L^2(S^{N-1})$-orthonormal basis $\{\phi_n\}_0^\infty \subseteq C^\infty(S^{N-1})$ such that $\hat{L}\phi_n = -\lambda_n\phi_n$, $n \ge 0$. Because ϕ_0 may be chosen to be 1, (2.15′) with $2K = 1/\lambda_1$ will follow once it is shown that $\lambda_1 > 0$. To show that $\lambda_1 > 0$, suppose not. Then $\|\phi_1\|_{L^2(m)} = 1$, $\bar{\phi}_1 = 0$, and $\exp(\bar{\hat{L}})\phi_1 = \phi_1$. But if ϕ_1 achieves its maximum value at θ_0, then, from $\phi_1(\theta_0) = \int \phi_1(\eta) p(1, \theta_0, \eta) m(d\eta)$, one has $\phi_1 \equiv \phi_1(\theta_0)$, which clearly contradicts $\|\phi_1\|_{L^2(m)} = 1$ and $\bar{\phi}_1 = 0$. □

Before proceeding, some more notation is required. For $\lambda \in R^1$, define

$$L_\lambda = L + \lambda \tilde{W}$$

on $C^\infty(S^{N-1})$ and

$$J_\lambda(\mu) = -\inf\left\{ \int \frac{L_\lambda u}{u} \, d\mu : u \in C^\infty(S^{N-1}) \text{ and } u > 0 \right\}$$

for $\mu \in M_1(S^{N-1})$. Writing $u = e^\phi$, one sees that an equivalent expression for $J_\lambda(\mu)$ is

$$J_\lambda(\mu) = \sup\left\{ -\int \left(\tfrac{1}{2}\langle \phi, \phi \rangle + L_\lambda \phi \right) d\mu : \phi \in C^\infty(S^{N-1}) \right\}. \tag{2.16}$$

LEMMA 2.17. *$J_0(m) = 0$ and for each $\eta \in C^\infty(S^{N-1})$ there is an $A_\eta \in (0, \infty)$ such that*

$$J_0(\mu) \geq A_\eta\left(\int \eta d_\mu - \bar{\eta}\right)^2, \qquad \mu \in M_1(S^{N-1}). \tag{2.18}$$

Moreover, if $\mu \in M_1(S^{N-1})$ is given by $\mu(d\theta) = g(\theta)m(d\theta)$ where g is a positive element of $C^\infty(S^{N-1})$, then

$$J_0(\mu) \leq K\left\|\frac{1}{\psi}L^*(\psi g)\right\|^2_{L^2(m)}\Big/ 2\min\{g(\theta)\colon \theta \in S^{N-1}\}, \tag{2.19}$$

where K is the constant in (2.15) *and L^* is the adjoint of L in $L^2(S^{N-1})$.*

PROOF. First note that by (2.16)

$$0 \leq J_0(m) = \sup\left(-\tfrac{1}{2}\int \langle \phi, \phi\rangle\, dm\right) \leq 0.$$

Next, given $\eta \in C^\infty(S^{N-1})$, let h be the unique element of $C^\infty(S^{N-1})$ satisfying $Lh = \bar{\eta} - \eta$ and $\bar{h} = 0$. Then, by (2.16)

$$J_0(\mu) \geq -\frac{\lambda^2}{2}\int \langle h, h\rangle\, d\mu + \lambda\left(\int \eta\, d\mu - \bar{\eta}\right)$$

for all $\lambda \in R^1$. In particular,

$$J_0(\mu) \geq \left(\int \eta\, d\mu - \bar{\eta}\right)^2 \Big/ 2\int \langle h, h\rangle\, d\mu$$

$$\geq \left(\int \eta\, d\mu - \bar{\eta}\right)^2 \Big/ 2\|\langle h, h\rangle\|_{C(S^{N-1})}.$$

Thus (2.18) is proved.

To prove (2.19), let $\phi \in C^\infty(S^{N-1})$ be given. Then, by (2.15)

$$\left|\int L\phi\, d\mu\right| = \left|\int L^*(\psi g)\cdot(\phi - \bar{\phi})\, d\eta\right| \leq \|\phi - \bar{\phi}\|_{L^2(m)}\left\|\frac{1}{\psi}\circ L^*(\psi g)\right\|_{L^2(m)}$$

$$\leq K^{1/2}\left\|\frac{1}{\psi}\circ L^*(\psi g)\right\|_{L^2(m)}\left(\int \langle \phi, \phi\rangle\, dm\right)^{1/2}.$$

Hence, if $\varepsilon = \min\{g(\theta)\colon \theta \in S^{N-1}\}$, then

$$-\int\left(\frac{1}{2}\langle \phi, \phi\rangle + L\phi\right) d\mu$$

$$\leq -\frac{\varepsilon}{2}\int \langle \phi, \phi\rangle\, dm + K^{1/2}\left\|\frac{1}{\psi}\circ L^*(\psi g)\right\|_{L^2(m)}\left(\int \langle \phi, \phi\rangle\, dm\right)^{1/2}$$

$$\leq K\left\|\frac{1}{\psi}\circ L^*(\psi g)\right\|^2_{L^2(m)}\Big/ 2\varepsilon,$$

and so (2.19) follows from (2.16). □

REMARK 2.20. Although it will not be used in the present article, one may want to note that if $\mu \in M_1(S^{N-1})$ is given by $\mu(d\theta) = g(\theta)m(d\theta)$ with $g \in C^\infty(S^{N-1})$, then $J_0(\mu) < \infty$ as soon as there exists an $A_\mu \in (0,\infty)$ for which

$$\left|\int W_0\phi\, d\mu\right| \le A_\mu\left(\int\langle\phi,\phi\rangle\, d\mu\right)^{1/2}, \qquad \phi \in C^\infty(S^{N-1}) \tag{2.21}$$

holds. Before proving this, observe that if $g \ge \varepsilon > 0$, then, by (2.15),

$$\begin{aligned}\left|\int W_0\phi\, d\mu\right| &= \left|\int(\hat{W}_0 g)\cdot(\phi - \bar{\phi})\, dm\right| \\ &\le \|\hat{W}_0 g\|_{L^2(m)}\|\phi - \bar{\phi}\|_{L^2(m)} \\ &\le K^{1/2}\|\hat{W}_0 g\|_{L^2(m)}\left(\int\langle\phi,\phi\rangle\, dm\right)^{1/2} \\ &\le (K/\varepsilon)^{1/2}\|\hat{W}_0 g\|_{L^2(m)}\left(\int\langle\phi,\phi\rangle\, d\mu\right)^{1/2},\end{aligned}$$

where $\hat{W}_0$ is defined as in the proof of Lemma 2.14. Thus, (2.21) holds with $A_\mu = (K/\varepsilon)^{1/2}\|\hat{W}_0 g\|_{L^2(m)}$ if $g \ge \varepsilon$. Also, note that if $W_0 = \sum_1^d b_k W_k$ where $\{b_k\}_1^d \subseteq C(S^{N-1})$, then (2.21) holds with $A_\mu = \|(\sum_1^d b_k^2)^{1/2}\|_{L^2(\mu)}$ for every $\mu \in M_1(S^{N-1})$.

To prove that $J_0(\mu) < \infty$ when $\mu(d\theta) = g(\theta)m(d\theta)$ with $g \in C^\infty(S^{N-1})^+$ and (2.21) holds, observe that:

$$\left|\int L\phi\, d\mu\right| \le \left|\sum_1^d\int\hat{W}_k g\cdot W_k\phi\, dm\right| + \left|\int W_0\phi\, d\mu\right|,$$

where $\hat{W}_k$ is defined as in the proof of Lemma 2.14. Since $\hat{W}_k = W_k + c_k$, where $c_k \in C^\infty(S^{N-1})$:

$$\left|\sum_1^d\int\hat{W}_k g\cdot W_k\phi\, dm\right| \le B_1\int g\langle\phi,\phi\rangle^{1/2}\, dm + \int\langle g,g\rangle^{1/2}\langle\phi,\phi\rangle^{1/2}\, dm,$$

where $B_1 = \|(\sum_1^d c_k^2)^{1/2}\|_{C(S^{N-1})}$. Because $g \ge 0$, $(W_k g)^2 \le 2\|W_k^2 g\|_{C(S^{N-1})}g$; and so

$$\langle g,g\rangle^{1/2} \le B_2 g^{1/2},$$

where $B_2 = (2\sum_1^d\|W_k^2 g\|_{C(S^{N-1})})^{1/2}$. Combining these with (2.21), one easily arrives at

$$\begin{aligned}-\int\left(\tfrac{1}{2}\langle\phi,\phi\rangle + L\phi\right) d\mu &\le -\tfrac{1}{2}\int\langle\phi,\phi\rangle\, d\mu + (B_1 + B_2 + A_\mu)\left(\int\langle\phi,\phi\rangle\, d\mu\right)^{1/2} \\ &\le (B_1 + B_2 + A_\mu)^2/2.\end{aligned}$$

LEMMA 2.22. *For each $\lambda \in R^1$ and $H \in C(S^{N-1})$,*

$$\lim_{T\to\infty}\sup_x\left|\frac{1}{T}\log\left(E\left[\exp\left(\lambda\sum_1^d\int_0^T\sigma_k(\theta(t,x))\, d\beta_k(t) + \int_0^T H(\theta(t,x))\, dt\right)\right]\right) - \sup_\mu\left[\int\left(H + \frac{\lambda^2}{2}a\right)d\mu - J_\lambda(\mu)\right]\right| = 0. \tag{2.23}$$

PROOF. Define $\theta_\lambda(\cdot, x)$, $x \in R^N \setminus \{0\}$, by

$$\theta_\lambda(T, x) = \frac{x}{|x|} + \sum_{k=1}^{d} \int_0^T W_k(\theta_\lambda(t, x)) \circ d\beta_k(t) + \int_0^T (W_0 + \lambda \tilde{W})(\theta_\lambda(t, x))\, dt, \qquad T \geq 0.$$

Then, by the Cameron–Martin formula,

$$E\left[\exp\left(\lambda \sum_1^d \int_0^T \sigma_k(\theta(t, x))\, d\beta_k(t) + \int_0^T H(\theta(t, x))\, dt\right)\right] = E\left[\exp\left(\int_0^T H_\lambda(\theta_\lambda(t, x))\, dt\right)\right],$$

where $H_\lambda = H + \lambda^2 a/2$. At the same time, $\theta_\lambda(\cdot, x)$ is the diffusion starting at $x/|x|$ and generated by L_λ; and, because of (1.6), Hörmander's theorem, and the maximum principle, the transition probability function $P_\lambda(T, \theta, d\eta)$ for this diffusion is given by $p_\lambda(T, \theta, \eta)\, d\eta$ with p_λ a positive element of $C^\infty((0, \infty) \times S^{N-1} \times S^{N-1})$. Hence, the theory of Donsker and Varadhan [1] applies and yields (2.23). [See Chapters 6 and 7 of [4], in particular Corollary (7.21), for details.] □

PROOF OF THEOREM 1.11. Assume that $\alpha > 0$ [cf. (1.10)]. Applying (2.23) with $H = \lambda Q$, one sees from (2.12) and (2.16) that

$$\Lambda(\lambda) = \sup_\mu \left[\int \left(\lambda Q + \frac{\lambda^2}{2} a\right) d\mu - J_\lambda(\mu)\right] = \sup_\mu \inf_\phi \left[\int \left(\lambda Q + \frac{\lambda^2}{2} a\right) d\mu + \int \left(\frac{1}{2}\langle \phi, \phi \rangle + L_\lambda \phi\right) d\mu\right].$$

Hence, by (2.13),

$$I(\rho) = \sup_\lambda \inf_\mu \sup_\phi \left[\lambda\rho - \int \left(\lambda Q + \frac{\lambda^2}{2} a\right) d\mu - \int \left(\frac{1}{2}\langle \phi, \phi \rangle + L_\lambda \phi\right) d\mu\right].$$

If $\lambda \neq 0$ (after replacing ϕ by $\lambda\phi$)

$$\inf_\mu \sup_\phi \left[\lambda\rho - \int \left(\lambda Q + \frac{\lambda^2}{2} a\right) d\mu - \int \left(\frac{1}{2}\langle \phi, \phi \rangle + L_\lambda \phi\right) d\mu\right] = \inf_\mu \sup_\phi \left[\lambda\left(\rho - \int (Q - L\phi)\, d\mu\right) - \frac{\lambda^2}{2} \sum_1^d \int (\sigma_k - W_k\phi)^2\, d\mu\right].$$

At the same time

$$0 \leq \inf_\mu \sup_\phi \left[-\int \left(\tfrac{1}{2}\langle \phi, \phi \rangle + L\phi\right) d\mu\right] \leq \sup_\phi \left[-\tfrac{1}{2} \int \langle \phi, \phi \rangle\, dm\right] = 0.$$

Thus

$$I(\rho) = \sup_\lambda \inf_\mu \sup_\phi \left[\lambda\left(\rho - \int (Q - L\phi)\, d\mu\right) - \frac{\lambda^2}{2}\sum_1^d \int (\sigma_k - W_k\phi)^2\, d\mu\right];$$

and so, after two applications of the minimax theorem,

$$I(\rho) = \sup_\phi \inf_\mu \sup_\lambda \left[\lambda\left(\rho - \int (Q - L\phi)\, d\mu\right) - \frac{\lambda^2}{2}\sum_1^d \int (\sigma_k - W_k\phi)^2\, d\mu\right].$$

The expression for $I(\rho)$ given in (1.12) follows immediately from the preceding one.

Starting from (1.12), one has

$$I(\rho) \le \sup_\phi \left[\left(\rho - \int (Q - L\phi)\, d\mu\right)^2 \Big/ 2\sum_1^d \int (\sigma_k - W_k\phi)^2\, d\mu\right]$$
$$\le (\rho - \bar{Q})^2/2\alpha.$$

On the other hand, choosing h for Q as in the proof of (2.18), we see that

$$I(\rho) \ge \inf_\mu \left[(\rho - \bar{Q})^2/2\sum_1^d \int (\sigma_k + W_k h)^2\, d\mu\right] \ge A(\rho - \bar{Q})^2,$$

where $1/A \equiv \|\sum_1^d (\sigma_k + W_k h)^2\|_{C(S^{N-1})} \in (0, \infty)$. Thus, (1.13) has now been proved.

The rest of Theorem 1.11 now follows immediately from (1.13) and standard facts about lower semicontinuous convex function. □

LEMMA 2.24. *If* $\alpha = 0$ [*cf.* (1.10)], *then there is a unique* $f \in C^\infty(S^{N-1})$ *satisfying* $\bar{f} = 0$ *and* $W_k f = \sigma_k$, $1 \le k \le d$.

PROOF. The uniqueness is immediate from (2.15). To prove existence, choose $\{f_n\}_1^\infty \subseteq C^\infty(S^{N-1})$ so that $\bar{f}_n = 0$ and $\sum_1^d \int (\sigma_k - W_k f_n)^2\, dm \to 0$ as $n \to \infty$. By (2.15), there exists an $f \in L^2(m)$ such that $f_n \to f$ in $L^2(m)$. Define $\hat{W}_k$ as in the proof of Lemma 2.14 and note that

$$\int (\hat{W}_k\phi) \cdot f\, dm = \lim_{n\to\infty} \int (\hat{W}_k\phi) \cdot f_n\, dm = \lim_{n\to\infty} - \int \phi W_k f_n\, dm = -\int \phi\sigma_k\, dm$$

for each $1 \le k \le d$ and $\phi \in C^\infty(S^{N-1})$. In particular, if $f \in C^\infty(S^{N-1})$, then $W_k f = \sigma_k$, $1 \le k \le d$. To prove that $f \in C^\infty(S^{N-1})$, define $\hat{L}$ as in the proof of Lemma 2.14 and observe that $\int (\hat{L}\phi) \cdot f\, dm = \int \phi \cdot g\, dm$ for all $\phi \in C^\infty(S^{N-1})$, where $g = \sum_1^d \hat{W}_k(\sigma_k) \in C^\infty(S^{N-1})$. Hence $\hat{L}f = g$ in the sense of distributions, and so by Hörmander's theorem applied to $\hat{L}$, $f \in C^\infty(S^{N-1})$. □

PROOF OF THEOREM 1.14. By Lemma 2.24, f exists and is unique. Hence, by Itô's formula,

$$\rho(T, x) = f(\theta(T, x)) - f\left(\frac{x}{|x|}\right) + \int_0^T \hat{Q}(\theta(t, x))\, dt.$$

Applying (2.23) (with $\lambda = 0$) to (2.12) and using the above expression for $\rho(T, x)$, one sees that

$$\Lambda(\lambda) = \sup_{\mu}\left[\lambda \int \hat{Q}\, d\mu - J_0(\mu)\right].$$

Hence, by (2.13) and the minimax theorem,

$$\begin{aligned} I(\rho) &= \sup_{\lambda}\inf_{\mu}\left[\lambda\left(\rho - \int \hat{Q}\, d\mu\right) + J_0(\mu)\right] \\ &= \inf_{\mu}\sup_{\lambda}\left[\lambda\left(\rho - \int \hat{Q}\, d\mu\right) + J_0(\mu)\right] \\ &= \inf\left\{J_0(\mu)\colon \int \hat{Q}\, d\mu = \rho\right\}. \end{aligned}$$

Thus, (1.15) has been proved. To prove that $I(\rho) \geq A(\rho - \overline{Q})^2$ for some $A \in (0, \infty)$, let h be chosen as in the proof of Lemma 2.17 and set $\hat{h} = h + f$. Then, since $\int \hat{Q}\, dm = \overline{Q}$, $L\hat{h} = \overline{Q} - \hat{Q}$. By repeating the argument used to prove (2.18), only this time using $\hat{h}$ in place of h, one sees that $J_0(\mu) \geq A(\int \hat{Q}\, d\mu - \overline{Q})^2$, $\mu \in M_1(S^{N-1})$, for some $A \in (0, \infty)$. In view of (1.15), this proves that $I(\rho) \geq A(\rho - \overline{Q})^2$.

Next, suppose that $\rho \notin [\hat{q}_-, \hat{q}_+]$. Then there is no $\mu \in M_1(S^{N-1})$ such that $\int \hat{Q}\, d\mu = \rho$ and so $I(\rho) = \infty$. On the other hand, if $\rho \in (\hat{q}_-, \hat{q}_+)$, then there is a positive $g \in C^\infty(S^{N-1})$ such that $\int g\, dm = 1$ and $\int \hat{Q} g\, dm = \rho$. Hence, by (2.19), $J_0(\mu) < \infty$ when $\mu(d\theta) = g(\theta)m(d\theta)$. In particular, by (1.15), $I(\rho) < \infty$.

To complete the proof of Theorem 1.14, it suffices to recall (cf. Lemma 2.8) that $I(\overline{Q}) = 0$. □

Proof of Corollary 1.18. Let $R\colon (0, \infty) \to (0, \infty)$ satisfying $\lim_{T\to\infty} 1/T \log R(T) = 0$ be given. For $\delta > 0$ choose $T_\delta > 0$ so that $|1/T \log R(T)| < \delta$ when $T \geq T_\delta$. Then, for any $x \in R^N \setminus \{0\}$ and $T \geq T_\delta$

$$P(\rho(T, x)/T > \delta) \leq P(|X(T, x)|/|x| \geq R(T)) \leq P(\rho(T, x)/T \geq -\delta);$$

and so, by (1.8) and (1.9),

$$\begin{aligned} -\inf_{\rho > \delta} I(\rho) &\leq \liminf_{T\to\infty} \frac{1}{T}\log\left(\inf_x P(|X(T, x)|/|x| \geq R(T))\right) \\ &\leq \limsup_{T\to\infty} \frac{1}{T}\log\left(\sup_x P(|X(T, x)|/|x| \geq R(T))\right) \qquad (2.25)\\ &\leq -\inf_{\rho \geq -\delta} I(\rho). \end{aligned}$$

Since this is true for every $\delta > 0$ and because, when either $\alpha > 0$ or $\alpha = 0$ and $0 \in (\hat{q}_-, \hat{q}_+)$, I is continuous at 0, one concludes that

$$\lim_{T\to\infty} \sup_x \left|\frac{1}{T}\log(P(|X(T, x)|/|x| \geq R(T))) - \Pi(\overline{Q})\right| = 0,$$

where $\Pi(\bar{Q}) = -\inf I(\rho)$. But, if $\bar{Q} < 0$, then I is increasing on $[0, \infty)$ and so, in this case, $\Pi(\bar{Q}) = -I(0)$. On the other hand, if $\bar{Q} \geq 0$, then $0 \leq \inf_{\rho \geq 0} I(\rho) \leq I(\bar{Q}) = 0$; and so $\Pi(\bar{Q}) = 0$ when $\bar{Q} \geq 0$. Thus (1.19) is proved.

Finally, suppose that $\alpha = 0$. If $\bar{Q} > 0$, then (2.25), with $0 < \delta < \bar{Q}$, implies that

$$0 = -\inf_{\rho > \delta} I(\rho) \leq \liminf_{T \to \infty} \frac{1}{T} \log\Big(\inf_x P\big(|X(T, x)|/|x| \geq R(T)\big)\Big);$$

and so (1.20) follows. On the other hand, if $0 > \hat{q}_+$, then (2.25), with $0 < \delta < -\hat{q}_+$, implies that

$$\limsup_{T \to \infty} \frac{1}{T} \log\Big(\sup_x P\big(|X(T, x)|/|x| \geq R(T)\big)\Big) \leq -\inf_{\rho \geq -\delta} I(\rho) = -\infty;$$

from which (1.21) is immediate. □

REMARK 2.26. It is seldom true that $\alpha = 0$. For example, $\alpha = 0$ implies both that there is *no* $\theta \in S^{N-1}$ for which $\{V_1(\theta), \ldots, V_d(\theta)\}$ spans R^N and that there is *some* $\theta \in S^{N-1}$ at which a vanishes. To see these, first suppose that $\alpha = 0$ and that $\mathrm{span}(\{V_1(\theta_0), \ldots, V_d(\theta_0)\}) = R^N$ for some $\theta_0 \in S^{N-1}$. Then by Lemma 2.24, there is an $f \in C^\infty(S^{N-1})$ satisfying $W_k f = \sigma_k$, $1 \leq k \leq d$. Define $\tilde{f}(x) = f(x/|x|)$ for $x \in R^N \setminus \{0\}$ and note that $(\eta, V_k(\theta_0)) = W_k f(\theta_0) = \sigma_k(\theta_0) = (\theta_0, V_k(\theta_0))$, $1 < k \leq d$, where $\eta = \mathrm{grad}\, \tilde{f}(\theta_0) \in T_{\theta_0}(S^{N-1})$. But, since $\{V_1(\theta_0), \ldots, V_d(\theta_0)\}$ spans R^N, this means that $\eta = \theta_0$ and that $(\eta, \theta_0)_{R^N} = 0$, which is obviously impossible. Second, assuming that $\alpha = 0$, again use Lemma 2.24 to find $f \in C^\infty(S^{N-1})$ with $\sigma_k = W_k f$, $1 \leq k \leq d$. Let $\theta_0 \in S^{N-1}$ be a point at which f is maximal. Then, $W_k f(\theta_0) = 0$, $1 \leq k \leq d$, and so $a(\theta_0) = 0$.

REMARK 2.27. In [3] Pinsky dealt with vector fields V_k given by $V_k(x) = B_k x$, $0 \leq k \leq d$ and $x \in R^N \setminus \{0\}$, where the B_k are $N \times N$ matrices. The additional structure in this case gives rise to several interesting features. In the first place, the condition (1.6) becomes the condition that

$$\mathrm{span}(\{B\theta - (\theta, B\theta)\theta \colon B \in \mathrm{Lie}(B_1, \ldots, B_d)\}) = T_\theta(S^{N-1}), \qquad \theta \in S^{N-1},$$

where $\mathrm{Lie}(B_1, \ldots, B_d)$ is the Lie algebra generated by the matrices B_k, $1 \leq k \leq d$ (i.e., the Lie product here is the commutator corresponding to matrix multiplication). Secondly, and more important, is the observation that the $X(\cdot, x)$ of (1.2) is now given by

$$X(T, x) = A(T)x, \qquad (T, x) \in [0, \infty) \times (R^N \setminus \{0\}),$$

where $A(\cdot)$ is the matrix valued stochastic process determined by

$$A(T) = I + \sum_{k=0}^{d} \int_0^T B_k A(t) \circ d\beta_k(t), \qquad T \geq 0. \tag{2.28}$$

It is therefore natural to transfer questions about $|X(T, x)|/|x|$ to ones about the norm of $A(T)$. Because, for the present purposes, the choice of norm is inconsequential, let $\|A(T)\|$ denote the Hilbert–Schmidt norm of $A(T)$ and set

(2.29) $$K(T) = \log\|A(T)\|.$$

Fix an o.n. basis $\{\theta_1, \dots, \theta_N\}$ in R^N and observe that

$$\rho(T, \theta_1) \le K(T) \le \tfrac{1}{2}\log N + \max_{1 \le i \le N} \rho(T, \theta_i).$$

Hence, by (1.3) and the ergodic theorem

(2.30) $$\lim_{T\to\infty} K(T)/T = \overline{Q} \quad \text{a.s. } P;$$

and, by Theorem 1.7,

(2.31) $$\begin{aligned} -\inf_{\rho > \delta} I(\rho) &\le \liminf_{T\to\infty} \frac{1}{T}\log P(K(T)/T > \delta) \\ &\le \limsup_{T\to\infty} \frac{1}{T}\log P(K(T)/T \ge \delta) \le -\inf_{\rho \ge \delta} I(\rho), \qquad \delta \in R^1. \end{aligned}$$

In particular, by Corollary 1.18, if $\alpha > 0$ or $\alpha = 0$ and $0 \in (\hat{q}_-, \hat{q}_+)$, then for any R: $(0, \infty) \to (0, \infty)$ satisfying $\lim_{T\to\infty} 1/T \log R(T) = 0$

(2.32) $$\lim_{T\to\infty} \frac{1}{T}\log P(K(T)/T \ge R(T)) = \Pi(\overline{Q}),$$

where $\Pi(\overline{Q})$ is the same as it was in that corollary.

For purposes of comparison, it is interesting to look at $\Delta(T) \equiv \log(\det(A(T)))$. Indeed, by Itô's formula for Stratonovich integrals,

$$\det(A(T)) = 1 + \sum_{k=0}^{d} \int_0^T b_k \det(A(t)) \circ d\beta_k(t), \qquad T \ge 0,$$

where $b_k \equiv \operatorname{Trace} B_k$. Hence

$$\det(A(T)) = \exp\left(\sum_{k=1}^{d} b_k \beta_k(T)\right), \qquad T \ge 0,$$

and so

$$\Delta(T) = \sum_{k=1}^{d} b_k \beta_k(T) + b_0 T, \qquad T \ge 0.$$

In particular,

(2.33) $$\lim_{T\to\infty} \Delta(T)/T = b_0 \quad \text{a.s. } P,$$

and, after an elementary computation

(2.34) $$\lim_{T\to\infty} \frac{1}{T}\log P(\Delta(T)/T \ge \delta) = -(\delta - b_0)^2/2H, \qquad \delta \ge b_0,$$

if

$$H \equiv \sum_{k=1}^{d} b_k^2 > 0.$$

Noting that

$$\Delta(T)/N \le K(T), \qquad T \ge 0,$$

one concludes from (2.30) and (2.33) that

$$\overline{Q} \ge b_0/N \tag{2.35}$$

and, so long as $H > 0$, from (2.31) and (2.34)

$$I(\rho) \le (N\rho - b_0)^2/2H, \qquad \rho > \overline{Q}. \tag{2.36}$$

[In the derivation of (2.36), recall that I is increasing on $[\overline{Q}, \infty)$.] In particular, if $H > 0$, then $I(\rho) < \infty$ for all $\rho \ge \overline{Q}$ and so, by Theorem 1.14,

$$\alpha > 0 \quad \text{if } H > 0. \tag{2.37}$$

Note that (2.37) leads to the following statement about matrices: if $\{B\theta - (\theta, B\theta)\theta \colon B \in \text{Lie}(B_1, \dots, B_d)\}$ spans $T_\theta(S^{N-1})$ for each $\theta \in S^{N-1}$ and if $\text{Trace}\, B_k \ne 0$ for some $1 \le k \le d$, then there is no $f \in C^\infty(S^{N-1})$ such that $(\theta, B_k\theta)_{R^N} = (\text{grad}\, \tilde{f}(\theta), B_k\theta)_{R^N}$ for all $\theta \in S^N$ [where $\tilde{f}(x) = f(x/|x|)$, $x \in R^N \setminus \{0\}$]. Surely there is a more direct route to this fact than the one given above. [Such a proof has been found by Peter Baxendale.]

REMARK 2.38. Assume that $\overline{Q} < 0$ and that either $\alpha > 0$ or $\alpha = 0$ and $0 \in (\hat{q}\ , \hat{q}_+)$. Let $R\colon (0, \infty) \to (0, \infty)$ with $\lim_{T\to\infty} 1/T \log R(T) = 0$ be given. Then

$$\lim_{T\to\infty} \sup_x \left| \frac{1}{T} \log\left(P\left(\sup_{t \ge T} |X(t,x)|/|x| \ge R(T) \right) + I(0) \right) \right| = 0. \tag{2.39}$$

In view of (1.19), checking (2.39) comes down to showing that

$$\limsup_{T\to\infty} \sup_x \frac{1}{T} \log\left(P\left(\sup_{t > T} |X(t,x)|/|x| \ge R(T) \right) \right) \le -I(0).$$

To this end, note that

$$P\left(\sup_{t \ge T} |X(t,x)|/|x| \ge R(T) \right) \le \sum_0^\infty J_n(T,x),$$

where

$$J_n(T,x) = P\left(\sup_{T \le t \le T+1} |X(t+n,x)|/|x| \ge R(T) \right).$$

Clearly,

$$\begin{aligned} J_n(T,x) \le{} & P\left(\rho(T+n,x) \ge \log R(T) - (T+n)^{3/4} \right) \\ & + P\left(\sup_{T+n \le t \le T+n+1} \rho(t,x) - \rho(T+n,x) \ge (T+n)^{3/4} \right); \end{aligned}$$

and, by standard estimates, there exist $C \in (0, \infty)$ and $A \in (0, \infty)$ such that

$$P\left(\sup_{0 \le t \le 1} \rho(s+t,x) - \rho(s,x) \ge M \right) \le C \exp(-M^2/2A)$$

for all $(s, x) \in [0, \infty) \times (R^N \setminus \{0\})$ and $M > 0$. Now let $\lambda \in (0, I(0))$ be given and choose $\delta_\lambda > 0$ so that $I(-\delta_\lambda) > \lambda$. Next, choose $T_\lambda \geq (2\lambda A)^2$ so that

$$\left(\frac{1}{T}|\log R(T)|\right) \vee (1/T^{1/4}) < \delta_\lambda/2$$

and [cf. (1.19)]

$$\sup_x P(\rho(T, x)/T \geq -\delta_\lambda) \leq e^{-\lambda T}$$

for all $T \geq T_\lambda$. Then, so long as $T \geq T_\lambda$,

$$\begin{aligned} J_n(T, x) &\leq e^{-\lambda(T+n)} + C\exp\left(-(T+n)^{3/4}/2A\right) \\ &\leq (C+1)e^{-\lambda(T+n)} \end{aligned}$$

for all $n \geq 0$. Hence,

$$\sup_x P\Big(\sup_{t > T} |X(t, x)|/|x| \geq R(T)\Big) \leq \left[(C+1)/(1-e^{-\lambda})\right]e^{-\lambda T}, \qquad T \geq T_\lambda.$$

Since λ was any element of $(0, I(0))$, (2.38) has now been proved.

REMARK 2.40. It must be clear that the analysis given in this article applies equally well in a much broader setting. For example, let M be a connected, compact, Riemannian manifold and let $W_0, \ldots, W_d$ be smooth vector fields on M satisfying $\mathrm{Lie}(W_1, \ldots, W_d) = T(M)$. Next, let $(\beta_0(\cdot), \ldots, \beta_d(\cdot))$ be as before and, for $\theta \in M$, let $\theta(\cdot, \theta)$ be the solution to $d\theta(t, \theta) = \Sigma_0^d W_k(\theta(t, \theta)) \circ d\beta_k(t)$ with $\theta(0, \theta) = \theta$ and denote by $P(t, \theta, \cdot)$ the transition probability function determined by $\{\theta(\cdot, \theta): \theta \in M\}$. Finally, let $\sigma_0, \ldots, \sigma_d \in C^\infty(M)$ and set

$$\begin{aligned} \rho(T, \theta) &= \sum_{k=0}^{d} \int_0^T \sigma_k(\theta(t, \theta)) \circ d\beta_k(t) \\ &= \sum_{k=1}^{d} \int_0^T \sigma_k(\theta(t, \theta))\, d\beta_k(t) + \int_0^T Q(\theta(t, \theta))\, dt, \qquad T \geq 0, \end{aligned}$$

where $Q = \sigma_0 + \frac{1}{2}\Sigma_1^d W_k \sigma_k$. Then, with no essential changes, the analysis given and conclusions drawn in this article can be transferred to the study of $\log P(\rho(T, x)/T \in \Gamma)$ as $T \to \infty$.

Actually, with more work, it is possible to get away from the compact case if one is willing to impose a sufficiently strong ergodicity assumption (e.g., something on the order of hypercontractivity). Such extensions allow one to study the analogue of Pinsky's problem even when the vector fields are not homogeneous.

Acknowledgment. The origin of this paper was a question posed to me by Mark Pinsky. What he wanted to know is whether, at least in the case when $\{V_1(\theta), \ldots, V_d(\theta)\}$ span R^N for each $\theta \in S^{N-1}$, $\lim_{T \to \infty} 1/T \log(P(|X(T, x)| \geq R))$ exists and is independent of $x \in R^N \setminus \{0\}$ and $R > 0$. I profitted greatly from Pinsky's own work [3] on this problem; and it is a pleasure to acknowledge here his contribution to the present article.

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