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VARIANCE OF SET-INDEXED SUMS OF MIXING RANDOM VARIABLES AND WEAK CONVERGENCE OF SET-INDEXED PROCESSES

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A uniform bound is found for the variance of a partial-sum set-indexed process under a mixing condition. Sufficient conditions are given for a sequence of partial-sum set-indexed processes to converge to Brownian motion. The requisite tightness follows from hypotheses on the metric entropy of the class of sets and moment and mixing conditions on the summands. The proof uses a construction of Bass [2]. Convergence of finite-dimensional laws in this context is studied in [16].

1. Introduction. Let $\{\xi_{n,\mathbf{j}}\}_{\mathbf{j}\in J_n}$ be a triangular array which in each level is indexed by the $1/n$ lattice in $(0,1]^d$, namely by

$$J_n := \{(j_1/n,\dots,j_d/n)\colon j_1,\dots,j_d \in \{1,2,\dots,n\}\}.$$

Think of $\xi_{n,\mathbf{j}}$ as spread uniformly over the $1/n$ cube

$$C_{n,\mathbf{j}} := (\mathbf{j} - n^{-1}\mathbf{1}, \mathbf{j}),$$

where $\mathbf{1} := (1,\dots,1)$ and we use the notation $(\mathbf{a},\mathbf{b}]$ for the half-open interval

$$\{(x_1,\dots,x_d)\colon a_i < x_i \le b_i,\ i = 1,\dots,d\}$$

with endpoints $\mathbf{a} = (a_1,\dots,a_d)$, $\mathbf{b} = (b_1,\dots,b_d)$. It then is natural to define the partial-sum set-indexed process based on the nth level as

$$Z_n(A) := \sum_{\mathbf{j}\in J_n} \frac{|A \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\xi_{n,\mathbf{j}}, \qquad A \in \mathscr{B}([0,1]^d),$$

where $|\ |$ is Lebesgue measure and $\mathscr{B}(\cdot)$ is the class of Borel sets in the space $(\cdot)$. In this paper we prove weak convergence of Z_n to Brownian motion, having first restricted its domain of definition to a subset $A \subseteq \mathscr{B}[0,1]^d$ satisfying a metric-entropy bound. We also impose moment and mixing conditions on the $\xi_{n,\mathbf{j}}$. The ingredients of our result are

1. convergence of finite-dimensional laws,
2. variance bounds on arbitrary sums of $\mathbb{Z}^d$-indexed mixing r.v.s,
3. tightness.

Topic 1 is dealt with in [16]. Item 2 forms Sections 2–4 of the present paper and contains results of independent interest which we now pause to describe. From an

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array $\{X_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$ of r.v.s we form the partial-sum process

$$Z(A) := \sum_{\mathbf{i}\in\mathbb{Z}^d} |A \cap C_{\mathbf{i}}| X_{\mathbf{i}}, \qquad A \in \mathscr{B}(\mathbb{R}^d), \tag{1.0.1}$$

where $C_{\mathbf{i}} := (\mathbf{i} - \mathbf{1}, \mathbf{i}]$. Let $\sigma^2 := \sup_{\mathbf{i}} \operatorname{var} X_{\mathbf{i}}$, assumed finite. In Section 3 we prove that under a mixing condition there exists $C < \infty$, depending only on d and the mixing rate, such that

$$\operatorname{var} Z(A) \le C\sigma^2 |A|, \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

We note that an alternative formulation is

$$\operatorname{var} \sum_{\mathbf{i}\in\mathbb{Z}^d} c_{\mathbf{i}} X_{\mathbf{i}} \le C\sigma^2 \sum_{\mathbf{i}\in\mathbb{Z}^d} c_{\mathbf{i}}$$

whatever the weights $c_{\mathbf{i}} \in [0,1]$, and in particular

$$\operatorname{var} \sum_{\mathbf{i}\in I} X_{\mathbf{i}} \le C\sigma^2 \operatorname{card}(I), \qquad I \subseteq \mathbb{Z}^d.$$

Only a logarithmic rate of decay is needed on the mixing coefficient, so that the strength of our result is near to that with the slowest known rate (Peligrad [18]) for the case $d = 1$. But when $d > 1$, geometrical considerations arise which are absent when $d = 1$, and in Section 2 we prove a result in n-dimensional geometry which we call a "bisection lemma," on which our proofs in Sections 3 and 4 depend.

Section 4 extends the foregoing by deriving a uniform integrability result for $Z^2(A)/|A|$ based on uniform integrability of the r.v.s $X_{\mathbf{i}}^2$. The results of Sections 3 and 4 are applied to each level of a triangular array in [16] and in Section 5 of the present paper.

Tightness, item 3, is established in Section 5, on a suitable function space which we shall shortly describe. Here we extend the method of Bass [2] to the dependent case by means of a novel form of blocking. Bass's result (proved earlier by a more intricate method by Alexander and Pyke [1]) needs only second-moment assumptions on i.i.d. summands. Our tightness proof needs $2 + \varepsilon$ moments, where ε can be as small as desired at the cost of requiring a severe mixing rate. The mixing rate also has to be related to the metric entropy bound. With low metric entropy exponent, and moments of high order available, the mixing rate is less. It is, though, for the tightness proof always a *polynomial* mixing rate, and a question we leave open is whether a logarithmic rate, as in our Sections 3 and 4 for variance considerations, would suffice.

We conclude this section by setting up the mixing, weak convergence, and metric entropy apparatus needed subsequently, and stating our main theorem. In terms of *separation distance*

$$\rho(E,F) := \inf_{\mathbf{x}\in E,\, \mathbf{y}\in F} \|\mathbf{x} - \mathbf{y}\|, \qquad E, F \subseteq \mathbb{R}^d,$$

where $\|\ \|$ is supremum norm and $\inf \phi = +\infty$, we shall need three mixing coefficients, of which the first two were used in [16, Section 4]. The *maximal* *correlation coefficient* is, for $n = 1, 2, \ldots$ and $x > 0$,

$$\rho_n(x) := \sup_{\substack{I, J \subseteq J_n \\ \rho(I, J) \geq x}} \sup_{\substack{X \in L_2(\sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in I)) \\ Y \in L_2(\sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in J))}} |\mathrm{corr}(X, Y)|,$$

where $L_2(\cdot)$ is the set of L_2 r.v.s measurable with respect to $(\cdot)$, and degenerate r.v.s have zero correlations with every r.v. The *symmetric* ϕ*-mixing coefficient* is

$$\phi_n(x) := \sup_{\substack{I, J \subset J_n \\ \rho(I, J) > x}} \sup_{\substack{E \in \sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in I) \\ F \in \sigma(\xi_{n,\mathbf{j}}, \mathbf{j} \in J) \\ P(E) > 0,\, P(F) > 0}} \max(|P(E|F) - P(E)|, |P(F|E) - P(F)|).$$

The *absolute regularity coefficient* is

$$\beta_n(x) := \sup_{I, J \subset J_n,\, \rho(I, J) \geq x} \left\| \mathscr{L}\left(\{\xi_{n,\mathbf{j}}\}_{\mathbf{j} \in I \cup J}\right) - \mathscr{L}\left(\{\xi_{n,\mathbf{j}}\}_{\mathbf{j} \in I}\right) \times \mathscr{L}\left(\{\xi_{n,\mathbf{j}}\}_{\mathbf{j} \in J}\right) \right\|_{\mathrm{var}},$$

where $\| \ \|_{\mathrm{var}}$ is variation norm and $\mathscr{L}(\cdot)$ denotes probability law. Again, in the case where the $\xi_{n,\mathbf{j}}$ arise as a normed sum $\xi_{n,\mathbf{j}} := n^{-d/2} X_{(nj_1, \ldots, nj_d)}$, this coefficient is related to the absolute regularity coefficient for the Xs,

$$\beta(x) := \sup_{I, J \subseteq \mathbb{Z}^d,\, \rho(I, J) > x} \| \mathscr{L}(\{X_{\mathbf{i}}\}_{\mathbf{i} \in I \cup J}) - \mathscr{L}(\{X_{\mathbf{i}}\}_{\mathbf{i} \in I}) \times \mathscr{L}(\{X_{\mathbf{i}}\}_{\mathbf{i} \in J}) \|_{\mathrm{var}}, \tag{1.0.2}$$

by $\beta_n(x) \leq \beta(nx)$, so that polynomial decay of $\beta(\cdot)$ corresponds to

$$\beta_n(x) = o\left((nx)^{-b}\right), \qquad nx \to \infty. \tag{1.0.3}$$

When (1.0.3) holds, for the general $\{\xi_{n,\mathbf{j}}\}$ array, we call b the *exponent of absolute regularity*.

Relations among these mixing coefficients and the strong mixing coefficient $\alpha_n(x)$ used in [16, Section 5.8], are as follows. By [18], $\rho_n(x) \leq 2\phi_n(x)$, and clearly $\alpha_n(x) \leq \rho_n(x)$. Since absolute regularity is an L_1 form of ϕ-mixing (cf., e.g., [8]), $\beta_n(x) \leq \phi_n(x)$, and again, clearly $\alpha_n(x) \leq \beta_n(x)$. There is no general relation between $\rho_n(x)$ and $\beta_n(x)$.

We say that Borel sets A, B in $[0,1]^d$ are equivalent if $|A \triangle B| = 0$, and denote by $\mathscr{E}$ the set of equivalence classes. Set-theoretic operations on $\mathscr{E}$ will be performed by first selecting representatives. Thus the Lebesgue disjunction, $d_L(A, B) := |A \triangle B|$, is a metric on $\mathscr{E}$. Since the processes Z_n have d_L-continuous paths we may regard them as $\mathscr{E}$-indexed. The results of [16] continue to provide sufficient conditions for finite-dimensional weak convergence of the Z_n, indexed by $\mathscr{E}$ or some subset, to a correspondingly indexed Wiener limit.

The set $\mathscr{E}$ forms a complete metric space under d_L ([22], II, Section 2). Let $\mathscr{A}$ be a totally bounded subset of $\mathscr{E}$. Its closure $\bar{\mathscr{A}}$ is complete and totally bounded, hence compact. Let $C(\bar{\mathscr{A}})$ be the space of continuous real-valued functions on $\bar{\mathscr{A}}$ with the sup norm$\| \ \|$. Because $\bar{\mathscr{A}}$ is compact, $C(\bar{\mathscr{A}})$ is separable ([11], page 437). Thus $C(\bar{\mathscr{A}})$ is a complete, separable metric space. Let $CA(\bar{\mathscr{A}})$ be the set of *everywhere additive* elements of $C(\bar{\mathscr{A}})$, namely, elements f such that $f(A \cup B) = f(A) + f(B) - f(A \cap B)$ whenever $A, B, A \cup B, A \cap B \in \bar{\mathscr{A}}$. It is a closed subset of $C(\bar{\mathscr{A}})$, and since the Z_n are random elements of $CA(\bar{\mathscr{A}})$, it will be on that space that our weak convergence occurs. A *standard Wiener process* on $\mathscr{A}$ is a random element W of $CA(\mathscr{A})$ whose finite-dimensional laws are Gaussian with $EW(A) \equiv 0$, $E(W(A)W(B)) \equiv |A \cap B|$. In order that W should exist it is necessary (Dudley [10]) that $\mathscr{A}$ satisfy a metric entropy condition. For our tightness proof in Section 5 we need $\mathscr{A}$ to be totally-bounded-with-inclusion, i.e., for every $\varepsilon > 0$ there is a finite set $\mathscr{N}(\mathscr{A}, \varepsilon) \subset \mathscr{E}$, which we take to have minimal cardinality $e^{H(\varepsilon)}$, such that for every $A \in \mathscr{A}$ there exist $A^+, A^- \in \mathscr{N}(\mathscr{A}, \varepsilon)$ with $A^- \subseteq A \subseteq A^+$ and $|A^+ \setminus A^-| \le \varepsilon$. The function H is called the d_L-metric entropy (with inclusion). Its *exponent* is

$$r := \limsup_{\varepsilon \downarrow 0} \frac{\log H(\varepsilon)}{\log(1/\varepsilon)}.$$

We now state our weak convergence theorem. Recall from [16] the notation $\mathscr{J}$ for the set of all half-open intervals $(\mathbf{a}, \mathbf{b}] \subset [0,1]^d$. A *null family* is a collection $\{D_h\}_{0 < h \le h_0} \subset \mathscr{J}$ such that $D_h \subseteq D_{h'}$ for $h \le h'$ and $|D_h| = h$ for each h.

1.1 THEOREM. *Assume*

(i) $E\xi_{n,\mathbf{j}} = 0 \; \forall \; n, \mathbf{j}$;

(ii) *for some* $s > 2$, *the set* $\{|n^{d/2}\xi_{n,\mathbf{j}}|^s\}_{\mathbf{j} \in J_n,\, n = 1,2,\dots}$ *is uniformly integrable*;

(iii) *the exponent r of metric entropy* (*with inclusion*) *satisfies* $r < 1$;

(iv) *the exponent* b *of absolute regularity satisfies* $b \ge ds/(s-2)$ *and* $b > d(1 + r)/(1 - r)$;

(v) *the symmetric* ϕ*-mixing coefficients satisfy* $\sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} \phi_n^{1/2}(n^{-1}2^j) < \infty$;

(vi) *for any null family* $\{D_h\}_{0 < h \le h_0}$ *in* $\mathscr{J}$,

$$\lim_{h \downarrow 0} \limsup_{n \to \infty} \left| \frac{EZ_n^2(D_h)}{|D_h|} - 1 \right| = 0.$$

Then Z_n *converges weakly in* $CA(\mathscr{A})$ *to a standard Wiener process.*

1.2 REMARK. The proof of this result will be concluded in Section 5. For conciseness we have selected only Theorem 4.3 of [16] as the finite-dimensional ingredient of the above result. However, [16, Theorem 4.1] can be used equally well: replace (v) by the weaker

(v′) $$\sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} \rho_n^{1/2}(n^{-1}2^j) < \infty$$

and (vi) by [16, 2.2(ii) or 2.6(i)].

1.3 REMARK. Tightness (Section 5) can be proved by extending Pyke's [20] method instead of Bass's, provided we replace (iv) by

(iv′) $$b > ds \frac{1 + r}{s(1 - r) - 2}.$$

This necessitates $s > 2/(1 - r)$, and is stronger than (iv) for every such s and $0 < r < 1$. However, metric entropy without inclusion can then be used.

1.4. *The stationary normed sum case.* Consider a random field $\{X_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d}$. Its absolute regularity coefficient is given by (1.0.2) and its *maximal correlation* *coefficient* (at separation distance x) by

$$(1.4.1) \qquad \rho(x) := \sup_{\substack{I, J \subseteq \mathbb{Z}^d \\ \rho(I, J) \geq x}} \sup_{\substack{X \in L_2(\sigma(X_{\mathbf{i}}, \mathbf{i} \in I)) \\ Y \in L_2(\sigma(X_{\mathbf{j}}, \mathbf{j} \in J))}} |\mathrm{corr}(X, Y)|.$$

For the stationary case, with a very slight strengthening of 1.1(iv) we arrive at the following simplification of our main result. For proof see Section 5.8.

COROLLARY. *Let* $\xi_{n,\mathbf{j}} := n^{-d/2} X_{(nj_1, \dots, nj_d)}$ *where* $\mathbf{X} := \{X_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^d}$ *is a strictly stationary real random field. Assume*

(i) $EX_{\mathbf{0}} = 0$;
(ii) $E|X_{\mathbf{0}}|^s < \infty$ *for some* $s > 2$;
(iii) $\mathscr{A}$ *has exponent of metric entropy* (*with inclusion*) $r < 1$;
(iv) $\mathbf{X}$ *has absolute regularity coefficient* $\beta(x) = O(x^{-b})$ $(x \to \infty)$ *for some* $b > \max(ds/(s-2), d(1+r)/(1-r))$;
(v) $\mathbf{X}$ *has maximal correlation coefficient* $\rho(\cdot)$ *satisfying* $\sum_{j=1}^{\infty} \rho^{1/2}(2^j) < \infty$;
(vi) $\sum_{\mathbf{i} \in \mathbb{Z}^d} E(X_{\mathbf{0}} X_{\mathbf{i}}) = 1$.

Then Z_n *converges weakly in* $CA(\bar{\mathscr{A}})$ *to a standard Wiener process.*

1.5 REMARK. For a list of the metric entropy exponents of various families $\mathscr{A}$ see, e.g., Pyke [20]. One important case is $\mathscr{A} = \{(\mathbf{0}, \mathbf{x}]: \mathbf{x} \in (0,1]^d\}$, which obviously has $r = 0$. Here the limit process $\{W(\mathbf{0}, \mathbf{x}]\}_{\mathbf{x} \in (0,1]^d}$ has paths which are continuous real-valued functions on $(0,1]^d$, so it can be identified with the Brownian sheet on $(0,1]^d$. For existing results in this setting see Berkes and Morrow [4], Deo [9], Eberlein [12], and Eberlein and Csenki [14]. Our result, specialised to this setting, is not contained in these.

2. Minimal slices of *d*-dimensional sets. Let $|\ |$ denote the Euclidean norm in $\mathbb{R}^m$, for any $m \geq 2$, and denote the ball and sphere of radius r by $B_m(r) := \{\mathbf{x} \in \mathbb{R}^m: |\mathbf{x}| \leq r\}$, $S_m(r) := \{\mathbf{x} \in \mathbb{R}^m: |\mathbf{x}| = r\}$, respectively. We use $|\ |$ also for Lebesgue measure in $\mathbb{R}^m$ and for $(m-1)$-dimensional measure of subsets of $S_m(r)$. A *slice* in $\mathbb{R}^d$ is a set

$$S(\mathbf{c}, a, \eta) := \{\mathbf{x} \in \mathbb{R}^d: a \leq \mathbf{c}'\mathbf{x} \leq a + \eta\},$$

where $\mathbf{c} \in \mathbb{R}^d$, $|\mathbf{c}| = 1$, $a \in \mathbb{R}$, and $\eta > 0$. The *thickness* is η, the *direction* (of the normal to the two bounding hyperplanes) is $\mathbf{c}$, and the *displacement* a. The slice splits a set $A \subseteq \mathbb{R}^d$ into three parts, namely $A \cap S(\mathbf{c}, a, \eta)$ and the two sets

$$A_+ := A \cap \{\mathbf{x} \in \mathbb{R}^d: \mathbf{c}'\mathbf{x} > a + \eta\}, \qquad A_- := A \cap \{\mathbf{x} \in \mathbb{R}^d: \mathbf{c}'\mathbf{x} < a\}.$$

If A is measurable and $|A_+| = |A_-|$ we say the slice *bisects* A. The aim of this section is to prove

2.1 BISECTION LEMMA. *There exist positive constants* C_0, q, *depending only on* d, *such that for all* p *satisfying* $0 < p < 1/d$, *and for every measurable* $A \subset \mathbb{R}^d$ *of finite measure, we can find a slice* S *that bisects* A, *has thickness* $(\frac{1}{2}|A|)^p$, *and is such that*

$$|A \cap S| \leq C_0 \left(\tfrac{1}{2}|A|\right)^{(q+pd)/(q+1)}.$$

2.2 REMARK. The case $d = 1$ is immediate. For the case $d = 2$ we can by exact evaluations reduce the bound to $C_1 m^{1/2+p} L(m)$ where $m := \frac{1}{2}|A|$ and L is slowly varying. In relation to the exponent of m this is sharp, because for a square the exact bound is obviously $m^p\sqrt{2m}$. In dimension $d \geq 3$ we expect similar sharp bounds to be obtainable by refinement of our methods.

2.3 REMARK. Divide the scale on all axes in $\mathbb{R}^d$ by $|A|^{1/d}$; then the following reformulation is obtained:

There exist constants $C, r > 0$, depending only on d, such that for every measurable $A \subset \mathbb{R}^d$ of measure 1, and for every $t \in (0,1)$, we can find a bisecting slice S of thickness t such that $|A \cap S| \leq Ct^r$.

2.4. Assume $d \geq 2$. The proof of 2.1 is obtained by a sequence of lemmas. We have $|B_d(r)| = K_d r^d$ where $K_d = \pi^{d/2}/\Gamma(1 + \frac{1}{2}d)$ ([21]), so $|S_d(r)| = dK_d r^{d-1}$. For any $A \subseteq \mathbb{R}^m$ we denote $l(\mathbf{x}, A) := \inf_{\mathbf{y} \in A}|\mathbf{x} - \mathbf{y}|$ and $N_\varepsilon(A) := \{\mathbf{x} \in \mathbb{R}^m\colon l(\mathbf{x}, A) \leq \varepsilon\}$.

2.5 LEMMA. *Let V_m be any m-dimensional subspace $(1 \leq m < d)$ of $\mathbb{R}^d$; then for $0 < \varepsilon < 1$ the intersection of $N_\varepsilon(V_m)$ with the unit sphere satisfies*

$$mK_m K_{d-m}\varepsilon^{d-m}(1-\varepsilon^2)^{m/2-1} < |N_\varepsilon(V_m) \cap S_d(1)| < mK_m K_{d-m}\varepsilon^{d-m},$$

for $1 < m < d$, and

$$K_1 K_{d-1}\varepsilon^{d-1} < |N_\varepsilon(V_1) \cap S_d(1)| < K_1 K_{d-1}\varepsilon^{d-1}(1-\varepsilon^2)^{-1/2}.$$

2.6 PROOF OF LEMMA 2.5. By a rigid rotation in $\mathbb{R}^d$ we can change V_m into the subspace $\{\mathbf{x} \in \mathbb{R}^d\colon x_1 = \cdots = x_{d-m} = 0\}$, leaving $S_d(r)$ and $B_d(r)$ unaltered. Then clearly

$$N_\varepsilon(V_m) = \left\{\mathbf{x} \in \mathbb{R}^d\colon x_1^2 + \cdots + x_{d-m}^2 \leq \varepsilon^2\right\}.$$

So for $r > \varepsilon$,

$$\begin{aligned}
&|N_\varepsilon(V_m) \cap B_d(r)| \\
&\quad = \int_{-\varepsilon}^{\varepsilon} dx_1 \int_{-(\varepsilon^2 - x_1^2)^{1/2}}^{(\varepsilon^2 - x_1^2)^{1/2}} dx_2 \cdots \int_{-(\varepsilon^2 - x_1^2 - \cdots - x_{d-m-1}^2)^{1/2}}^{(\varepsilon^2 - x_1^2 - \cdots - x_{d-m-1}^2)^{1/2}} dx_{d-m} \\
&\qquad \times \int_{-(r^2 - x_1^2 - \cdots - x_{d-m}^2)^{1/2}}^{(r^2 - x_1^2 - \cdots - x_{d-m}^2)^{1/2}} dx_{d-m+1} \cdots \int_{-(r^2 - x_1^2 - \cdots - x_{d-1}^2)^{1/2}}^{(r^2 - x_1^2 - \cdots - x_{d-1}^2)^{1/2}} dx_d.
\end{aligned}$$

The part of the integral on the second line is $|B_m(\sqrt{r^2 - x_1^2 - \cdots - x_{d-m}^2})| = K_m(r^2 - x_1^2 - \cdots - x_{d-m}^2)^{m/2}$. So

$$|N_\varepsilon(V_m) \cap B_d(r)| = \int_{B_{d-m}(\varepsilon)} K_m(r^2 - |\mathbf{x}|^2)^{m/2}\, d\mathbf{x}$$

and the derivative in r at $r = 1$ is

$$|N_\varepsilon(V_m) \cap S_d(1)| = mK_m \int_{B_{d-m}(\varepsilon)} (1 - |\mathbf{x}|^2)^{m/2-1}\, d\mathbf{x}.$$

Since $1 - |\mathbf{x}|^2$ is bounded above and below by 1 and $1 - \varepsilon^2$, the result follows. □

2.7. If $\mathbf{c}_1, \dots, \mathbf{c}_d$ are linearly independent directions then the intersection $\bigcap_{i=1}^d S(\mathbf{c}_i, a_i, \eta)$ will have finite volume $V_\eta(\mathbf{c}_1, \dots, \mathbf{c}_d)$ which does not depend on the displacements a_i. (We may translate the origin to make the displacements zero.) Given directions $\mathbf{c}_1, \dots, \mathbf{c}_N$, where $N \geq d$, we define

$$V_\eta(\mathbf{c}_1, \dots, \mathbf{c}_N) := \textstyle\sum' V_\eta(\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_d}) \leq \infty,$$

the sum Σ' being over all integers $i_1, \dots, i_d$ with $1 \leq i_1 < i_2 < \cdots < i_d \leq N$. Then define

$$V_{\eta, N} := \inf V_\eta(\mathbf{c}_1, \dots, \mathbf{c}_N),$$

where the infimum is taken over all collections $\mathbf{c}_1, \dots, \mathbf{c}_N$ of N directions.

2.8 Lemma. *There exist positive constants K, q depending on d, such that*

$$V_{\eta, N} < KN^q \eta^d, \qquad \eta > 0,\ N = d, d+1, \dots.$$

2.9 Proof of Lemma 2.8. As in [6, Section X.7], for $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{R}^d$ the set

$$\Pi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m) := \{t_1\boldsymbol{\alpha}_1 + \cdots + t_m\boldsymbol{\alpha}_m : 0 \leq t_i \leq 1, i = 1, \dots, m\}$$

is the *parallelepiped* spanned by *edges* $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m$. If $m = d$ its volume

$$|\Pi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_d)| = |\det(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_d)|, \tag{2.9.1}$$

where $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_d)$ is the matrix with columns $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_d$. If $m < n$ then $\Pi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m)$ lies in the subspace $U(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m)$ spanned by $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m$, and has m-dimensional volume $|\Pi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m)|$ which is zero if $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m$ are linearly dependent, and otherwise is its m-dimensional Lebesgue measure in $U(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m)$. This volume may be obtained recursively as follows. For any $\mathbf{x} \in \mathbb{R}^d$ and subspace U of $\mathbb{R}^d$ the *perpendicular distance* of $\mathbf{x}$ from U is $l(\mathbf{x}, U)$ as defined above, and may also be obtained by expressing $\mathbf{x}$ as $\boldsymbol{\gamma} + \boldsymbol{\beta}$ where $\boldsymbol{\gamma} \in U$ and $\boldsymbol{\beta} \perp U$, and taking $l(\mathbf{x}, U) := |\boldsymbol{\beta}|$. Then

$$\begin{cases} |\Pi(\boldsymbol{\alpha}_1)| = |\boldsymbol{\alpha}_1|, \\ |\Pi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m)| = |\Pi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{m-1})| l(\boldsymbol{\alpha}_m, U(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{m-1})). \end{cases} \tag{2.9.2}$$

Let $C := (\mathbf{c}_1, \dots, \mathbf{c}_d)'$. The set $\bigcap_1^d S(\mathbf{c}_i, 0, \eta)$ consists of all $\mathbf{x}$ for which $\mathbf{c}_i'\mathbf{x} = \eta_i \in [0, \eta]$ for all $i = 1, \dots, d$. So if $\mathbf{c}_1, \dots, \mathbf{c}_d$ are linearly independent this is the set of points $C^{-1}\boldsymbol{\eta}$ where $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)'$ and $0 \leq \eta_i \leq \eta$. It therefore is the parallelepiped $\Pi(C^{-1}\eta\mathbf{e}_1, \dots, C^{-1}\eta\mathbf{e}_d)$ where $\mathbf{e}_i, \dots, \mathbf{e}_d$ are unit vectors in the coordinate directions. So

$$\begin{aligned} V_\eta(\mathbf{c}_1, \dots \mathbf{c}_d) &= |\det(C^{-1}\eta\mathbf{e}_1, \dots, C^{-1}\eta\mathbf{e}_d)| \\ &= |\det(\eta C^{-1})| \\ &= \frac{\eta^d}{|\det(\mathbf{c}_1, \dots, \mathbf{c}_d)|}, \end{aligned} \tag{2.9.3}$$

whence

$$V_\eta(\mathbf{c}_1, \dots, \mathbf{c}_N) = \eta^d \textstyle\sum' \dfrac{1}{|\det(\mathbf{c}_{i_1}, \dots, \mathbf{c}_{i_d})|}.$$

Fix $N > d$ and let

$$\varepsilon := N^{-(d-1)}\frac{1}{2}\,dK_d \Big/ \sum_{p=1}^{d-1} K_p K_{d-p}. \tag{2.9.4}$$

We assume N is large enough so that

$$(1-\varepsilon^2)^{-1/2} < 2, \tag{2.9.5}$$

that is, $\varepsilon < (\frac{3}{4})^{1/2}$. We construct directions $\mathbf{c}_1, \ldots, \mathbf{c}_N$ such that for each $n = d$, $d+1, \ldots, N$ the direction $\mathbf{c}_N$ has the property

$$(P_n) \qquad \begin{cases} \text{for each } p = 1, \ldots, d-1, \\ \text{and for every } p \text{ distinct elements } \mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p} \\ \text{from } \mathbf{c}_1, \ldots, \mathbf{c}_{n-1},\ l\big(\mathbf{c}_n, U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p})\big) > \varepsilon. \end{cases}$$

Take $\mathbf{c}_1, \ldots, \mathbf{c}_d$ to be the coordinate directions $\mathbf{e}_1, \ldots, \mathbf{e}_d$. Then (P_d) is satisfied. Suppose $\mathbf{c}_1, \ldots, \mathbf{c}_{n-1}$ have been found, then the requirement on $\mathbf{c}_n$ that $l(\mathbf{c}_n, U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p})) > \varepsilon$ may be reexpressed as $\mathbf{c}_n \notin N_\varepsilon(U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p}))$, so that $\mathbf{c}_n$, as a point of $S_1(1)$, has to belong to the subset $S_d(1) \setminus N_\varepsilon(U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p}))$. Thus the requirement (P_n) is equivalent to

$$(P_n') \qquad \mathbf{c}_n \in S_d(1) \setminus \bigcup_{p=1}^{d-1} \cup N_\varepsilon\big(U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p})\big),$$

where the inner union is over all selections $\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p}$ of distinct elements of $\{\mathbf{c}_1, \ldots, \mathbf{c}_{n-1}\}$. Now by Lemma 2.5, (2.9.5) and (2.9.4),

$$\begin{aligned} &\left| S_d(1) \cap \left(\bigcup_{p=1}^{d-1} \cup N_\varepsilon\big(U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p})\big)\right)\right| \\ &\quad \le \sum_{p=1}^{d-1} \sum \left| S_d(1) \cap N_\varepsilon\big(U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_p})\big)\right| \\ &\quad = \sum_{p=1}^{d-1} \binom{n-1}{p} \left| S_d(1) \cap N_\varepsilon\big(U(\mathbf{c}_1, \ldots, \mathbf{c}_p)\big)\right| \\ &\quad < \sum_{p=1}^{d-1} \binom{n-1}{p} pK_p K_{d-p} \varepsilon^{d-p} (1-\varepsilon^2)^{-1/2} \\ &\quad < \sum_{p=1}^{d-1} N^{d-1} K_p K_{d-p} \varepsilon . 2 \\ &\quad = dK_d = |S_d(1)|. \end{aligned}$$

Thus, in (P_n') the set to which $\mathbf{c}_n$ is required to belong has positive measure and so a choice of $\mathbf{c}_n$ is possible. We have established that $\mathbf{c}_1, \ldots, \mathbf{c}_N$ can be constructed as claimed.

Now for every selection $\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_d}$ from $\mathbf{c}_1, \ldots, \mathbf{c}_N$, taking the subscripts in increasing order we find from (2.9.1) and (2.9.2) that

$$|\det(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_d})| = |\mathbf{c}_{i_1}| \prod_{m=2}^{d} l\big(\mathbf{c}_{i_m}, U(\mathbf{c}_{i_1}, \ldots, \mathbf{c}_{i_{m-1}})\big)$$
$$> \varepsilon^{d-1};$$

hence, by (2.9.3),

$$V_\eta(\mathbf{c}_1, \ldots, \mathbf{c}_N) < \eta^d {\sum}' \varepsilon^{-(d-1)}$$
$$= \eta^d \binom{N}{d} N^{(d-1)^2} \left(2 \sum_1^{d-1} \frac{K_p K_{d-p}}{dK_d} \right)^{d-1}$$
$$< K \eta^d N^{(d-1)^2 + d}.$$

This holds for all N large enough that (2.9.5) holds, and then, by enlargement of K_0 if necessary, for all $N \geq d$. Lemma 2.8 is proved. □

2.10 LEMMA. *Let $A_1, \ldots, A_N$ be events in some probability space $(\Omega, \mathscr{A}, Q)$ and let $l \in \mathbb{N}$. Then there exists $A_i \in \{A_1, \ldots, A_N\}$ such that*

$$\frac{1}{l} Q(A_i) \leq \frac{1}{N} Q\left(\bigcup_{j=1}^N A_j \right) + \sum_{\substack{j_1 < j_2 < \cdots < j_l \\ j_1, j_2, \ldots, j_l \neq i}} \cdots \sum Q(A_i \cap A_{j_1} \cap \cdots \cap A_{j_l}).$$

2.11 PROOF. This is trivial for $l \geq N$ so assume otherwise; likewise, assume all $Q(A_j) > 0$. Now

$$Q\left(\bigcup_{j=1}^N A_j \right)$$
$$= \sum_{j=1}^N Q(A_j) - \sum_{j_1 < j_2}\sum Q(A_{j_1} \cap A_{j_2}) + \sum_{j_1 < j_2 < j_3}\sum\sum Q(A_{j_1} \cap A_{j_2} \cap A_{j_3}) - \cdots$$
$$= \sum_{j=1}^N \left\{ Q(A_j) - \tfrac{1}{2} \sum_{j_1 \neq j} Q(A_j \cap A_{j_1}) + \tfrac{1}{3} \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \neq j}}\sum Q(A_j \cap A_{j_1} \cap A_{j_2}) - \cdots \right\},$$

so there exists some value of j, say i, for which

$$\frac{1}{N} Q\left(\bigcup_{j=1}^N A_j \right) \geq Q(A_i) - \frac{1}{2} \sum_{j_1 \neq i} Q(A_i \cap A_{j_1})$$
$$+ \frac{1}{3} \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \neq i}}\sum Q(A_i \cap A_{j_1} \cap A_{j_2}) - \cdots.$$

Write $P(\cdot) := Q(\cdot | A_i)$ and $I := \{1, \ldots, N\} \setminus \{i\}$. Define $S_1 := \sum_{j_1 \in J} P(A_{j_1})$ and generally

$$S_k := \sum_{\substack{j_1 < \cdots < j_k \\ j_1, \dots, j_k \in I}} \cdots \sum P(A_{j_1} \cap \cdots \cap A_{j_k}).$$

This is the notation of Feller [15, page 110] from which we also use

$$P_{[r]} := P(\text{exactly } r \text{ of the events } A_j,\ j \in I, \text{ occur}),$$

$$P_r := P(\text{ at least } r \text{ of the events } A_j,\ j \in I, \text{ occur}).$$

The above may now be written

$$(2.11.1) \quad \frac{1}{N} Q\left(\bigcup_{j=1}^{N} A_j \right) \ge Q(A_i)\left(1 - \frac{1}{2}S_1 + \frac{1}{3}S_2 - \cdots + \frac{(-1)^N}{N} S_{N-1}\right)$$

and

$$\begin{aligned}
&\frac{1}{2}S_1 - \frac{1}{3}S_2 + \cdots + \frac{(-1)^N}{N} S_{N-1} \\
&= \sum_{\nu=2}^{N} \frac{(-1)^\nu}{\nu} \sum_{r=\nu-1}^{N-1} \binom{r}{\nu - 1} P_{[r]} \qquad [15, \text{page } 110] \\
&= \sum_{r=1}^{N-1} P_{[r]} \frac{1}{r+1} \sum_{\nu=2}^{r+1} (-1)^\nu \binom{r+1}{\nu} \\
&= \sum_{r=1}^{N-1} \frac{r}{r+1} P_{[r]} \\
&\le \frac{l-1}{l} \sum_{r=1}^{l-1} P_{[r]} + \sum_{r=l}^{N-1} P_{[r]} \\
&\le 1 - l^{-1} + P_l \\
&\le 1 - l^{-1} + S_l \qquad [15, \text{page } 110].
\end{aligned}$$

Thus, from (2.11.1),

$$\begin{aligned}
\frac{1}{N} Q\left(\bigcup_{j=1}^{N} A_j \right) &\ge Q(A_\iota)(l^{-1} - S_l) \\
&= l^{-1} Q(A_i) - \sum_{\substack{j_1 < \cdots < j_l \\ j_1, \dots, j_l \ne \iota}} \cdots \sum Q(A_\iota \cap A_{j_1} \cap \cdots \cap A_{j_l}),
\end{aligned}$$

as claimed. □

2.12 Proof of Lemma 2.1. If $|A| \le 1$ then $|A|^{(q+pd)/(q+1)} \ge |A|$ so the result is trivial. Set $m := \frac{1}{2}|A|$ and $N := [m^{(1-pd)/(q+1)}]$ where [ ] denotes integer part. We shall assume $|A|$ is so large that $N \ge d$, for remaining cases are then covered by increasing C_0.

Take $\eta := m^p$ and let $\mathbf{c}_1, \ldots, \mathbf{c}_N$ be directions for which $V_\eta(\mathbf{c}_1, \ldots, \mathbf{c}_N)$ is so close to its infimum that, by Lemma 2.8,

$$V_\eta(\mathbf{c}_1, \ldots, \mathbf{c}_N) \leq KN^q \eta^d.$$

For each slice $S(\mathbf{c}_i, a, \eta)$ we choose $a = a_i$ so that $|A_+| = |A_-|$. This is practicable since $|A_+| - |A_-|$ is a continuous function of a, tending to $-|A|$ as $a \to +\infty$ and to $|A|$ as $a \to -\infty$. In Lemma 2.10 take $l := d$, take $A_j := A \cap S(\mathbf{c}_j, a_j, \eta)$, and let Q be the uniform probability law on A. Lemma 2.10 yields the existence of one of the bisecting slices, $S(\mathbf{c}_i, a_i, \eta)$, such that

$$\begin{aligned} |A \cap S(\mathbf{c}_i, a_i, \eta)| &\leq \frac{d}{N}|A| + dV_\eta(\mathbf{c}_1, \ldots, \mathbf{c}_N) \\ &\leq \frac{d}{N} 2m + dKN^q m^{pd} \\ &< d(4 + K) m^{(q+pd)/(q+1)}, \end{aligned}$$

as required. □

3. Variance of arbitrarily indexed sums of mixing random variables. This section is concerned with the process Z defined by (1.0.1) on the array $\{X_\mathbf{i}\}_{\mathbf{i} \in \mathbb{Z}^d}$ whose maximal correlation coefficient is given by (1.4.1). We assume the $X_\mathbf{i}$ are centred at means, and denote $\rho := \sum_{j=0}^{\infty} \rho(2^j)$, $\sigma := \sup_\mathbf{i} \|X_\mathbf{i}\|_2$.

3.1 THEOREM. *There exist constants a, b, depending only on d, such that*

$$\|Z(A)\|_2 \leq a e^{b\rho} \sigma |A|^{1/2}, \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

3.2 PROOF. We may assume ρ, σ, and $|A|$ are finite, otherwise there is nothing to prove. Then we may also assume $\sigma = 1$. The proof will use a repeated splitting of a large set A, imitating the one-dimensional version of Peligrad [18]; the idea goes back to Ibragimov [17]. Some new aspects arise because the bisection lemma does not cut the set as simply as in its trite one-dimensional version.

Set

$$\sigma(h) := \sup_{A \in \mathscr{B}(\mathbb{R}^d),\, |A| = h} \|Z(A)\|_2, \qquad \bar{\sigma}(h) := \sup_{h' \leq h} \sigma(h').$$

Observe

$$\|Z(A)\|_2 \leq \sum_{\mathbf{j} \in \mathbb{Z}^d} |A \cap C_\mathbf{j}| \, \|X_\mathbf{j}\|_2 \leq \sum_{\mathbf{j} \in \mathbb{Z}^d} |A \cap C_\mathbf{j}| = |A|,$$

so

$$\sigma(h) \leq h. \tag{3.2.1}$$

Take p in the bisection lemma to be such that the exponent $r := (q + pd)/(q + 1)$ does not have any positive power equal to $\frac{1}{2}$. This is to avoid a minor technicality later. Pick A of positive finite measure $2m$, say. The slice S in the bisection lemma is $S(\mathbf{c}, a, m^p)$ for some $\mathbf{c}$, a. We refer to $\{\mathbf{x}\colon \mathbf{c}'\mathbf{x} > a + m^p\}$ and $\{\mathbf{x}\colon \mathbf{c}'\mathbf{x} < a\}$ as the "sides" of S containing A_+ and A_-, respectively. Since

$|A_+| = |A_-|$ we know

$$m \geq |A^+| = |A^-| \geq m - |A \cap S| \geq m - C_0 m^r.$$

We may find A'_+ such that A'_+ is in the side of S containing A_+, is disjoint from A_+, and has measure $|A'_+| = m - |A_+|$. Thus $|A'_+| \leq C_0 m^r$. Let $A''_+ := A_+ \cup A'_+$; then $|A''_+| = m$. Similarly we construct A'_- and A''_- on the side of S containing A_-. Now if $\mathbf{x} \in A''_+$, $\mathbf{y} \in A''_-$ then (recall $\|\ \|$ is sup norm, $|\ |$ Euclidean)

$$\|\mathbf{x} - \mathbf{y}\| \geq d^{-1/2}|\mathbf{x} - \mathbf{y}| \geq d^{-1/2} m^p.$$

Hence if $C_\mathbf{i}$ and $C_\mathbf{j}$ intersect A''_+ and A''_-, respectively, then

$$\|\mathbf{i} - \mathbf{j}\| \geq d^{-1/2} m^p - 2,$$

whence

$$\begin{aligned} E\big(Z(A''_+) + Z(A''_-)\big)^2 &\leq \big(1 + \rho(d^{-1/2}m^p - 2)\big)\big(E\big(Z^2(A''_+)\big) + E\big(Z^2(A''_-)\big)\big) \\ &\leq \big(1 + \rho(d^{-1/2}m^p - 2)\big) 2\sigma(m)^2. \end{aligned}$$

Since

$$(3.2.2) \qquad Z(A) = Z(A''_+) + Z(A''_-) - Z(A'_+) - Z(A'_-) + Z(A \cap S)$$

the triangle inequality gives

$$\sigma(2m) \leq 2^{1/2}\big(1 + \rho(d^{-1/2}m^p - 2)\big)^{1/2}\sigma(m) + 3\bar{\sigma}(C_0 m^r).$$

Choose $h > 1$, then $h = 2^k m$ where $k \in \mathbb{N}$ and $\frac{1}{2} < m \leq 1$, and we have

$$\sigma(m) \leq 1, \qquad \sigma(2^{j+1}m) \leq \alpha_j \sigma(2^j m) + \beta_j,$$

where

$$\alpha_j := 2^{1/2}\big(1 + \rho(d^{-1/2}2^{jp}m^p - 2)\big)^{1/2}, \qquad \beta_j := 3\bar{\sigma}\big(C_0 2^{jr}\big).$$

Iterating,

$$(3.2.3) \qquad \sigma(h) \leq \prod_{j=0}^{k-1} \alpha_j + \sum_{j=0}^{k-1} \beta_j \prod_{i=j+1}^{k-1} \alpha_i.$$

Let $l \in \mathbb{N}$ satisfy $p > 1/l$. Now $d^{1/2}2^{jp}2^{-p} - 2 > 2^{j/l}$ for $j \geq j_0 \geq 1$, so

$$\begin{aligned} \sum_{j=j_0}^{\infty} \rho(d^{-1/2}2^{jp}m^p - 2) &\leq \sum_{j=1}^{\infty} \rho(2^{j/l}) \\ &= \sum_{j=0}^{\infty} \sum_{i=1}^{l} \rho(2^j 2^{i/l}) \\ &\leq \sum_{j=0}^{\infty} \sum_{i=1}^{l} \rho(2^j) = l\rho. \end{aligned}$$

Thus

$$(3.2.4) \qquad \prod_{j=0}^{\infty} \big(1 + \rho(d^{-1/2}2^{jp}m^p - 2)\big)^{1/2} \leq 2^{j_0/2} e^{l\rho/2} =: a_1 e^{b_1 \rho},$$

and (3.2.3) yields

$$\sigma(h) \leq a_1 e^{b_1\rho}\left(2^{k/2} + 3\sum_{j=0}^{k-1} 2^{(k-1-j)/2}\bar{\sigma}(C_0 2^{jr})\right). \tag{3.2.5}$$

Let ν be the integer such that $r^\nu < \frac{1}{2} < r^{\nu-1}$ (recall that $r^\nu = \frac{1}{2}$ was excluded by choice of r). We shall use (3.2.5) iteratively, ν times, to obtain the result of the theorem,

$$\sigma(h) \leq ae^{b\rho}h^{1/2}. \tag{3.2.6}$$

Applying (3.2.1) to the right-hand side of (3.2.5), if $\nu = 1$ we find

$$\sigma(h) \leq a_1 e^{b_1\rho} 2^{k/2}\left\{1 + 3C_0\left(1 - 2^{-(1/2-r)}\right)^{-1}\right\},$$

and since $2^{k/2} \leq 2^{1/2}h^{1/2}$ the proof is concluded. In the other case, when $r > \frac{1}{2}$, we obtain instead

$$\sigma(h) \leq ae^{b_1\rho}\left\{2^{k/2} + \tfrac{3}{2}C_0\left(2^{r-1/2} - 1\right)^{-1}2^{kr}\right\}$$

whence $\bar{\sigma}(h) \leq a_1' e^{b_1\rho}h^r$ for some a_1'. Substituting this in (3.2.5), if $r^2 < \frac{1}{2}$ we obtain (3.2.6) and otherwise $\sigma(h) \leq a_2 e^{b_2\rho}h^{r^2}$. After a total of ν uses of (3.2.5) we obtain (3.2.6). □

4. Uniform integrability of arbitrarily indexed sums. The setting is the same as Section 3 except that the mixing and moment conditions are strengthened. Again assume the $X_{\mathbf{i}}$ are centred at means, and set

$$\rho' := \sum_{j=0}^{\infty} \rho^{1/2}(2^j), \qquad g(y) := \sup_{\mathbf{i}} E\left(X_{\mathbf{i}}^2 1\{|X_{\mathbf{i}}| > y\}\right).$$

4.1 THEOREM. *If the set of r.v.s* $\{X_{\mathbf{i}}^2\}_{\mathbf{i}\in\mathbb{Z}^d}$ *is uniformly integrable and* $\rho' < \infty$ *then the set of r.v.s* $\{Z^2(A)/|A|\}_{A\in\mathscr{B}(\mathbb{R}^d),\,|A|<\infty}$ *is uniformly integrable. Explicitly, there exist constants* c_1, c_2, c_3, *depending only on* d, *such that*

$$E\left(\frac{Z^2(A)}{|A|}1\left\{\frac{Z^2(A)}{|A|} > y\right\}\right) < \left\{c_1\min(1, y^{-1}) + c_2 g(y^{1/4})\right\}e^{c_3\rho'}, \qquad y > 0.$$

4.2. As in [16] the convention is $0/0 = 0$. When $|A| = 0$ we have $Z(A) = 0$ a.s., consistent with the above. The proof is by truncation and fourth moments. The latter are dealt with by the following lemma, for which we define $\tau := \sup_{\mathbf{i}}\|X_{\mathbf{i}}\|_4$.

4.3 LEMMA. *There exist constants* a', b', *depending only on* d, *such that if* $\tau < \infty$ *and* $\rho' < \infty$,

$$\|Z(A)\|_4 \leq a'e^{b'\rho'}\tau|A|^{1/2}, \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

4.4 PROOF. Again we may assume τ, ρ, and $|A|$ are finite. Divide through by τ, and thus assume $\tau = 1$. Then $\sigma \leq 1$ by Hölder's inequality. Define $\sigma(h)$ as in

Section 3.2 and set

$$\tau(h) := \sup_{A\in\mathscr{B}(\mathbb{R}^d),\,|A|=h} \|Z(A)\|_4, \qquad \bar\tau(h) := \sup_{h'\le h}\tau(h').$$

Similarly to (3.2.1),

$$\bar\tau(h) \le h. \tag{4.4.1}$$

Take $p, r, A, S, A_+, A_-, A'_+, A'_-, A''_+, A''_-$ as in Section 3.2 and write $\rho_0 := \rho^{1/2}(d^{-1/2}m^p - 2)$. Following Peligrad [18], proof of Lemma 3.6, we find

$$\begin{aligned} E(Z(A''_+) + Z(A''_-))^4 &\le \{2^{1/4}(1+7\rho_0)^{1/4}\tau(m) + 2\sigma(m)\}^4 \\ &\le \{2^{1/4}(1+7\rho_0)^{1/4}\tau(m) + 2ae^{b\rho'}m^{1/2}\}^4, \end{aligned}$$

the latter inequality by Theorem 3.1 and since $\rho \le \rho'$. By (3.2.2),

$$\tau(2m) \le 2^{1/4}(1+7\rho_0)^{1/4}\tau(m) + 2ae^{b\rho'}m^{1/2} + 3\bar\tau(C_0m^r).$$

Choose $h > 1$ and take $k \in \mathbb{N}$, $\frac12 < m \le 1$ such that $h = 2^k m$; then

$$\tau(m) \le 1, \qquad \tau(2^{j+1}m) \le \alpha'_j\tau(2^jm) + \beta'_j, \tag{4.4.2}$$

where

$$\begin{aligned} \alpha'_j &:= 2^{1/4}(1 + 7\rho^{1/2}(d^{-1/2}2^{jp}m^p - 2))^{1/4}, \\ \beta'_j &:= 2ae^{b\rho'}2^{j/2}m^{1/2} + 3\bar\tau(C_02^{jr}m^r). \end{aligned}$$

The assumptions made for the present proof imply, similarly to (3.2.4), that

$$\prod_{j=0}^{\infty}(1 + 7\rho^{1/2}(d^{-1/2}2^{jp}m^p - 2))^{1/4} \le a'_1e^{b'_1\rho'}$$

for some constants a'_1, b'_1 depending only on d. Then iteration of (4.4.2) yields

$$\begin{aligned} \tau(h) \le a'_1e^{b'_1\rho'}\Bigg\{2^{k/4} &+ 2ae^{b\rho'}\sum_{j=0}^{k-1}2^{j/2}2^{(k-1-j)/4} \\ &+ 3\sum_{j=0}^{k-1}\bar\tau(C_02^{jr})2^{(k-1-j)/4}\Bigg\}. \end{aligned} \tag{4.4.3}$$

Inserting (4.4.1) in the right-hand side yields

$$\tau(h) \le a_2e^{b_2\rho'}2^{k\max(1/2,\,r)} \le 2a_2e^{b_2\rho'}h^{\max(1/2,\,r)}.$$

If $r < \frac12$ the proof is done; otherwise we insert $\tau(h) \le 2a_2e^{b_2\rho'}h^r$ into (4.4.3) in place of (4.4.1) and conclude in the same way as in Section 3.2. □

4.5 Proof of Theorem 4.1. We follow Peligrad [18] Lemma 3.5. For $y > 0$ put

$$\begin{aligned} X_{\mathbf{i}}^y &:= X_{\mathbf{i}}1\{|X_{\mathbf{i}}| \le y^{1/4}\} - E(X_{\mathbf{i}}1\{|X_{\mathbf{i}}| \le y^{1/4}\}), \\ \bar X_{\mathbf{i}}^y &:= X_{\mathbf{i}}1\{|X_{\mathbf{i}}| > y^{1/4}\} - E(X_{\mathbf{i}}1\{|X_{\mathbf{i}}| > y^{1/4}\}), \end{aligned}$$

$$Z_y(A) := \sum_{\mathbf{i}\in\mathbb{Z}^d}|A\cap C_{\mathbf{i}}|X_{\mathbf{i}}^y, \qquad \bar Z_y(A) := \sum_{\mathbf{i}\in\mathbb{Z}^d}|A\cap C_{\mathbf{i}}|\bar X_{\mathbf{i}}^y, \qquad A \in \mathscr{B}(\mathbb{R}^d).$$

Then $Z(A) = Z_y(A) + \bar{Z}_y(A)$ and $Z^2(A) \le 2Z_y^2(A) + 2\bar{Z}_y^2(A)$, so

$$\begin{aligned}
&\frac{Z^2(A)}{|A|} 1\left\{\frac{Z^2(A)}{|A|} > y\right\} \\
&\quad \le \frac{Z^2(A)}{|A|} 1\left\{\frac{Z_y^2(A)}{|A|} > \frac{y}{4}\right\} + \frac{Z^2(A)}{|A|} 1\left\{\frac{Z_y^2(A)}{|A|} \le \frac{y}{4}, \frac{\bar{Z}_y^2(A)}{|A|} > \frac{y}{4}\right\} \\
&\quad \le \frac{Z^2(A)}{|A|} 1\left\{\frac{Z_y^2(A)}{|A|} > \frac{y}{4}\right\} + 4\frac{\bar{Z}_y^2(A)}{|A|} \\
&\quad \le 2\frac{Z_y^2(A)}{|A|} 1\left\{\frac{Z_y^2(A)}{|A|} > \frac{y}{4}\right\} + 6\frac{\bar{Z}_y^2(A)}{|A|}.
\end{aligned}$$

For $y > 1$,

$$\begin{aligned}
&E\left(\frac{Z^2(A)}{|A|} 1\left\{\frac{Z^2(A)}{|A|} > y\right\}\right) \\
&\quad \le 2\left(\frac{4}{y}\right)^2 \left(\frac{\|Z_y(A)\|_4}{|A|^{1/2}}\right)^4 + 6\left(\frac{\|\bar{Z}_y(A)\|_2}{|A|^{1/2}}\right)^2 \\
&\quad \le 2\left(\frac{4}{y}\right)^2 (a' e^{b'\rho'})^4 \sup_{\mathbf{i}} E\left((X_{\mathbf{i}}^y)^4\right) + 6(a e^{b\rho})^2 \sup_{\mathbf{i}} E\left((\bar{X}_{\mathbf{i}}^y)^2\right) \\
&\quad \le 2\left(\frac{4}{y}\right)^2 (a' e^{b'\rho'})^4 (2y^{1/4})^4 + 6(a e^{b\rho})^2 g(y^{1/4}), \\
&\quad \le (c_1 y^{-1} + c_2 g(y^{1/4})) e^{c_3 \rho'},
\end{aligned}$$

since $\rho \le \rho'$, where $c_3 := \max(4b', 2b)$. For $y < 1$ the above expectation is bounded by $(ae^{b\rho})^2 g(0)$, so by enlarging the c_i if necessary we obtain the claimed bound. □

5. Tightness and weak convergence. We identify which considerations are needed for the tightness component of Theorem 1.1 by stating that component as a separate lemma, 5.1. Its proof occupies Sections 5.3–6. The proof of Theorem 1.1 is concluded in Section 5.7.

We need first the following lemma, a consequence of the contraction property $\|\mathscr{L}(X) \times \mathscr{L}(Z) - \mathscr{L}(Y) \times \mathscr{L}(Z)\|_{\mathrm{var}} \le \|\mathscr{L}(X) - \mathscr{L}(Y)\|_{\mathrm{var}}$ for variation norm.

5.0 LEMMA (Eberlein [13]). *Let $X_1, \dots, X_n$ be random elements of some space. Suppose that for all $k = 1, \dots, n-1$,*

$$\|\mathscr{L}(\{X_1, \dots, X_k\}) \times \mathscr{L}(\{X_{k+1}, \dots, X_n\}) - \mathscr{L}(\{X_1, \dots, X_n\})\|_{\mathrm{var}} \le \varepsilon.$$

Then

$$\|\mathscr{L}(\{X_1, \dots, X_n\}) - \mathscr{L}(X_1) \times \cdots \times \mathscr{L}(X_n)\|_{\mathrm{var}} \le (n-1)\varepsilon.$$

5.1 TIGHTNESS LEMMA. *Assume* 1.1(i), (ii), (iii), (iv) *and*

(B) *there exists* $C < \infty$ *such that for all* $n \geq 1$, $A \in \mathscr{E}$, *and all functions* g: $\mathbb{R} \to \mathbb{R}$ *with the property* $|g(x)| \leq |x| \; \forall \, x$,

$$\operatorname{var}\left(\sum_{j \in J_n} \frac{|A \cap C_{n,\mathrm{j}}|}{|C_{n,\mathrm{j}}|} g(\xi_{n,\mathrm{j}}) \right) \leq C|A|.$$

Then for every $\lambda > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P(w_\delta(Z_n) > \lambda) = 0,$$

where $w_\delta(f) := \sup_{A, B \in \mathscr{A}, |A \triangle B| \leq \delta} |f(A) - f(B)|$ *for* f: $A \to \mathbb{R}$.

5.2 REMARK. We will use g to "layer" the $\xi_{n,\mathrm{j}}$. Thus g will be of the form $g_1(x) := x\, 1\{\alpha \leq |x| < \beta\}$, or $g_2(x) := |g_1(x)|$, for certain constants α, β.

5.3 TIGHTNESS PROOF: FIRST STEPS. Choose r' satisfying both $r < r' < 1$ and $b > d(1 + r')/(1 - r')$. If $r' \geq 1/(s - 1)$ then it is easy to see we can find s' satisfying $2 < s' < s$ and

(5.3.1) $$r' < 1/(s' - 1),$$

(5.3.2) $$b \geq ds'/(s' - 2).$$

Otherwise set $s' := s$. In both cases we have (5.3.1), (5.3.2), and, from 1.1(ii),

$$h(x) := \sup_n \max_{\mathrm{j} \in J_n} E\left(|n^{d/2}\xi_{n,\mathrm{j}}|^{s'} 1\left\{ |n^{d/2}\xi_{n,\mathrm{j}}| \geq x \right\} \right) \to 0, \qquad x \to \infty.$$

For $0 \leq u \leq v \leq \infty$ define

$$\xi_{n,\mathrm{j}}(u, v) := \xi_{n,\mathrm{j}} 1\left\{ u \leq n^{d(s'-2)/(2(s'-1))} |\xi_{n,\mathrm{j}}| < v \right\},$$

$$Z_n(A, u, v) := \sum_{\mathrm{j} \in J_n} \frac{|A \cap C_{n,\mathrm{j}}|}{|C_{n,\mathrm{j}}|} \left(\xi_{n,\mathrm{j}}(u, v) - E\xi_{n,\mathrm{j}}(u, v) \right),$$

$$U_n(A, u, v) := \sum_{\mathrm{j} \in J_n} \frac{|A \cap C_{n,\mathrm{j}}|}{|C_{n,\mathrm{j}}|} |\xi_{n,\mathrm{j}}(u, v)|.$$

Then

(5.3.3)
$$\begin{aligned} EU_n(A, u, v) &\leq \sum_{\mathrm{j} \in J_n} \frac{|A \cap C_{n,\mathrm{j}}|}{|C_{n,\mathrm{j}}|} n^{-d} u^{-(s'-1)} h\left(u n^{d/(2(s'-1))} \right) \\ &= |A| u^{-(s'-1)} h\left(u n^{d/(2(s'-1))} \right). \end{aligned}$$

For $0 < \delta \leq 1$ let $\mathscr{A}_\delta := \{A \setminus B: \; A, B \in \mathscr{A}, |A \setminus B| \leq \delta\}$, and $\|f\|_{\mathscr{A}_\delta} := \sup_{A \in \mathscr{A}_\delta} |f(A)|$. By additivity, $|Z_n(A) - Z_n(B)| \leq |Z_n(A \setminus B)| + |Z_n(B \setminus A)|$; hence, to prove the theorem it suffices to show that for all $\lambda > 0$,

(5.3.4) $$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P(\|Z_n\|_{\mathscr{A}_\delta} > \lambda) = 0.$$

For the metric entropy of $\mathscr{A}_\delta$ (for fixed δ), observe that

$$\operatorname{card}(\mathscr{N}(\mathscr{A}_\delta, \varepsilon)) \le \operatorname{card}(\mathscr{N}(\mathscr{A}_1, \varepsilon)) \le \{\operatorname{card}(\mathscr{N}(\mathscr{A}, \tfrac{1}{2}\varepsilon))\}^2,$$

since if $A_i^- \subseteq A_i \subseteq A_i^+$ and $|A_i^+ \setminus A_i^-| \le \frac{1}{2}\varepsilon$ for $i = 1,2$, then $A_1^- \setminus A_2^+ \subseteq A_1 \setminus A_2 \subseteq A_1^+ \setminus A_2^-$ and

$$(A_1^+ \setminus A_2^-) \setminus (A_1^- \setminus A_2^+) \subseteq (A_1^+ \setminus A_1^-) \cup (A_2^+ \setminus A_2^-)$$

so the left has measure at most ε. Thus $\mathscr{A}_\delta$ has metric entropy at most $2H(\frac{1}{2}\,\cdot\,)$. Since $r' > r$, there exists K such that

$$2H(\tfrac{1}{2}\varepsilon) \le K\varepsilon^{-r'}, \qquad 0 < \varepsilon \le 1. \tag{5.3.5}$$

5.4 TIGHTNESS PROOF: TRUNCATION. Since $Z_n(A) = Z_n(A, 0, a) + Z_n(A, a, \infty)$ and $|Z_n(A, a, \infty)| \le U_n([0,1]^d, a, \infty) + EU_n([0,1]^d, a, \infty)$, we see that

$$\|Z_n\|_{\mathscr{A}_\delta} \le \|Z_n(\cdot, 0, a)\|_{\mathscr{A}_\delta} + U_n([0,1]^d, a, \infty) + EU_n([0,1]^d, a, \infty).$$

By (5.3.3), $U_n([0,1]^d, a, \infty)$ tends to 0 as $n \to \infty$, in L_1 and hence in probability. So instead of (5.3.4) it suffices to prove that for all $\lambda > 0$,

$$\lim_{a,\delta \downarrow 0} \limsup_{n\to\infty} P(\|Z_n(\cdot, 0, a)\|_{\mathscr{A}_\delta} > \lambda) = 0. \tag{5.4.1}$$

5.5 TIGHTNESS PROOF: BLOCKING. Define $m_n := \frac{1}{2}n/[n^{s'/(2(s'-1))}]$, where [ ] is integer part, and set $p_n := n/(2m_n)$. We decompose $[0,1]^d$ into the union of the cubes $C_{p_n,\mathbf{l}}$, $\mathbf{l} \in J_{p_n}$, and then further decompose each $C_{p_n,\mathbf{l}}$ into the disjoint union of those cubes $C_{2p_n,\mathbf{j}}$, $\mathbf{j} \in J_{2p_n}$, it contains. Denote these by $I_{n,\mathbf{l},i}$, $i = 1, 2, \dots, 2^d$, indexed the same way in each $C_{p_n,\mathbf{l}}$ cube. Set $I_{n,i} := \bigcup_{\mathbf{l} \in J_{p_n}} I_{n,\mathbf{l},i}$. Then each $I_{n,i}$ is a regular lattice of cubes of side m_n/n, with separation distances $\ge m_n/n$ in between. For instance, in two dimensions each $C_{p_n,\mathbf{l}}$ is cut into four similar quarters, and $I_{n,1}$ might represent the union of the southwest quarters. Now

$$Z_n(\cdot, 0, a) = \sum_{i=1}^{2^d} Z_n(\cdot \cap I_{n,i}, 0, a);$$

hence, (5.4.1) will follow if we can show that for each $i = 1, 2, \dots, 2^d$ and $\lambda > 0$,

$$\lim_{a,\delta\downarrow 0} \limsup_{n\to\infty} P(\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{\mathscr{A}_\delta} > \lambda) = 0, \tag{5.5.1}$$

where $\lambda/2^d$ has been changed into λ.

Now

$$\begin{aligned} Z_n(\cdot \cap I_{n,i}, 0, a) &= \sum_{\mathbf{l} \in J_{p_n}} Z_n(\cdot \cap I_{n,\mathbf{l},i}, 0, a) \\ &= \sum_{\mathbf{l} \in J_{p_n}} \sum_{\mathbf{j} \in S(n,\mathbf{l},i)} \frac{|\cdot \cap I_{n,\mathbf{l},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\xi_{n,\mathbf{j}}(0, a) - E\xi_{n,\mathbf{j}}(0, a)), \end{aligned}$$

where $S(n,\mathbf{l},i) := \{\mathbf{j}\colon C_{n,\mathbf{j}} \cap I_{n,\mathbf{l},i} \ne \varnothing\}$. For fixed n and i, the sets $S(n,\mathbf{l},i)$ are at separation distances $\geq [m_n]/n$ from each other. On a new probability space construct mutually independent processes $\{\tilde{\xi}_{n,\mathbf{j}}\}_{\mathbf{j}\in S(n,\mathbf{l},i)}$, $\mathbf{l}\in J_{p_n}$, such that each process has the same law as the corresponding $\{\xi_{n,\mathbf{j}}\}_{\mathbf{j}\in S(n,\mathbf{l},i)}$. Keeping to a fixed i, and so omitting it as a subscript, let

$$\tilde{\xi}_{n,\mathbf{j}}(u,v) := \tilde{\xi}_{n,\mathbf{j}}1\left\{u \leq n^{d(s'-2)/(2(s'-1))}|\tilde{\xi}_{n,\mathbf{j}}| < v\right\},$$

$$V_{n,\mathbf{l}}(A,u,v) := \sum_{\mathbf{j}\in S(n,\mathbf{l},i)} \frac{|A\cap I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\left(\tilde{\xi}_{n,\mathbf{j}}(u,v) - E\tilde{\xi}_{n,\mathbf{j}}(u,v)\right),$$

$$\tilde{Z}_n(A,u,v) := \sum_{\mathbf{l}\in J_{p_n}} V_{n,\mathbf{l}}(A,u,v),$$

$$W_{n,\mathbf{l}}(A,u,v) := \sum_{\mathbf{j}\in S(n,\mathbf{l},i)} \frac{|A\cap I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}|\tilde{\xi}_{n,\mathbf{j}}(u,v)|,$$

$$\tilde{U}_n(A,u,v) := \sum_{\mathbf{l}\in J_{p_n}} W_{n,\mathbf{l}}(A,u,v).$$

Then by Lemma 5.0,

$$\begin{aligned}\|\mathscr{L}(Z_n(\cdot\cap I_{n,i},0,a)) - \mathscr{L}(\tilde{Z}_n(\cdot,0,a))\|_{\text{var}} &\leq (p_n^d - 1)\beta_n([m_n]/n)\\ &< n^{ds'/(2(s'-1))}\beta_n\left(\tfrac{1}{4}n^{-s'/(2(s'-1))}\right)\\ &= o\left(n^{(ds'-b(s'-2))/(2(s'-1))}\right) = o(1),\end{aligned}$$

using (5.3.2). In particular,

$$\left|P\left(\|Z_n(\cdot\cap I_{n,i},0,a)\|_{\mathscr{A}_\delta} > \lambda\right) - P\left(\|\tilde{Z}_n(\cdot,0,a)\|_{\mathscr{A}_\delta} > \lambda\right)\right| = o(1), \qquad n\to\infty,$$

and so for (5.5.1) it suffices to show that for all $\lambda > 0$,

$$\lim_{a,\delta\downarrow 0}\limsup_{n\to\infty} P\left(\|\tilde{Z}_n(\cdot,0,a)\|_{\mathscr{A}_\delta} > \lambda\right) = 0. \tag{5.5.2}$$

We do this by a modification of Bass's [2] technique. For the purpose of using Bernstein's inequality, note that $\tilde{Z}_n(A,u,v)$ is the sum of independent summands $V_{n,\mathbf{l}}(A,u,v)$, $\mathbf{l}\in J_{p_n}$, with a.s. bounds as follows:

$$\begin{aligned}|V_{n,\mathbf{l}}(A,u,v)| &\leq 2vn^{-d(s'-2)/(2(s'-1))}\sum_{\mathbf{j}\in S(n,\mathbf{l},i)} |I_{n,\mathbf{l},i}\cap C_{n,\mathbf{j}}|/|C_{n,\mathbf{j}}|\\ &= 2vn^{-d(s'-2)/(2(s'-1))}n^d|I_{n,\mathbf{l},i}| < 2v,\end{aligned}$$

while

$$\begin{aligned}\operatorname{var}\tilde{Z}_n(A,u,v) &= \sum_{\mathbf{l}\in J_{p_n}}\operatorname{var}V_{n,\mathbf{l}}(A,u,v)\\ &= \sum_{\mathbf{l}\in J_{p_n}}\operatorname{var}Z_n(A\cap I_{n,\mathbf{l},i},u,v)\\ &\leq \sum_{\mathbf{l}\in J_{p_n}} C|A\cap I_{n,\mathbf{l},i}| \leq C|A|.\end{aligned}$$

Similarly, $\tilde{U}_n(A, u, v) - E\tilde{U}_n(A, u, v)$ is the sum of independent summands $W_{n,1}(A, u, v) - EW_{n,1}(A, u, v)$, each with absolute value bounded by v, and again

$$\operatorname{var} \tilde{U}_n(A, u, v) \leq C|A|.$$

Lastly,

$$E\tilde{U}_n(A, u, v) = EU_n(A \cap I_{n,i}, u, v) \tag{5.5.3}$$
$$\leq |A| u^{-(s'-1)} h(0)$$

by (5.3.3).

5.6 Tightness proof: Nesting. We adapt Bass [2], proof of Theorem 6.1. We shall choose λ_i, δ_i, a_i later so that

$$\begin{cases} \lambda_i \downarrow 0, & \lambda \geq 4 \sum_{i=0}^{\infty} \lambda_i, \\ \delta_0 = \delta, & \delta_i \downarrow 0, \\ a_0 = a, & a_i \downarrow 0. \end{cases} \tag{5.6.1}$$

For any $A \in \mathscr{A}_\delta$ there exist $A_i, A_i^+ \in \mathscr{N}(\mathscr{A}_\delta, \delta_i)$ such that $A_i \subseteq A \subseteq A_i^+$ and $|A_i^+ \setminus A_i| \leq \delta_i$. Then

$$\tilde{Z}_n(A, 0, a) = \tilde{Z}_n(A_0, 0, a_0) + \sum_{i=0}^{\infty} \{\tilde{Z}_n(A_{i+1}, 0, a_i) - \tilde{Z}_n(A_i, 0, a_i)\}$$
$$+ \sum_{i=1}^{\infty} \{\tilde{Z}_n(A, a_i, a_{i-1}) - \tilde{Z}_n(A_i, a_i, a_{i-1})\}.$$

So if $\|Z_n(\cdot, 0, a)\|_{\mathscr{A}_\delta}$ is to exceed λ, at least one of the following must hold:

(a) for some $A_0 \in \mathscr{N}(\mathscr{A}_\delta, \delta_0)$, $|\tilde{Z}_n(A_0, 0, a_0)| > \lambda_0$;

(b) for some i, for some $A_i \in \mathscr{N}(\mathscr{A}_\delta, \delta_i)$, $A_{i+1} \in \mathscr{N}(\mathscr{A}_\delta, \delta_{i+1})$, $|A_{i+1} \triangle A_i| \leq 2\delta_i$,

$$|\tilde{Z}_n(A_{i+1}, 0, a_i) - \tilde{Z}_n(A_i, 0, a_i)| > 2\lambda_i;$$

(c) for some i, for some $A_i, A_i^+ \in \mathscr{N}(\mathscr{A}_\delta, \delta_i)$, $A \in A_\delta$, $A_i \subseteq A \subseteq A_i^+$, $|A_i^+ \setminus A_i| \leq \delta_i$,

$$|\tilde{Z}_n(A, a_i, a_{i-1}) - \tilde{Z}_n(A_i, a_i, a_{i-1})| > \lambda_i.$$

The number of pairs A_i, A_i^+ in $\mathscr{N}(\mathscr{A}_\delta, \delta_i)$ is $\leq \exp(4H(\frac{1}{2}\delta_i))$, while the number of pairs $A_i \in \mathscr{N}(\mathscr{A}_\delta, \delta_i)$, $A_{i+1} \in \mathscr{N}(\mathscr{A}_\delta, \delta_{i+1})$ is $\leq \exp(4H(\frac{1}{2}\delta_{i+1}))$ since $H(\cdot)$ is nonincreasing.

We have

$$P\big(\|\tilde{Z}_n(\cdot, 0, a)\|_{\mathscr{A}_\delta} > \lambda\big) \leq p_0 + \sum_{i=0}^{\infty} r_i + \sum_{i=1}^{\infty} s_i,$$

where

$$p_0 = \exp\left(2H\left(\tfrac{1}{2}\delta_0\right)\right) \max_{|A_0| \le 2\delta_0} P\left(|\tilde{Z}_n(A_0, 0, a_0)| > \lambda_0\right),$$

$$r_i = \exp\left(4H\left(\tfrac{1}{2}\delta_{i+1}\right)\right) \max_{|A_{i+1} \triangle A_i| \le 2\delta_i} \left\{P\left(|\tilde{Z}_n(A_{i+1} \setminus A_i, 0, a_i)| > \lambda_i\right)\right.$$
$$\left. + P\left(|\tilde{Z}_n(A_i \setminus A_{i+1}, 0, a_i)| > \lambda_i\right)\right\},$$

$$s_i = \exp\left(4H\left(\tfrac{1}{2}\delta_i\right)\right) \max_{A_i \subseteq A_i', \, |A_i' \setminus A_i| \le \delta_i} P\left(\sup_{A_i \subseteq A \subseteq A_i'} |\tilde{Z}_n(A, a_i, a_{i-1})\right.$$
$$\left. - \tilde{Z}_n(A_i, a_i, a_{i-1})| > \lambda_i\right).$$

To estimate, first, p_0, recall that $\tilde{Z}_n(A_0, 0, a_0)$ is a mean-zero r.v. of variance $\le C|A_0|$, and is the sum of independent summands each bounded in absolute value by $2a_0$. So by Bernstein's inequality ([3]),

$$p_0 \le 2\exp\left(2H\left(\frac{1}{2}\delta_0\right)\right)\exp\left(\frac{-\lambda_0^2}{4C\delta_0 + 4a_0\lambda_0/3}\right).$$

Similarly, bounding each probability in the expression for r_i, we obtain

$$r_i \le 4\exp\left(4H\left(\frac{1}{2}\delta_{i+1}\right)\right)\exp\left(\frac{-\lambda_i^2}{4C\delta_i + 4a_i\lambda_i/3}\right).$$

To estimate s_i, observe that

$$\sup_{A_i \subseteq A \subseteq A_i'} |\tilde{Z}_n(A, a_i, a_{i-1}) - \tilde{Z}_n(A_i, a_i, a_{i-1})|$$
$$\le \tilde{U}_n(A_i^+ \setminus A_i, a_i, a_{i-1}) + E\tilde{U}_n(A_i^+ \setminus A_i, a_i, a_{i-1})$$
$$\le \tilde{U}_n(A_i^+ \setminus A_i, a_i, a_{i-1}) - E\tilde{U}_n(A_i^+ \setminus A_i, a_i, a_{i-1})$$
$$+ 2|A_i^+ \setminus A_i| a_i^{-(s'-1)} h(0)$$

by (5.5.3). We insist that

$$2\delta_i a_i^{-(s'-1)} h(0) \le \lambda_i/3, \qquad i = 0, 1, 2, \ldots. \tag{5.6.2}$$

Then

$$s_i \le \exp\left(4H\left(\frac{1}{2}\delta_i\right)\right) \max_{|B| \le \delta_i} P\left(\tilde{U}_n(B, a_i, a_{i-1}) - E\tilde{U}_n(B, a_i, a_{i-1}) > \frac{\lambda_i}{3}\right)$$
$$\le \exp\left(4H\left(\frac{1}{2}\delta_i\right)\right)\exp\left(\frac{-(\lambda_i/3)^2}{2C\delta_i + 2a_{i-1}(\lambda_i/3)/3}\right).$$

Let us set $\delta_i := 2^{-i}\delta$ and $\lambda_i := \lambda_0 2^{-i(1+r'-r's')/s'}$ where $\lambda_0 := \frac{1}{4}\lambda(1 - 2^{-(1+r'-r's')/s'})$. Take $a_i := 2^{-i(1+r')/s'}a$ where $a := c\delta^{1/(s'-1)}$ and $c := (6h(0)/\lambda_0)^{1/(s'-1)}$. Then (5.6.1) and (5.6.2) are satisfied. We now have

$$r_i \le 4\exp\left(4K\delta_i^{-r'} - \frac{3\lambda_i^2}{12C\delta_i + 4a_i\lambda_i}\right)$$

by (5.3.5), while

$$s_\iota \leq \exp\left(2K\delta_\iota^{-r'} - \frac{\lambda_\iota^2}{18C\delta_i + 2.2^{(1+r')/s'}a_\iota\lambda_\iota}\right).$$

Thus

$$p_0 + \sum_0^\infty r_\iota + \sum_1^\infty s_\iota \leq 6\sum_{i=0}^\infty \exp\left(4K\delta_\iota^{-r'} - \frac{\lambda_\iota^2}{18C\delta_\iota + 4.2^{(1+r')/s'}a_\iota\lambda_\iota}\right).$$

Now

$$\begin{aligned} 18C\delta_\iota + 4\cdot 2^{(1+r')/s'}a_\iota\lambda_\iota &= a_\iota\lambda_\iota\left\{4.2^{(1+r')/s'} + 18C\delta\lambda_0^{-1}2^{-\iota(1+r')(s'-2)/s'}\right\} \\ &\leq c'a_\iota\lambda_\iota, \end{aligned}$$

where $c' := 4\cdot 2^{(1+r')/s'} + 18C/\lambda_0$. So

$$\begin{aligned} \sup_n P\big(\|\tilde{Z}_n(\cdot,0,a)\|_{\mathscr{A}_\delta} > \lambda\big) &\leq 6\sum_{\iota=0}^\infty \exp\left(4K\delta_\iota^{-r'} - \frac{\lambda_\iota}{c'a_\iota}\right) \\ &= 6\sum_{i=0}^\infty \exp\left\{\left(4K\delta^{-r'} - \frac{\lambda_0}{cc'}\delta^{-1/(s'-1)}\right)2^{\iota r'}\right\}. \end{aligned}$$

Because of (5.3.1) the coefficient of $2^{\iota r'}$ may be made negative as large as desired, by choosing δ small enough, and (5.5.2) follows, concluding the proof of Lemma 5.1. □

5.7 **Proof of Theorem 1.1.** For $f \in C(\bar{\mathscr{A}})$ we can use w_δ as defined in Lemma 5.1 as a modulus of continuity, since $w_\delta(f) = \sup_{A,B\in\bar{\mathscr{A}},|A\triangle B|\leq\delta}|f(A) - f(B)|$. Then, since $\bar{\mathscr{A}}$ is compact, we have available a version of the Arzelá–Ascoli theorem: a subset U of $C(\bar{\mathscr{A}})$ has compact closure if and only if it is equibounded ($\sup_{f\in U}\sup_{A\in\bar{\mathscr{A}}}|f(A)| < \infty$) and equicontinuous ($\lim_{\delta\downarrow 0}\sup_{f\in U}w_\delta(f) = 0$). Using this, the proof of [5] Theorem 8.2 essentially carries over to give the following characterisation of relative compactness of a sequence $\{S_n\}$ of random elements of $C(\bar{\mathscr{A}})$: every subsequence of $\{S_n\}$ contains a weakly convergent subsequence if and only if

(a) for each element A of some countable dense set in $\mathscr{A}$, the family $\{S_n(A)\}_{n\geq 1}$ is tight, and
(b) for every $\lambda > 0$, $\lim_{\delta\downarrow 0}\limsup_{n\to\infty}P(w_\delta(S_n) > \lambda) = 0$.

Under the conditions of Theorem 1.1 we have convergence of finite-dimensional laws of Z_n, to standard Wiener laws, by [16] Theorem 4.3. Since this also gives (a), while (b) is the conclusion of Lemma 5.1, the theorem is complete once the remaining condition ((B)) of Lemma 5.1 is checked. Fix n and f and write temporarily $X_{(nj_1,\ldots,nj_d)} := n^{d/2}(f(\xi_{n,\mathbf{j}}) - Ef(\xi_{n,\mathbf{j}}))$; then for $A \in \mathscr{E}$,

$$\sum_{\mathbf{j}\in J_n}\frac{|A\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\big(f(\xi_{n,\mathbf{j}}) - Ef(\xi_{n,\mathbf{j}})\big) = n^{-d/2}\sum_{\mathbf{i}\in\{1,\ldots,n\}^d}|(nA)\cap C_{\mathbf{i}}|X_{\mathbf{i}}$$

and by Theorem 3.1 the right-hand side has variance at most

$$n^{-d}(ae^{b\rho_n})^2\Big(\sup_{\mathbf{j}}\operatorname{var}\big(n^{d/2}f(\xi_{n,\mathbf{j}})\big)\Big)|nA|,$$

where $\rho_n := \sum_{j=0}^{\infty}\rho_n(n^{-1}2^j)$. Now $\operatorname{var}(n^{d/2}f(\xi_{n,\mathbf{j}})) \le E|n^{d/2}\xi_{n,\mathbf{j}}|^2$ which by 1.1(ii) is bounded in $\mathbf{j}$ and n. Also, by 1.1(v′), the quantities ρ_n are bounded. So we have 5.1(B), with

$$C := a^2e^{2b\sup\rho_n}\sup_{n,\mathbf{j}}E|n^{d/2}\xi_{n,\mathbf{j}}|^2. \qquad \square$$

5.8 PROOF OF COROLLARY 1.4. The random field has strong-mixing coefficient

$$\alpha(x) := \sup_{\substack{I,J\subseteq\mathbb{Z}^d\\ \rho(I,J)\ge x}}\ \sup_{\substack{E\in\sigma(X_\mathbf{i},\mathbf{i}\in I)\\ F\in\sigma(X_\mathbf{i},\mathbf{i}\in J)}}|P(E\cap F) - P(E)P(F)|,$$

and $\alpha(x) \le \beta(x)$. Thus $\sum_{k=1}^{\infty}k^{d-1}\alpha(k)^{(s-2)/s} < \infty$. Let $\gamma(\mathbf{i}) := E(X_\mathbf{0}X_\mathbf{i})$ for $\mathbf{i} \in \mathbb{Z}^d$. Then by Davydov's lemma ([7]) $|\gamma(\mathbf{i})| \le 8\|X_\mathbf{0}\|_s^2\alpha(\|\mathbf{i}\|)^{(s-2)/s}$; hence, $\sum_{\mathbf{i}\in\mathbb{Z}^d}|\gamma(\mathbf{i})| < \infty$. (The sum $\sum_\mathbf{i}\gamma(\mathbf{i})$ therefore converges, and assumption (vi) can be achieved by normalisation if we know $\sum_\mathbf{i}\gamma(\mathbf{i}) \neq 0$.)

Choose $C \in \mathscr{J}$, with $|C| > 0$. We show that $EZ_n^2(C) \to |C|$ as $n \to \infty$, whence the result follows by Remark 1.2. Now

$$EZ_n^2(C) = \sum_{\mathbf{k}\in J_n}\frac{|C\cap C_{n,\mathbf{k}}|}{|C_{n,\mathbf{k}}|}\sum_{\mathbf{j}\in J_n}\frac{|C\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}n^{-d}\gamma(n\mathbf{j} - n\mathbf{k}).$$

For each $\mathbf{x} \in (0,1]^d$ let $\mathbf{k}_n(x)$ be the unique $\mathbf{k}$ such that $\mathbf{x} \in C_{n,\mathbf{k}}$. We then have

$$EZ_n^2(C) = \int_C h_n(\mathbf{x})\,d\mathbf{x}, \tag{5.8.1}$$

where

$$h_n(\mathbf{x}) := \sum_{\mathbf{j}\in J_n}n^d|C\cap C_{n,\mathbf{j}}|\gamma(n\mathbf{j} - nk_n(\mathbf{x})).$$

In fact, $h_n(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbb{Z}^d}w_{n,\mathbf{i}}\gamma(\mathbf{i})$ where the weights $w_{n,\mathbf{i}}$ lie between 0 and 1, and for each fixed $\mathbf{i}$, $w_{n,\mathbf{i}}$ equals 1 for all large n. Since $\sum_\mathbf{i}|\gamma(\mathbf{i})| < \infty$, dominated convergence gives $h_n(\mathbf{x}) \to \sum_{\mathbf{i}\in\mathbb{Z}^d}\gamma(\mathbf{i}) = 1$ as $n \to \infty$, for each $\mathbf{x}$. Also $|h_n(\mathbf{x})| \le \sum_\mathbf{i}|\gamma(\mathbf{i})|$, so dominated convergence in (5.8.1) yields $EZ_n^2(C) \to |C|$ as required. $\square$

One should notice that although the above shows that $EZ_n^2(C)/|C| \to 1$ for each fixed C, the convergence has not been shown uniform in C, and condition (v) and the results of Section 3 are still needed to establish the uniformity. Without a condition such as (v), uniform convergence is problematic (see [4]).

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