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STOCHASTIC DETERMINATION OF MODULI OF ANNULAR REGIONS AND TORI

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Let $A = A(r, 1)$ be an annulus $\{z: r < |z| < 1\}$ with the Poincaré metric g on A. Let $\mathbf{Z} = (Z_t, P_a)$ be a Brownian motion on A corresponding to g. If we take a geodesic disc D centered at c in A, then the probability $P_a(\exists t, Z_t \in \partial D$ such that Z_s, $0 < s < t$, winds around the origin in the positive direction) is a function of r, $|c|$, and the radius ρ of D. In the present paper we shall calculate the value S of the supremum of these winding probabilities. Then it will turn out that there exists a 1 to 1 correspondence between S and r. Noting that r is called the modulus of A, we have an explicit formula of moduli of annular regions. Further we shall give an explicit formula of moduli of tori in a similar way.

1. Introduction. Let M_1 and M_2 be two Riemann surfaces and suppose that there exists a conformal homeomorphism $f: M_1 \to M_2$. Then it is a theorem of Lévy that if Z_t, $t \geq 0$, is a Brownian motion on M_1, then $f(Z_t)$ is also a Brownian motion on M_2, although moving with varying speed. In other words Brownian motions are conformally invariant except for time-change.

Let M be a Riemann surface whose fundamental group is commutative. Then M is conformally equivalent to one of the following seven surfaces, the unit disc $\Delta = \{z: |z| < 1\}$, the complex plane $\mathbf{C}$, the Riemann sphere (the extended complex plane) $\hat{\mathbf{C}}$, the punctured plane $\mathbf{C} - \{0\}$, the punctured disc $\Delta - \{0\}$, the annulus $A(r, 1) = \{z: r < |z| < 1\}$ with $0 < r < 1$, and the torus. See Farkas and Kra (1980, page 192). In the present paper we shall give explicit formulae concerning the modulus of M in terms of the winding probability of a given Brownian motion.

2. Preliminaries. Let M be a Riemann surface with a Hermitian metric $g = g(z)\, dz\, d\bar{z}$. The corresponding Laplace–Beltrami operator $L(g) = 4g(z)^{-1}\, \partial^2/\partial z\, \partial\bar{z}$ defines a Brownian motion $\mathbf{X} = (X_t, P_a, t \geq 0, a \in M)$. Since a conformal change of metric merely changes the time scale and we study only pathwise behaviour of $\mathbf{X}$, the metric may be chosen arbitrarily. See Lyons and McKean (1984). We introduce the standard metric on M as follows. Let $\tilde{M}$ be the universal covering surface of M with natural projection π. Then we may assume that $\tilde{M}$ is one of the following three surfaces, the unit disc Δ, the complex plane $\mathbf{C}$ and the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. See Farkas and Kra (1980, page 180). We introduce Hermitian metrics $\tilde{g}_0 = \tilde{g}_0(z)\, dz\, d\bar{z}$ as

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follows. We set

$$\tilde{g}_0(z) = \frac{4}{(1 - |z|^2)^2} \quad \text{for } \tilde{M} = \Delta,$$

$$\tilde{g}_0(z) = 1 \quad \text{for } \tilde{M} = \mathbf{C},$$

$$\tilde{g}_0(z) = \frac{4}{(1 + |z|^2)^2} \quad \text{for } \tilde{M} = \hat{\mathbf{C}}.$$

Then it is well known that there exists a Hermitian metric $g_0 = g_0(\zeta)\, d\zeta\, d\bar{\zeta}$ on M, where ζ is a local coordinate in M, such that the pull-back of g_0 is $\tilde{g}_0$. See Farkas and Kra (1980, page 198). Then the corresponding Laplace–Beltrami operators $L(\tilde{g}_0)$ and $L(g_0)$ define Brownian motions $\tilde{\mathbf{Z}}$ and $\mathbf{Z}$, respectively. Since the projection of $\tilde{\mathbf{Z}}$ onto M and $\mathbf{Z}$ have the same probability law, we call $\tilde{\mathbf{Z}}$ the covering motion of $\mathbf{Z}$. Further we note that if $\varphi(t) = \int_0^t (g_0/g)(X_s)\, ds$ and $\hat{\mathbf{X}} = (\hat{X}_t) = (X_{\varphi_t - 1})$, then $\hat{\mathbf{X}}$ and $\mathbf{Z}$ have the same probability law.

REMARK. It is well known that spherical and flat Brownian motions are conservative. Further the Brownian motion corresponding to the Poincaré metric is also conservative. One way to see this is to show $\int_0^\sigma (1 - |B_t|^2)^{-2}\, dt = \infty$, where $\mathbf{B} = (B_t)$ is a flat Brownian motion in Δ with an absorbing barrier $\partial\Delta$ and σ is its lifetime.

The results of the present paper will be proved by studying explicit formulae for the harmonic functions $h(z)$ on the complement of two discs D_1 and D_2 in the Riemann sphere with boundary values 0 and 1 on ∂D_1 and ∂D_2, respectively. Such a function is called the harmonic measure of ∂D_2 in $\hat{\mathbf{C}} - D_1 \cup D_2$. (Note that $h(z)$ is the probability that a Brownian motion starting at z hits on ∂D_2 before hitting on ∂D_1.) We summarize as follows.

LEMMA 1. *If* $D_1 = \{z\colon \operatorname{Im} z < 0\}$ *and* $D_2 = \{z\colon |z - iR| < \rho\}$ *with* $0 < \rho < R$, *then*

$$h(z) = \frac{\log|(ik + z)/(ik - z)|}{\log|(k + R - \rho)/(k - R + \rho)|},$$

where $k = (R^2 - \rho^2)^{1/2}$.

LEMMA 2. *If* $D_1 = \{z\colon |z| < \rho\}$ *and* $D_2 = \{z\colon |z - R| < \rho\}$, *then*

$$h(z) = \frac{\log|(z - \kappa\rho)/(\rho - \kappa z)|}{\log|(R - \rho - \kappa\rho)/(\rho - \kappa R + \kappa\rho)|},$$

where $\kappa = (2\rho)^{-1}(R - (R^2 - 4\rho^2)^{1/2})$.

3. Main results. Let $\mathbf{X} = (X_t, P_\zeta, t \geq 0, \zeta \in M)$ be a Brownian motion on M with lifetime τ. We shall use the same notation as in Section 2.

Definition. We call an open set D in M a geodesic disc centered at $\zeta \in D$, provided that $\tilde{M} = \Delta$, $\mathbf{C}$ or $\hat{\mathbf{C}}$ and that there exists a disc $D_0 \subset \tilde{M}$ centered at $\zeta_0 \in D_0$ in the same sense of Poincaré, Euclidean or spherical metric on $\tilde{M}$, respectively, such that π: $\overline{D}_0 \to \overline{D}$ is bijective and $\pi(\zeta_0) = \zeta$, where $\overline{D}_0$ and $\overline{D}$ denote the closure of D_0 and D, respectively.

Throughout this and next sections we assume that $\pi_1(M)$, the fundamental group of M, is commutative. Then $\pi_1(M)$ is isomorphic to one of the following three groups, $\{1\}$, the group of all integers $\mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Z}$ (the free Abelian group on two generators). See Farkas and Kra (1980, page 192).

Case 1. Suppose that $\pi_1(M) = \{1\}$. In this case M is conformally equivalent to $\hat{\mathbf{C}}$ if and only if M is compact. If M is not compact and the Brownian motion $\mathbf{X} = (X_t, P_a)$ is recurrent, then $M \simeq \mathbf{C}$ (conformally equivalent). If M is not compact and $\mathbf{X}$ is transient, then $M \simeq \Delta$.

Case 2. Suppose that $\pi_1(M)$ is isomorphic to the group of all integers Z. Take $\zeta \in M$ and a geodesic disc D centered at ζ. Let c be a closed curve starting and ending at ζ such that c generates $\pi_1(M)$. Let $M/\overline{D}$ be the quotient space obtained by identifying all the points in $\overline{D}$. If $\gamma(t)$, $0 \le t \le T$, is a continuous curve in M, $\gamma^*(t)$, $0 \le t \le T$ will be the curve in $M/\overline{D}$ which results from the projection of γ. If γ_1^* and γ_2^* are two curves such that γ_2^* begins where γ_1^* ends, we let $\gamma_1^* + \gamma_2^*$ stand for the curve γ_1^* followed by γ_2^*. We write $\gamma_1^* \sim \gamma_2^*$, if γ_1^* is homotopic to γ_2^*. See Davis (1975).

Define a stopping time σ by

$$\sigma = \inf\{t\colon X^*_{[0,t]} \sim c^*\},$$

where $X_{[0,t]}$ is the restriction of sample paths of $\mathbf{X}$ in $[0, t]$. Then we have the following.

Theorem 1. *If for some $\zeta \in M$ and a geodesic disc D centered at ζ*

$$P_\zeta(\sigma < \pi) = 1 \tag{3.1}$$

holds, then it does for all ζ and D and $M \simeq \mathbf{C} - \{0\}$.

If for some $\zeta \in M$ and a geodesic disc D with center ζ, (3.1) *does not hold, define S by*

$$S = \sup P_\zeta(\sigma < \pi), \tag{3.2}$$

where the supremum is taken over all $\zeta \in M$ and geodesic discs D centered at ζ. If $S = 1$, then $M \simeq \Delta - \{0\}$. If $S < 1$, then $M \simeq A(r,1) = \{z \in \mathbf{C}\colon r < |z| < 1\}$, where r is uniquely determined by

$$S = \frac{\log(\lambda + 1)/(\lambda - 1)}{\log(\sqrt{\lambda} + 1)/(\sqrt{\lambda} - 1)} \tag{3.3}$$

and

$$r = \exp(-2\pi^2/\log\lambda). \tag{3.4}$$

CASE 3. Suppose that $\pi_1(M)$ is isomorphic to $Z \oplus Z$ (the free Abelian group on two generators). In this case M is a torus, i.e., there exist $a, b \in \mathbf{C}$ such that a/b is not real and M is conformally equivalent to the quotient surface $\mathbf{C}/G$ where G is the group of automorphisms on $\mathbf{C}$ generated on two elements $z \to z + a$ and $z \to z + b$. We may assume that $a > 0$, $|b| \geq a$ and $\operatorname{Im} b > 0$. Recall that b/a is called the modulus of M.

REMARK 3.1. The modulus b/a is not unique. But if b/a is given, then the conformal equivalence class to which M belongs is uniquely determined. See Farkas and Kra (1980, page 196).

Take a geodesic disc D centered at ζ and two closed curves c_1 and c_2 starting and ending at ζ such that they generate $\pi_1(M)$. Without loss of generality we shall assume $\pi(0) = \zeta$. Let $\tilde{c}_1$ and $\tilde{c}_2$ be lifted curves in $\mathbf{C}$ starting at 0 such that $\pi(\tilde{c}_i) = c_i$, $i = 1, 2$. Let a_0 and b_0 be endpoints of $\tilde{c}_1$ and $\tilde{c}_2$, respectively; then we may assume $a_0 > 0$, $|b_0| \geq a_0$ and $\operatorname{Im} b_0 \neq 0$. For integers m, n define $\sigma_{(m,n)}$ by

$$\sigma_{(m,n)} = \inf\{t\colon X^*_{[0,t]} \sim mc_1^* + nc_2^*\}. \tag{3.5}$$

THEOREM 2. *Let* $s_1 = P_\zeta(\sigma_{(1,0)} < \sigma_{(2,0)})$, $s_2 = P_\zeta(\sigma_{(0,1)} < \sigma_{(0,2)})$ *and* $s_3 = P_\zeta(\sigma_{(-1,1)} < \sigma_{(-2,2)})$. *Then the modulus of* M *is equal to one of* $e^{i\theta}H(s_2)/H(s_1)$ *and* $e^{i(\pi-\theta)}H(s_2)/H(s_1)$, *where* $H(x)$ *and* θ *are defined by*

$$H^{-1}(x) = \frac{\log|(1 - 2kx)/(2x - k)|}{\log|(1 + k - kx)/(x - k - 1)|},$$

$$k = 2^{-1}\left(x - \sqrt{x^2 - 4}\right),$$

$$0 < \theta = \cos^{-1}\left(\frac{h_1^2 + h_2^2 - h_3^2}{2h_1h_2}\right) < \pi,$$

and $h_i = H(S_i)$, $i = 1, 2, 3$.

4. Proofs. We may identify M with one of the standard surfaces, i.e., $\mathbf{C} - \{0\}$, $\Delta - \{0\}$, $A(r, 1)$ for r, $0 < r < 1$, in Case 2 and $\mathbf{C}/G$ for $a, b \in \mathbf{C}$ with $a > 0$, $|b| \geq a$, $\operatorname{Im} b \neq 0$ in Case 3. Further we may consider the standard conservative Brownian motion $\mathbf{Z}$ instead of a given general Brownian motion $\mathbf{X}$ with lifetime τ.

PROOF OF THEOREM 1. For a geodesic disc D centered at ζ in M there exists a disc D_0 with center ζ_0 in $\tilde{M}$ in the sense of Euclidean or Poincaré metric, accordingly as $\tilde{M} = \mathbf{C}$ or Δ. Let α be the conformal automorphism on $\tilde{M}$ such that $\alpha(\zeta_0)$ is the endpoint of the lift of c starting at ζ_0. For covering motion $\tilde{\mathbf{Z}}$, define σ_0 by

$$\sigma_0 = \inf\{t\colon \tilde{Z}_t \in \overline{\alpha(D_0)}\}. \tag{4.1}$$

Since the projection of $\tilde{\mathbf{Z}}$ onto M and $\mathbf{Z}$ have the same probability law, we have

$$P_\zeta(\sigma < \infty) = \tilde{P}_{\zeta_0}(\sigma_0 < \infty). \tag{4.2}$$

If $M = \mathbf{C} - \{0\}$, then $\tilde{M} = \mathbf{C}$ and the recurrence property of one-dimensional complex Brownian motion shows that the right-hand side of (4.2) is equal to 1 for all geodesic discs D centered at ζ in M.

If $M = \Delta - \{0\}$ or $A(r,1)$ with $0 < r < 1$, then $\tilde{M} = \Delta$. For technical reasons we shall consider the upper half plane $U = \{z\colon \operatorname{Im} z > 0\}$ with the Poincaré metric $(\operatorname{Im} z)^{-2}\, dz\, d\bar{z}$ on U instead of the unit disc Δ with its Poincaré metric $(1 - |z|^2)^{-2}\, dz\, d\bar{z}$ and identify that $\tilde{M}$ with U. It is easy to see that the corresponding Brownian motion $\tilde{\mathbf{Z}}_1$ on U is obtained by setting

$$\tilde{\mathbf{Z}}_1 = \left(\beta \cdot \tilde{Z}_t, \tilde{P}_{\beta^{-1}(\zeta)}, t \geq 0, \zeta \in U\right),$$

where $\beta(z) = i(1 - z)/(1 + z)$. We shall write $\tilde{\mathbf{Z}}$ for $\tilde{\mathbf{Z}}_1$ and define α, D_0, ζ_0, σ_0 in those contexts. Now Kakutani's theorem (see Itô (1960, page 103)) enables us to calculate the value of $u(z) = \tilde{P}_z(\sigma_0 < \infty)$. Note that $u(z)$ is the harmonic measure of $\partial\alpha(D_0)$ in $U - \overline{\alpha(D_0)}$.

If $M = \Delta - \{0\}$, consider the universal covering map $f\colon U \to \Delta - \{0\}$ given by $f(z) = \exp(2\pi i z)$. From the translation invariant property of harmonic measure we may assume $\alpha(D_0) = \{z\colon |z - iR| < \rho\}$ with $0 < \rho < R$. Then by Lemma 1 we have for fixed ρ

$$\tilde{P}_{\zeta_0}(\sigma < \infty) = \frac{\log\left|2\sqrt{R^2 - \rho^2} + i\right|}{\log\left(1 + \rho^{-1}\sqrt{R^2 - \rho^2}\right)} \to 1 \quad \text{as } R \to \infty.$$

This shows that $S = \sup P_\zeta(\sigma < \infty) = 1$.

If $M = A(r,1)$ for r, $0 < r < 1$, consider the covering map $g\colon U \to A(r,1)$ given by $g(z) = \exp(2\pi i \log z/\log\lambda)$, where $\lambda = \exp(-2\pi^2/\log r)$. In this case we may assume that $\alpha(D_0) = \Delta(Re^{i\theta}; \rho)$ with $0 < \theta < \pi$, $0 < \rho < R\sin\theta$, $(R + \rho)/(R - \rho) < \lambda$ and $\zeta_0 = \lambda(R\cos\theta + i(R^2\sin^2\theta - \rho^2)^{1/2})$. Note that $\lambda^{-1}\zeta_0$ is the center of $\Delta(Re^{i\theta}; \rho)$ in the sense of the Poincaré metric on U. Let $u(z)$ be the harmonic measure of $\partial\Delta(Re^{i\theta}; \rho)$ in $U - \Delta(Re^{i\theta}; \rho)$. To obtain the value of $S = \sup P_\zeta(\sigma < \infty)$, we shall show

$$u(\zeta_0) < u\left(Re^{i\theta} + i\lambda\sqrt{R^2 - \rho^2} - iR\right).$$

Since $\Gamma = \{z\colon u(z) \geq u(\zeta_0)\} \cup \overline{\Delta(Re^{i\theta}; \rho)}$ is a closed disc with center $\lambda^{-1}\zeta_0$ in the sense of the Poincaré metric and $\Delta(iRe^{i\theta}; \rho) \subset \Gamma$, the Euclidean center p of Γ can be written as $p = Re^{i\theta} + i\varepsilon$ with $\varepsilon > 0$. Thus $\lambda(R^2 - \rho^2)^{1/2}e^{i\theta}$ belongs to Γ and so does $Re^{i\theta} + i(\lambda(R^2 - \rho^2)^{1/2} - R)$.

Let $u_{R,\rho}(z)$ be the harmonic measure of $\partial\Delta(iR; \rho)$ in $U - \overline{\Delta(iR; \rho)}$. It is clear that $u(Re^{i\theta} + i\lambda(R^2 - \rho^2)^{1/2} - iR) < u_{R,\rho}(i\lambda(R^2 - \rho^2)^{1/2})$. We note that $u_{R,\rho}(i\lambda(R^2 - \rho^2)^{1/2}) = \tilde{P}_{\zeta_0}(\sigma < \infty)$ for $\alpha(D_0) = \Delta(iR; \rho)$ and $\zeta_0 = i\lambda(R^2 - \rho^2)^{1/2}$. Thus to obtain the value of S we may only consider the case that $\theta = \pi/2$. In this case we have by Lemma 1,

$$\begin{aligned}\tilde{P}_{\zeta_0}(\sigma < \infty) &= u_{R,\rho}(i\lambda k)\\ &= \frac{\log(\lambda+1)/(\lambda-1)}{\log(|k+R-\rho|)/(|k-R+\rho|)},\end{aligned}$$

where $k = (R^2 - \rho^2)^{1/2}$. Put $\rho_0 = (\lambda - 1)/(\lambda + 1)R$. Then it is easy to see that

$$\begin{aligned}S &= \lim_{\rho \uparrow \rho_0} u_{R,\rho}\left(i\lambda\sqrt{R^2-\rho^2}\right)\\ &= \frac{\log(\lambda+1)/(\lambda-1)}{\log(\sqrt{\lambda}+1)/(\sqrt{\lambda}-1)}. \qquad \square\end{aligned}$$

PROOF OF THEOREM 2. We shall assume that M is the quotient surface $\mathbf{C}/G$ where G is the group of conformal automorphisms on $\mathbf{C}$ generated on two elements $z \to z + a$, $z \to z + b$ with $a, b \in \mathbf{C}$, $a > 0$, $|b| \geq a$, $\operatorname{Im} b \neq 0$ such that the lifts of c_1 and c_2 starting at 0 end at a and b, respectively.

If $\operatorname{Im} b > 0$, take a geodesic disc D centered at ζ in M. Then there exists an Euclidean disc D_0 centered at ζ_0 in $\mathbf{C}$. Without loss of generality we shall assume that $D_0 = \Delta(0; \rho)$ for ρ, $0 < \rho < 2^{-1}a$ and $\zeta_0 = 0$. For the covering motion $\tilde{\mathbf{Z}}$ we set

$$\tilde{\sigma}_{(m,n)} = \inf\left\{t\colon \tilde{Z}_t \in \overline{\Delta(ma+nb;\rho)}\right\},$$

where m, n are integers. Then we have for integers m, n, p, q with $(m, n) \neq (p, q)$

$$\begin{aligned}P_\zeta\left(\sigma_{(m,n)} < \sigma_{(p,q)}\right) &= \tilde{P}_0\left(\tilde{\sigma}_{(m,n)} < \tilde{\sigma}_{(p,q)}\right) \qquad (4.4)\\ &= h(0),\end{aligned}$$

where $h(z)$ is the harmonic measure of $\partial\Delta(ma + nb; \rho)$ in $\mathbf{C} - \overline{\Delta(ma+nb,\rho)} \cup \overline{\Delta(pa+qb;\rho)}$.

Let $v_{R,\rho}(z)$ be the harmonic measure of $\partial\Delta(R; \rho)$ in $\mathbf{C} - \overline{\Delta(0;\rho)} \cup \overline{\Delta(R;\rho)}$, with $0 < 2\rho < R$; then we have by Lemma 2,

$$v_{R,\rho}(z) = \frac{\log|(z-\kappa\rho)/(\rho-\kappa z)|}{\log|(R-\rho-\kappa\rho)/(\rho-\kappa R+\kappa\rho)|},$$

where $\kappa = 1/2\rho(R - (R^2 - 4\rho^2)^{1/2})$. If $|ma + nb| = R$, $p = 2m$ and $q = 2n$, then by the translation and rotation invariant properties of harmonic measure we have $h(0) = v_{R,\rho}(2R)$. We note that $v_{R,\rho}(2R)$ is a function in $x = R/\rho \in (\frac{1}{2}, \infty)$. Set $K(x) = v_{R,\rho}(2R)$. We shall show that $K(x)$ is strictly decreasing in x and $K(x) \downarrow 2^{-1}$ as $x \uparrow \infty$. To show this it is sufficient to show $v_{R,\rho_1}(2R) < v_{R,\rho_2}(2R)$ for $0 < \rho_1 < \rho_2$. Since $\{z\colon v_{R,\rho_1}(z) = 2^{-1}\} = \{z\colon v_{R,\rho_2}(z) = 2^{-1}\} = \{z\colon z = (R/2) + iy\}$ and $\Delta(R; \rho_1) \subset \Delta(R; \rho_2)$, we have $v_{R,\rho_1}(z) < v_{R,\rho_2}(z)$ for $z \in \{x + iy\colon x > R/2\}$.

Let $m = 1$ and $n = 0$; then from Lemma 2 we have $s_1 \equiv \tilde{P}_0(\tilde{\sigma}_{(1,0)} < \tilde{\sigma}_{(2,0)}) = K(|a|/\rho)$. Similarly we have $s_2 = K(|b|/\rho)$ and $s_3 = K(|b - a|/\rho)$. Hence if

Im $b > 0$, assertions of Theorem 2 follow. If Im $b < 0$, then a similar argument shows this fact. Since Brownian motions are anticonformally invariant, we cannot know whether Im $b > 0$ or Im $b < 0$ from the data $\{s_1, s_2, s_3\}$. □

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