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CONVERSE RESULTS FOR EXISTENCE OF MOMENTS AND UNIFORM INTEGRABILITY FOR STOPPED RANDOM WALKS

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Let $\{S_n, n \geq 1\}$ be a random walk and N a stopping time. The Burkholder–Gundy–Davis inequalities for martingales can be used to give conditions on the moments of N (and of $X = S_1$), which ensure the finiteness of the moments of the stopped random walk, S_N. We establish converses to these results, that is, we obtain conditions on the moments of the stopped random walk and X or N which imply the finiteness of the moments of N or X. We also study one-sided versions of these problems and corresponding questions concerning uniform integrability (of families of stopping times and families of stopped random walks).

1. Introduction. Throughout this paper, X and $\{X_n, n \geq 1\}$ are i.i.d. random variables and $S_n = \sum_{k=1}^n X_k$, $n \geq 1$ ($S_0 = 0$).

Suppose that $E|X|^r < \infty$ for some $r > 0$ and that $EX = 0$ when $r > 1$. It follows from the c_r inequalities (when $0 < r \leq 1$) and the moment inequalities by Marcinkiewicz and Zygmund (1937) and elementary computations (when $r > 1$) that

$$E|S_n|^r \leq \begin{cases} nE|X|^r & \text{for } 0 < r \leq 1, \\ B_r nE|X|^r & \text{for } 1 \leq r \leq 2, \\ B_r n^{r/2} E|X|^r & \text{for } r \geq 2, \end{cases} \tag{1.1}$$

where B_r is a numerical constant depending on r only. (For $r = 2$ we have, of course, $E(S_n)^2 = nEX^2$.) A more compact way of writing (1.1) is

$$E|S_n|^r \leq B_r n^{r/2 \vee 1} E|X|^r \quad \text{for } r > 0, \tag{1.1$'$}$$

where, thus, $B_r = 1$ when $0 < r \leq 1$ and $r = 2$.

Now, let N be a stopping time with respect to an increasing sequence of sub-σ-algebras $\{\mathscr{F}_n, n \geq 0\}$, where we set $\mathscr{F}_0 = \{\varnothing, \Omega\}$. Further, assume that X_n is $\mathscr{F}_n$-measurable and independent of $\mathscr{F}_{n-1}$ for all n. (A typical case is when $\mathscr{F}_n = \sigma\{X_1, \cdots, X_n\}$.) By applying moment inequalities for martingales (Davis (1970), when $r = 1$; Burkholder (1966), Theorem 9, when $1 < r \leq 2$; and Burkholder (1973), Theorem 21.1, when $r \geq 2$), elementary computations and Wald's lemma, it is possible to extend (1.1) to randomly indexed sums as follows:

$$E|S_N|^r \leq \begin{cases} E|X|^r EN & \text{for } 0 < r \leq 1, \\ B_r E|X|^r EN & \text{for } 1 \leq r \leq 2, \\ B_r E|X|^r EN^{r/2} & \text{for } r \geq 2, \end{cases} \tag{1.2}$$

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where, again, B_r is a numerical constant, which only depends on r. If N is deterministic, (1.2) reduces to (1.1).

The condensed version of (1.2) is

$$(1.2') \qquad E|S_N|^r \le B_r E|X|^r E(N^{r/2 \vee 1}) \quad \text{for } r \ge 0.$$

For a derivation of (1.2) when $r \ge 1$ and N is a first passage time we refer to Gut (1974a), Lemma 2.3 (see also Gut (1974b)) and for the more general case to Gut (1986), Chapter I.

Now, let $r \ge 1$. If we do not assume that $\mu = EX = 0$ and apply the elementary inequality

$$|S_N| \le |S_N - \mu N| + |\mu| N,$$

the c_r-inequality and the fact that $|\mu|^r \le E|X|^r$, it follows immediately that

$$(1.3) \qquad E|S_N|^r \le B_r' E|X|^r E N^r,$$

where B_r' is a numerical constant, depending on r only (cf. Gut (1986), Theorem I.5.2).

Let us summarize the above conclusions in the following (slightly weaker) form. Since our main concern throughout will be the case $r \ge 1$ we confine ourselves to that case here.

THEOREM 1.1. *Let $r \ge 1$ and suppose that $E|X|^r < \infty$. Then*

$$\text{(i)} \qquad EN^r < \infty \Rightarrow E|S_N|^r < \infty,$$

$$\text{(ii)} \qquad EX = 0 \quad \textit{and} \quad E(N^{r/2 \vee 1}) < \infty \Rightarrow E|S_N|^r < \infty.$$

Our first task will be to try to establish converses to Theorem 1.1, that is, we shall investigate to what extent (if at all) the arrows may be reversed. In the next section we shall state some general results and in Section 3 we shall look in more detail at how the positive and negative tails of the distribution of X influence the results.

As it turns out, the proofs of the results in Section 2 are very different for the cases $r = 1$ and $r > 1$, respectively. In fact, once we have established that the case $r = 1$ holds true it follows by an iterative procedure that the latter holds too. We therefore present those proofs in separate sections. The proofs of the (harder) case $r = 1$ are presented in Section 4 and the proofs for $r > 1$ are given in Section 5. The results from Section 3 are proved in Section 6. Sections 7 and 8 contain some examples and further remarks.

The second aim of this paper is to study uniform integrability for families of stopped random walks.

In the classical strong law of large numbers and in the central limit theorem it is possible to prove moment convergence by proving results about uniform integrability (see, e.g., Gut (1986), Section I.4). With this in mind it is reasonable to ask to what extent similar results exist for stopped random walks. Some results, corresponding to Theorem 1.1, will be given in Section 9. In that section we shall also present some results concerning converses, which are parallel to Theorems 2.1 and 2.2. For proofs of the direct results we refer to Gut (1986), Section I.6 (and to the original papers mentioned there). The proofs of the converse results for $r = 1$, which, again, are the hardest part, are given in Section 10 and the proofs for $r > 1$ are found in Section 11.

Let us finally remark that the results, apart from being interesting in their own right, are useful for proving existence of moments and moment convergence results for, e.g., first passage times and generalizations thereof; see, e.g., Gut (1986), Chapters III and IV and references given there.

2. Converses for moments. General results. Let us first consider the case of positive random variables. The following result shows that Theorem 1.1(i) is sharp in this case.

THEOREM 2.1. *Let $r \geq 1$ and suppose that $P(X \geq 0) = 1$ and that $P(X > 0) > 0$. Then*

$$ES_N^r < \infty \Rightarrow EX^r < \infty \quad \textit{and} \quad EN^r < \infty.$$

If we consider general random walks with nonzero mean it follows from the following example, which deals with the theory of first passage times, that Theorem 2.1 as stated does not hold in this generality.

EXAMPLE 2.1. Suppose that $EX > 0$ and let N_+ be the first (strong, ascending) ladder index, that is,

$$N_+ = \min\{n;\ S_n > 0\}.$$

It is then known (see Gut (1974a), Theorem 2.1, or Gut (1986), Theorem III.3.1) that, for $r \geq 1$,

$$E(S_{N_+})^r < \infty \Leftrightarrow E(X^+)^r < \infty \tag{2.1}$$

and

$$EN_+^r < \infty \Leftrightarrow E(X^-)^r < \infty. \tag{2.2}$$

Thus, by choosing X with $EX > 0$, $E(X^+)^r < \infty$, and $E(X^-)^r = \infty$ for all $r > 1$, $ES_{N_+}^r < \infty$ (for every r), but none of the conclusions of Theorem 2.1 holds.

However, the following weaker converse to Theorem 1.1(i) holds for random walks with nonzero mean.

THEOREM 2.2. *Let $r \geq 1$ and suppose that $EX \neq 0$. Then*

$$E|S_N|^r < \infty \quad \textit{and} \quad E|X|^r < \infty \Rightarrow EN^r < \infty.$$

With (2.1) and (2.2) in mind we shall see in the next section how this theorem can be generalized if one considers the two tails of the various quantities separately.

The case $EX = 0$ is more complicated as is seen by the following example.

EXAMPLE 2.2. Consider a symmetric simple random walk; that is, suppose that $P(X = 1) = P(X = -1) = \frac{1}{2}$. Also, let N_+ be as above, that is, let $N_+ = \min\{n; S_n = +1\}$. Here X and S_{N+} obviously have moments of all orders and yet it is well known that N_+ has no moment of order $\geq \frac{1}{2}$. (The modification $N = \min\{n \geq Z; S_n = 1\}$, where Z is a suitable random variable independent of $\{X_k\}$, yields an example where N has no finite moment of any positive order.)

In Theorem 2.2 we made additional assumptions on the moments of X which, together with the assumption that $E|S_N|^r < \infty$, implied that $EN^r < \infty$. Our next results show under what additional assumptions on the moments of N we can infer that $E|X|^r < \infty$.

THEOREM 2.3. *Let $r \geq 1$. Then*

$$E|S_N|^r < \infty \quad \text{and} \quad EN^r < \infty \Rightarrow E|X|^r < \infty.$$

Note that no assumption was made about the existence of EX. However, if we assume that $EX = 0$, the assumption on the moments of N can be weakened.

THEOREM 2.4. *Let $r > 1$ and suppose that $EX = 0$. Then*

$$E|S_N|^r < \infty \quad \text{and} \quad EN < \infty \Rightarrow E|X|^r < \infty.$$

Note that here we have a situation parallel to (i) and (ii) of Theorem 1.1. Moreover, by combining these four assertions we obtain the following.

COROLLARY 2.1. *Let $r \geq 1$.*

(i) *If $EN^r < \infty$, then*

$$E|S_N|^r < \infty \Leftrightarrow E|X|^r < \infty.$$

(ii) *If $EX = 0$ and $EN^{r/2 \vee 1} < \infty$, then*

$$E|S_N|^r < \infty \Leftrightarrow E|X|^r < \infty.$$

As for sharpness, the following example shows that $E|S_N|^r < \infty$ alone does not imply even the existence of EX.

EXAMPLE 2.3. Let $P(X = \pm k) = 3/\pi^2 k^2$, $k = 1, 2, \ldots$, and define

$$N_0 = \min\{n \geq 1;\ S_n = 0\}.$$

Since the random walk is recurrent (cf. Feller (1966), Section XVIII.7), $N_0 < \infty$ a.s. Thus S_{N_0} $(= 0)$ has moments of all orders, but $E|X| = \infty$.

If we assume the existence of EX, we observe that if $EX > 0$ and if we choose $N = N_+$ as in Example 2.1, then (2.1) and (2.2) show that $EN^r < \infty$ is a necessary extra requirement for Theorem 2.3 to hold. For the case $EX = 0$, we have the following variant of Example 2.3.

EXAMPLE 2.4. Let X be integer valued and such that $EX = 0$ and $E|X|^r = \infty$ for every $r > 1$, and let N_0 be defined as in Example 2.3. Again, $S_{N_0} = 0$ and hence the conclusion of Theorem 2.4 (and Corollary 2.1(ii)) does not necessarily hold if $E|S_N|^r < \infty$ only.

We do not know the best possible condition on N in Theorem 2.4; it seems possible that $EN^{1/2} < \infty$ would suffice. In fact, we can prove this under the extra assumption that $EX^2 < \infty$; see Remark 5.1.

3. Results for the positive and negative tails. Motivated by (2.1) and (2.2) we shall, in this section, study problems of the previous kind, but for the *positive and negative tails* of X and S_N *separately*. In our first result we present one-sided versions of Theorem 1.1(i). Further, we give an improvement for the negative tail when $EX > 0$ (and, symmetrically, for the positive tail when $EX < 0$). Unfortunately, we do not know whether a corresponding result for the case $EX = 0$ (cf. Theorem 1.1(ii)) holds true.

THEOREM 3.1. *Let* $r \geq 1$.

(i) $E(X^+)^r < \infty$ *and* $EN^r < \infty \Rightarrow E(S_N^+)^r < \infty$.
(ii) *If* $EX > 0$, *then* $E(X^-)^r < \infty$ *and* $EN^{r/2 \vee 1} < \infty \Rightarrow E(S_N^-)^r < \infty$.

As for converses to this result (corresponding to Theorems 2.1–2.4 in the two-sided case), we first note that Example 2.1 shows that the opposite implications above do not hold in general. However, the following results are true.

THEOREM 3.2. *Let* $r \geq 1$ *and suppose that* $E|X| < \infty$.

(i) *If* $EX > 0$, *then*

$$E(S_N^+)^r < \infty \Rightarrow E(X^+)^r < \infty.$$

(ii) *If* $EX = 0$, *then*

$$E(S_N^+)^r < \infty \quad \text{and} \quad EN < \infty \Rightarrow E(X^+)^r < \infty.$$

(iii) *If* $EX < 0$, *then*

$$E(S_N^+)^r < \infty \quad \text{and} \quad EN^r < \infty \Rightarrow E(X^+)^r < \infty.$$

THEOREM 3.3. *Let* $r \geq 1$ *and suppose that* $EX > 0$. *Then*

$$E(S_N^+)^r < \infty \quad \text{and} \quad E(X^-)^r < \infty \Rightarrow EN^r < \infty.$$

REMARK 3.1. We leave the formulation of the corresponding results for S_N^- to the reader.

REMARK 3.2. Theorems 3.2(i) and 3.3 actually hold if we include a priori the possibility that $EX = +\infty$, that is, if $E(X^+) = \infty$ and $E(X^-) < \infty$, although

TABLE 1
The possible combinations when $EX > 0$. $*$ signifies that the corresponding rth moment is finite, — that it is infinite. ($r > 1$ is fixed.)

	X^+	X^-	N	S^+	S^-
1	*	*	*	*	*
2	*	*	—	—	*
3	*	*	—	—	—
4	*	—	*	*	—
5	*	—	—	*	*
6	*	—	—	*	—
7	*	—	—	—	*
8	*	—	—	—	—
9	—	*	*	—	*
10	—	*	—	—	*
11	—	*	—	—	—
12	—	—	*	—	—
13	—	—	—	—	*
14	—	—	—	—	—

the conclusion shows that this case does not occur. The proofs remain the same, cf. Theorem 2.1, where no assumption on EX is made. On the contrary, Theorem 3.2(iii) does not hold for $EX = -\infty$, see Example 7.4.

We shall later see that these results are in some respects best possible, and that, e.g., (2.1) and (2.2) do not hold for general stopping times.

Further implications may be derived by combining the statements above. As an example we obtain the following refinement of Corollary 2.1(i) (for finite mean only, cf. Example 7.4).

COROLLARY 3.1. *Let $r \geq 1$ and suppose that $E|X| < \infty$ and $EN^r < \infty$. Then*

(i) $$E(S_N^+)^r < \infty \Leftrightarrow E(X^+)^r < \infty,$$

(ii) $$E(S_N^-)^r < \infty \Leftrightarrow E(X^-)^r < \infty.$$

We may also obtain the following intriguing equivalence: Let $r \geq 1$ and suppose that $EX > 0$. Then $E(S_N^+)^r < \infty$ and $E(X^-)^r < \infty \Leftrightarrow E(S_N^-)^r < \infty$, $E(X^+)^r < \infty$ and $EN^r < \infty$.

For the case $EX > 0$, the theorems above exclude 18 of the 32 conceivable combinations of finite rth moments (for a fixed $r > 1$). Table 1 exhibits the 14 remaining possibilities. Examples covering all 14 cases may be given, see Section 7.

4. Proofs of Theorems 2.1–2.3 when $r = 1$. As mentioned in the introduction, the proofs of these results (for $r > 1$) split into two natural parts; a first part (which is harder) in which it is shown that the expected values of the relevant quantities exist and a second part in which it is shown that the higher moments exist. In this section we present the proofs of the first parts, and thus also proofs of these theorems for the case $r = 1$.

PROOF OF THEOREM 2.1 WHEN $r = 1$. Since the summands are positive we have $S_N \geq X_1$ and thus that

$$\mu = EX_1 \leq ES_N < \infty. \tag{4.1}$$

Next, we apply Wald's lemma to obtain

$$ES_{N \wedge n} = \mu E(N \wedge n). \tag{4.2}$$

Furthermore, $N \wedge n \to N$ and $S_{N \wedge n} \to S_N$ as $n \to \infty$. It therefore follows from monotone convergence that

$$\mu EN = ES_N < \infty,$$

which completes the proof. □

PROOF OF THEOREM 2.2 WHEN $r = 1$. Suppose, without restriction, that $\mu > 0$. We use a trick due to Blackwell (1953), who used it in the context of ladder variables. Let $\{N_k, k \geq 1\}$ be independent copies of N, constructed as follows: Let $N_1 = N$. Restart after N_1, i.e., consider the sequence $X_{N_1+1}, X_{N_1+2}, \ldots$, and let N_2 be a stopping time for this sequence. Restart after $N_1 + N_2$ to obtain N_3, and so on. Thus, $\{N_k, k \geq 1\}$ is a sequence of i.i.d. random variables distributed as N, and $\{S_{N_1 + \cdots + N_k}, k \geq 1\}$ is a sequence of partial sums of i.i.d. random variables distributed as S_N and, by assumption, with finite mean, ES_N.

Now

$$\frac{N_1 + \cdots + N_k}{k} = \frac{S_{N_1 + \cdots + N_k}}{k} \Bigg/ \frac{S_{N_1 + \cdots + N_k}}{N_1 + \cdots + N_k}. \tag{4.3}$$

Clearly $N_1 + \cdots + N_k \to +\infty$ as $k \to \infty$. By the strong law of large numbers it thus follows that

$$\frac{S_{N_1 + \cdots + N_k}}{N_1 + \cdots + N_k} \to \mu \quad \text{a.s. as } k \to \infty \tag{4.4}$$

and that

$$\frac{S_{N_1 + \cdots + N_k}}{k} \to ES_N \quad \text{a.s. as } k \to \infty. \tag{4.5}$$

Consequently,

$$\frac{N_1 + \cdots + N_k}{k} \to \mu^{-1} ES_N \quad \text{a.s. as } k \to \infty, \tag{4.6}$$

from which it follows that

$$EN < \infty$$

by the converse of the Kolmogorov strong law of large numbers. □

PROOF OF THEOREM 2.3 WHEN $r = 1$. We can, and do, suppose $ES_N = 0$ (otherwise we replace X with $X - ES_N/EN$). As in the previous proof we let $\{N_k, k \geq 1\}$ be independent copies of N and let $\{M_k, k \geq 1\}$ denote their partial sums. Further, let $\{\tau_n, n \geq 1\}$ be the corresponding first passage times, that is,

$$\tau_n = \min\{k; M_k \geq n\} \tag{4.7}$$

and set $M(n) = M_{\tau_n}$ (the first renewal after time n).

It now follows from the strong law of large numbers (recall (4.5)) that

$$\frac{S_{M_k}}{k} \to 0 \quad \text{a.s. as } k \to \infty. \tag{4.8}$$

Moreover, since $\tau_n \to \infty$ as $n \to \infty$ we also have

$$\frac{S_{M(n)}}{\tau_n} \to 0 \quad \text{a.s. as } n \to \infty \tag{4.9}$$

and, since, by renewal theory, $\tau_n/n \to 1/EN$ a.s. as $n \to \infty$, we conclude that

$$\frac{S_{M(n)}}{n} \to 0 \quad \text{a.s. as } n \to \infty. \tag{4.10}$$

The next step is to prove that

$$\frac{S_n}{n} \to_p 0 \quad \text{as } n \to \infty. \tag{4.11}$$

For simplicity we assume that N is aperiodic, i.e., that there exists no integer $d > 1$ such that N a.s. is a multiple of d. (Otherwise, the argument below holds for n restricted to multiples of the largest such integer d, which suffices to prove (4.11).)

Now, the overshoot $M(n) - n$ converges in distribution as $n \to \infty$ towards some random variable Y, say (see Prabhu (1965), Chapter 5, Theorem 4.4 and Problem 11).

Let $\varepsilon > 0$ and $j \geq 1$ be arbitrary. Then

$$\begin{aligned}
P\left(\left|\frac{S_n}{n}\right| > 2\varepsilon\right) &= P(|S_n| > 2n\varepsilon) \\
&\leq P\big(\{|S_{M(n)}| > n\varepsilon\} \cup \{|S_{M(n)} - S_n| > n\varepsilon\}\big) \\
&\leq P\big(|S_{M(n)}| > n\varepsilon\big) + P\big(|S_{n+k} - S_n| > n\varepsilon \text{ for some } k \leq M(n) - n\big) \\
&\leq P\left(\left|\frac{S_{M(n)}}{n}\right| > \varepsilon\right) + P\Big(\max_{1 \leq k \leq j} |S_k| > n\varepsilon\Big) + P(M(n) - n > j) \\
&\to 0 + 0 + P(Y > j) \quad \text{as } n \to \infty
\end{aligned}$$

and, since j was arbitrary, (4.11) follows.

By using symmetrization and Lévy's inequality it now follows that

$$\frac{1}{n} \max_{1 \leq k \leq n} |S_k| \to_p 0 \quad \text{as } n \to \infty, \tag{4.12}$$

which implies that

(4.13) $$P\Big(\max_{1\le j\le k}|S_j| > k\Big) < \tfrac{1}{2} \quad \text{for } k \ge \text{some } n_0.$$

For $k \ge n_0$ we thus have

$$\begin{aligned} P\Big(\min_{1\le j\le k}|S_j| > k\Big) &\ge P\Big(\{|X_1| > 2k\} \cap \Big\{\max_{1\le j\le k}|S_j - S_1| \le k\Big\}\Big)\\ &= P(|X_1| > 2k)P\Big(\max_{1\le j\le k-1}|S_j| \le k\Big)\\ &\ge \tfrac{1}{2}P(|X| > 2k). \end{aligned}$$

On the other hand, we have, for all k,

$$\begin{aligned} P\Big(\min_{1\le j\le k}|S_j| > k\Big) &= P\Big(\Big\{\min_{1\le j\le k}|S_j| > k\Big\} \cap \{N > k\}\Big)\\ &\quad + P\Big(\Big\{\min_{1\le j\le k}|S_j| > k\Big\} \cap \{N \le k\}\Big)\\ &\le P(N > k) + P(|S_N| > k). \end{aligned}$$

Summation finally yields

$$\begin{aligned} \tfrac{1}{2}\sum_{k=n_0}^{\infty} P(|X| > 2k) &\le \sum_{k=1}^{\infty} P(N > k) + \sum_{k=1}^{\infty} P(|S_N| > k)\\ &\le EN + E|S_N| < \infty, \end{aligned}$$

and thus that $E|X| < \infty$. □

REMARK 4.1. The weak law of large numbers (4.11) is not by itself sufficient to guarantee that EX exists; see Feller (1966), Chapter VII.7 and Lemma 10.1.

5. Proofs of Theorems 2.1–2.4 when $r > 1$.

PROOF OF THEOREM 2.1. Since $ES_N \le (ES_N^r)^{1/r} < \infty$ we know from Section 4 that $\mu = EX < \infty$ and that $EN < \infty$. Moreover, the positivity of the summands implies that

(5.1) $$EX_1^r \le ES_N^r < \infty.$$

Next, we note that

(5.2) $$\mu N \le |S_N - N\mu| + S_N.$$

We claim

(a) $$E|S_N - N\mu|^r < \infty$$

(b) $$EN^r < \infty.$$

To prove this we proceed by induction on r through the powers of 2 (cf. Gut (1974a, b)).

Let $1 < r \le 2$. By (1.2) we have

$$E|S_N - N\mu|^r \le B_r E|X - \mu|^r EN < \infty, \tag{5.3}$$

so (a) holds for this case, from which (b) follows by (5.2) and Minkowski's inequality or the c_r-inequality.

Next, suppose that $2 < r \le 2^2$. Since $1 < r/2 \le 2$ we know from what has just been proved that (b) holds with r replaced by $r/2$. This together with (1.2) shows that

$$E|S_N - N\mu|^r \le B_r E|X - \mu|^r EN^{r/2} < \infty, \tag{5.4}$$

and another application of (5.2) shows that $EN^r < \infty$. Thus (a) and (b) hold again.

In general, if $2^k < r \le 2^{k+1}$ for some $k > 2$ we repeat the same procedure from $r/2^k$ to $r/2^{k-1}$, etc., until r is reached and the conclusion follows. □

PROOF OF THEOREM 2.2. Since the summands may take negative values we first have to replace (5.2) by

$$|\mu|N \le |S_N - N\mu| + |S_N|. \tag{5.5}$$

An inspection of the proof of Theorem 2.1 shows that the positivity there was only used to conclude that the summands had a finite moment of order r and that the stopping time had finite expectation. Now, in the present result the first fact was assumed and the second fact has been proved in Section 4. Therefore, the last part of the previous proof, with (5.2) replaced by (5.5) carries over verbatim to the present theorem. We can thus conclude that $EN^r < \infty$ and the proof is complete. □

PROOF OF THEOREM 2.3. Since $E|S_N|$ and EN both are finite, we know, from Section 4, that $\mu = EX$ is finite. The conclusion therefore follows immediately from Corollary 3.1. Alternatively, one can use Theorem 2.4 applied to $\{X_k - \mu\}$. □

PROOF OF THEOREM 2.4. Since $\{S_n\}$ is a martingale, it follows that $E(S_{N\wedge n}|\mathscr{F}_1) = X_1$ for all $n = 1, 2, \ldots$. Further, $\sup_n|S_{N\wedge n}| \le \sum_1^N|X_k|$, and

$$E\sum_1^N|X_k| = ENE|X| < \infty. \tag{5.6}$$

Hence, by dominated convergence,

$$E(S_N|\mathscr{F}_1) = \lim_{n\to\infty} E(S_{N\wedge n}|\mathscr{F}_1) = X_1, \tag{5.7}$$

whence

$$E|X_1|^r = E|E(S_N|\mathscr{F}_1)|^r \le E|S_N|^r < \infty. \tag{5.8}$$

□

REMARK 5.1. The crucial formula (5.7) may be written as $E(\sum_2^N X_k|\mathscr{F}_1) = 0$, and can thus be recognized as a conditional version of Wald's lemma, $E\sum_2^N X_k = 0$. In fact, we may derive (5.7) by restricting attention to an arbitrary subset $A \in \mathscr{F}_1$ with $P(A) > 0$ and applying Wald's lemma to obtain $E(\sum_2^N X_k|A) = 0$. Similarly, the extension of Wald's lemma by Burkholder and Gundy (1970) and

Chow, Robbins, and Siegmund (1971), implies that (5.7) and Theorem 2.4 hold as soon as $EN^\alpha < \infty$ for some $\alpha \geq \frac{1}{2}$, provided we assume that $E|X|^{1/\alpha} < \infty$.

6. Proofs of the results in Section 3.

PROOF OF THEOREM 3.1. Since

$$S_N^+ \leq \sum_1^N X_k^+ \tag{6.1}$$

we have

$$E(S_N^+)^r \leq E\left(\sum_1^N X_k^+\right)^r, \tag{6.2}$$

from which part (i) follows from Theorem 1.1(i) applied to $\{X_k^+\}$.

In order to prove part (ii), we construct an i.i.d. sequence $\{Y_k\}$ by truncating the positive tails of X_k, such that $Y_k \leq X_k$, $EY_k = 0$, and $E|Y_k|^r < \infty$. Then $S_N \geq \Sigma_1^N Y_k$, whence $S_N^- \leq |\Sigma_1^N Y_k|$, and the result follows by Theorem 1.1(ii) applied to $\{Y_k\}$. □

PROOF OF THEOREM 3.2(i). By the law of large numbers $S_n \to +\infty$ a.s. Thus $\min_{n\geq 0} S_n$ is an a.s. finite random variable and for some real number A we have $P(\min_{n\geq 0} S_n > -A) > \frac{1}{2}$. Since $S_N^+ \geq S_N \geq \min_{n\geq 1} S_n = X_1 + \min_{n\geq 1}\Sigma_2^n X_k$, where X_1 and $\min_{n\geq 1}\Sigma_2^n X_k$ are independent and the latter minimum is distributed as $\min_{n\geq 0} S_n$, it follows that

$$\begin{aligned} P(X_1 > t) &\leq 2P(X_1 > t)P\left(\min_{n\geq 0} S_n > -A\right) \\ &\leq 2P(S_N^+ > t - A) = 2P(S_N^+ + A > t). \end{aligned}$$

Consequently, by integrating over t,

$$E(X_1^+)^r \leq 2E(S_N^+ + A)^r < \infty. \qquad \square$$

PROOF OF THEOREM 3.2(ii). By (5.7) and convexity we have

$$X_1^+ \leq E(S_N^+ | \mathscr{F}_1) \tag{6.3}$$

and hence (cf. (5.8)) that

$$E(X_1^+)^r \leq E(S_N^+)^r < \infty. \tag{6.4}$$

□

PROOF OF THEOREM 3.2(iii). Set $EX = \mu$ and let, for $k \geq 1$, $Y_k = X_k - 2\mu$. Then $EY_k = -\mu > 0$ and

$$\sum_{k=1}^N Y_k = S_N - 2\mu N \leq S_N^+ + 2|\mu|N. \tag{6.5}$$

Consequently

$$E\left(\left(\sum_{k=1}^N Y_k\right)^+\right)^r < \infty, \tag{6.6}$$

and thus, by part (i), we conclude that

$$E(Y_1^+)^r < \infty. \tag{6.7}$$

Finally, since $X_k = 2|\mu| + Y_k \le 2|\mu| + Y_k^+$, it follows that $X_k^+ \le 2|\mu| + Y_k^+$ and the proof is complete. □

PROOF OF THEOREM 3.3. By Theorem 3.2(i) we may assume that $E|X|^r < \infty$. The first part of the proof of Theorem 2.2 ($EN < \infty$) now carries over without modification once we note that, since $ES_N^+ < \infty$, (4.5) holds with ES_N either finite or $-\infty$, the latter case being ruled out by (4.4) and the fact that $\mu > 0$. Similarly, the second part of the proof carries over, if we replace (5.5) by

$$\mu N = \mu N - S_N + S_N \le |S_N - N\mu| + S_N^+. \tag{6.8}$$ □

7. Examples. Here we collect some further examples relating to the sharpness of the theorems in Sections 2 and 3. We begin with two trivial cases.

EXAMPLE 7.1. Let $N = 1$. Thus $S_N = X_1$. This shows that we cannot, in general, obtain higher moments on S_N than on X and conversely. The same holds for the positive and negative tails.

EXAMPLE 7.2. Let $X = 1$. Thus $S_N = N$ (which is arbitrary) and we cannot, in general (when $EX \ne 0$), obtain higher moments on S_N than on N and conversely. However, note that when $EX = 0$, Theorem 1.1(ii) yields an improvement in the order of the moments, while the converse utterly fails by Example 2.2.

When $EX > 0$, the law of large numbers implies that $S_n \to +\infty$ a.s. as $n \to \infty$ and thus that $S_n^- \to 0$. One might therefore suspect that the only way to get S_N^- large is to let N be comparatively small, which, in particular, would indicate that the moment condition on N in Theorem 3.1(ii) might be superfluous, i.e., that $E(S_N^-)^r$ be finite as soon as $E(X^-)^r$ is. The following example shows that this is false, and that, indeed, no moment condition weaker than $EN < \infty$ is sufficient. However, when $r > 2$ we do not know whether $EN^{r/2} < \infty$ (as given in Theorem 3.1(ii)) really is required or whether, e.g., $EN < \infty$ suffices. Note also that

$$EX > 0 \quad \text{and} \quad E(X^-)^{r+1} < \infty \Rightarrow E\left|\min_{n\ge 0} S_n\right|^r; \tag{7.1}$$

see, e.g., Janson (1986), from which it follows that $E(S_N^-)^r < \infty$ for any N (stopping time or not).

EXAMPLE 7.3. Let $1 < r < s$ and let X be such that $EX = 1$ and $P(X < -t) = t^{-s}$, $t \ge t_0$. By the law of large numbers $P(S_n/n < 2) > \frac{1}{2}$ for $n \ge n_0$. Fix $n > \max(t_0, n_0)$ and let E_k denote the event $\{S_n < -n \text{ and } X_k < -3n\}$, $k = 1, \dots, n$. Then

$$\begin{aligned} P(S_n < -n) &\ge P\left(\bigcup_{k=1}^n E_k\right) \ge \sum_1^n P(E_k) - \sum_{j<k} P(E_j \cap E_k) \\ &= nP(E_n) - \tfrac{1}{2}n(n-1)P(E_1 \cap E_2). \end{aligned}$$

Further,

$$\begin{aligned} P(E_n) &\ge P(S_{n-1} < 2n \text{ and } X_n < -3n) \\ &= P(S_{n-1} < 2n)P(X_n < -3n) > \tfrac{1}{2}(3n)^{-s} \end{aligned}$$

and

$$P(E_1 \cap E_2) \le P(X_1 < -3n \text{ and } X_2 < -3n) = (3n)^{-2s}.$$

Consequently, for some positive numbers c_1, c_2, c_3,

$$P(S_n < -n) \ge c_1 n^{1-s} - c_2 n^{2-2s} \ge c_3 n^{1-s}$$

and

$$E(S_n^-)^r \ge c_3 n^{1+r-s}, \qquad n > \max(t_0, n_0).$$

It follows that if N is independent of $\{X_i\}$, and $EN^{1+r-s} = \infty$, then $E(S_N^-)^r = \infty$.

In particular, if $\varepsilon > 0$ and $s < r + \varepsilon$ we may have

$$EN^{1-\varepsilon} < \infty \quad \text{and} \quad E(X^-)^r < \infty \quad \text{but} \quad E(S_N^-)^r = \infty.$$

The next example shows that Corollary 3.1 (unlike the two-sided version Corollary 2.1(i)) may fail if $E|X| = \infty$.

EXAMPLE 7.4. Let $1 < r < 2$ and let $\{U_k\}$ and $\{Y_k\}$ be independent sequences of random variables, such that U_k, $k \ge 1$, are i.i.d. standard normal random variables and Y_k, $k \ge 1$ and i.i.d. symmetric and stable with index r.

Set $Z_k = U_k^{-2}$ (that is, Z_k is positive and stable with index $\frac{1}{2}$) and $X_k = Y_k + Z_k$. Thus

$$EX^+ = \infty, \qquad EX^- < \infty, \qquad E(X^+)^r = E(X^-)^r = \infty. \tag{7.2}$$

Define N_+ as in Example 2.1. Fix a number α, such that $1 + r^{-1} < \alpha < 2$. Then

$$\begin{aligned}
P(N_+ > n) \le P(S_n \le 0) &= P\left(\sum_{k=1}^n Z_k \le -\sum_{k=1}^n Y_k\right) \\
&= P\left(\left\{\sum_{k=1}^n Z_k \le -\sum_{k=1}^n Y_k\right\} \cap \left\{\sum_{k=1}^n Z_k \le n^\alpha\right\}\right) \\
&\quad + P\left(\left\{\sum_{k=1}^n Z_k \le -\sum_{k=1}^n Y_k\right\} \cap \left\{\sum_{k=1}^n Z_k > n^\alpha\right\}\right) \\
&\le P\left(\sum_{k=1}^n Z_k \le n^\alpha\right) + P\left(\sum_{k=1}^n Y_k \le -n^\alpha\right) \\
&= P\left(\sum_{k=1}^n Z_k \le n^\alpha\right) + P\left(\sum_{k=1}^n Y_k \ge n^\alpha\right) \\
&= P(Z \le n^{\alpha-2}) + P(Y \ge n^{\alpha-1/r}) \\
&= P(U^{-2} \le n^{\alpha-2}) + P(Y \ge n^{\alpha-1/r}) \\
&= P\left(|U|^{2/(2-\alpha)} \ge n\right) + \tfrac{1}{2}P\left(|Y|^{r/(\alpha r-1)} \ge n\right)
\end{aligned}$$

and it follows that

$$E(N_+ - 1)^r \le E|U|^{2r/(2-\alpha)} + E|Y|^{r^2/(\alpha r-1)} < \infty. \tag{7.3}$$

since $r^2/(\alpha r - 1) < r$.

Consequently, N_+ and $S^-_{N_+}$ $(= 0)$ have finite moments of order r, whereas X^- does not. Thus, Corollary 3.1 is false without the assumption that EX is finite, and (by a change of signs) Theorem 3.2(iii) fails for $EX = -\infty$.

Returning to the case $EX > 0$, $r > 1$, we note that Examples 2.1 and 7.1–7.3 yield examples of 9 of the 14 cases in Table 1. Examples of the remaining 5 cases may be obtained by simple modifications, e.g., $N = N_+ + 1$ and $N = \min\{n; S_n > Z\}$, where Z is a random variable independent of $\{X_k\}$.

8. Further remarks.

A. The case $r < 1$. We note, without any attempt at completeness, that some of the results above also hold for $r < 1$. For example, Theorems 2.1 and 2.2 still hold; we may prove them by a simple modification of the proof of Theorem 2.2 for $r = 1$, given in Section 4, by using the Marcinkiewicz strong law of large numbers and its converse. On the other hand, we shall now see that Theorem 1.1(i) is not true for $r < 1$. From (1.2) we obtain

$$E|X|^r < \infty \quad \text{and} \quad EN < \infty \Rightarrow E|S_N|^r < \infty, \qquad 0 < r < 1, \tag{8.1}$$

which, by the following example, is best possible. The converse to (8.1) is obviously false (take $X = 1$), and thus there is a gap between the conditions in the two directions.

EXAMPLE 8.1. Let $\{Y_k\}$ be i.i.d. random variables that are symmetric and stable with index $\alpha > 2r$ and let $X_k = Y_k^2$. Then by the Marcinkiewicz—Zygmund inequalities,

$$ES_n^r \sim E\left|\sum_1^n Y_k\right|^{2r} = n^{2r/\alpha}E|Y|^{2r}.$$

Hence, if N is independent of $\{X_k\}$,

$$ES_N^r \sim EN^{2r/\alpha}. \tag{8.2}$$

Here $2r/\alpha$ may be arbitrarily close to 1.

B. N independent of $\{X_k\}$. In Examples 7.1–7.3, N is independent of $\{X_k\}$. This is a rather trivial type of stopping time, and one may ask whether sharper results are true in this case. In fact, we have the following improvements of the results in Sections 2 and 3.

THEOREM 8.1. *Let $r > 0$. If N is independent of $\{X_k\}$, then*

(i) $$E(S_N^+)^r < \infty \Rightarrow E(X^+)^r < \infty,$$

(ii) $$E(S_N^-)^r < \infty \Rightarrow E(X^-)^r < \infty,$$

(iii) $$E|S_N|^r < \infty \Rightarrow E|X|^r < \infty.$$

PROOF. Assume that $E(S_N^+)^r < \infty$. Then $E(S_n^+)^r < \infty$ for some n (every n with $P(N = n) > 0$), and $E(X^+)^r < \infty$ follows. (ii) and (iii) follow immediately. □

THEOREM 8.2. *Let $r > 0$ and let N be independent of $\{X_k\}$. If $EX > 0$ and $E(S_N^+)^r < \infty$ then $EN^r < \infty$.*

PROOF. By the law of large numbers, $P(S_n > n\mu/2) > \frac{1}{2}$ for $n > n_0$. Consequently $E((S_N^+)^r | N = n) = E(S_n^+)^r > \frac{1}{2}(\mu/2)^r n^r$, $n > n_0$, and $N^r \le n_0^r + CE((S_N^+)^r | N)$. The conclusion follows. □

By combining these results with Theorem 3.1 we obtain

COROLLARY 8.1. *Let $r \ge 1$. If N is independent of $\{X_k\}$ and $EX > 0$, then*

(i) $$E(S_N^+)^r < \infty \Leftrightarrow E(X^+)^r < \infty \quad \text{and} \quad EN^r < \infty,$$

(ii) $$E|S_N|^r < \infty \Leftrightarrow E|X|^r < \infty \quad \text{and} \quad EN^r < \infty.$$

Looking at Table 1, we see that four cases (5, 6, 7, 13) are impossible; examples of the other ten cases may be given.

Furthermore, in this situation, Theorem 3.1(ii) may be sharpened to

$$EX > 0, \qquad E(X^-)^r < \infty \quad \text{and} \quad EN < \infty \Rightarrow E(S_N^-)^r < \infty.$$

The proof is omitted.

We repeat that we do not know whether this holds for arbitrary stopping times.

C. N not a stopping time. In the previous remark we specialized N (to be independent of the summands) and found that some of the results could be strengthened. In this remark we shall, conversely, see to what extent (if any) the results remain true if we only assume that N is a positive, integer valued random variable, that is, not necessarily a stopping time.

We first show that Theorem 1.1 fails. In fact, the next example yields a counterexample to both parts.

EXAMPLE 8.2. Let β, $s > 1$ and let X be a symmetric random variable with $P(|X| > t) = t^{-s}$, $t > 1$. Then, by standard arguments,

$$P(|S_n| > t) \sim nt^{-s}, \qquad t > n \ge 1.$$

Let $N = \max\{n;\, n = 2^k \text{ for some } k \ge 0 \text{ and } S_n > n^\beta\}$ (where we define $\max \varnothing = 1$). Thus $S_N > N^\beta$ unless $N = 1$, i.e.,

$$S_N^+ \ge N^\beta - 1. \tag{8.3}$$

Now, let $k \ge 1$. Then

$$P(S_{2^k} > 2^{\beta k}) \le P(N \ge 2^k) \le \sum_{m=k}^{\infty} P(S_{2^m} > 2^{\beta m}).$$

Thus

$$P(N \geq 2^k) \sim P\left(S_{2^k} > 2^{\beta k}\right) \sim 2^{k(1-\beta s)}.$$

Consequently $EN^p < \infty \Leftrightarrow p < \beta s - 1$.

Now, given $r \geq 1$, take s such that $r < s < r + 1$ and let $\beta = (s - r)^{-1}$. Then $r < \beta r = \beta s - 1$. Hence $EX = 0$, $E|X|^r < \infty$, and $EN^r < \infty$ but $EN^{\beta r} = \infty$ and it follows, in view of (8.3), that $E(S_N^+)^r = +\infty$.

Turning to the converses, we note that Theorem 2.1 still holds when X is nonnegative. This follows from the inequality $X_1 \leq S_N$ and the following analogue of Theorem 3.3.

THEOREM 8.3. *Suppose that N is an arbitrary random variable and $EX > 0$. If $r \geq 1$, then*

$$E(S_N^+)^r < \infty \quad \textit{and} \quad E(X^-)^{r+1} < \infty \Rightarrow EN^r < \infty.$$

PROOF. Let $Y_k = X_k - \mu/2$. Then $EY > 0$ and $E(Y^-)^{r+1} < \infty$ and thus $\min_{n \geq 0} \Sigma_1^n Y_k \in L^r$, see, e.g., Janson (1986). Since $S_N - N\mu/2 = \Sigma_1^N Y_k \geq \min_{n \geq 0} \Sigma_1^n Y_k$, the conclusion follows from the fact that

$$N\mu/2 \leq S_N^+ + (-S_N + N\mu/2)^+ \leq S_N^+ + \left|\min_{n \geq 0} \sum_1^n Y_k\right| \in L^r. \qquad \square$$

EXAMPLE 8.3. One interesting application of Theorem 8.3 is when N is the last exit time

$$N = N_t = \max\{n; S_n \leq t\}, \quad \text{where } t \geq 0.$$

Since S_N^+ is bounded, Theorem 8.3 shows that $EN^r < \infty$ provided $E(X^-)^{r+1} < \infty$. Moreover, in this case the converse holds, i.e., $EN^r < \infty \Rightarrow E(X^-)^{r+1} < \infty$; see, e.g., Janson (1986).

This shows that we really do need one extra moment on X^- here in contrast to when N is a stopping time; cf. Theorem 3.3.

9. Uniform integrability. In the remainder of this paper we consider the random walk $\{S_n, n \geq 1\}$ and the sequence $\{\mathscr{F}_n, n \geq 1\}$ of σ-algebras as before, but, instead of a single stopping time, we have a family of stopping times, $\{N_\alpha, \alpha \in I\}$, where I is an arbitrary index set. A typical case with applications, e.g., in renewal theory is $I = R^+$.

We shall extend some of the above results about existence of moments to results about uniform integrability.

Let $\{b_\alpha, \alpha \in I\}$ be an arbitrary family of positive, normalizing constants. We begin by stating a result corresponding to Theorem 1.1.

THEOREM 9.1. *Let $r > 0$ and suppose that $E|X|^r < \infty$.*

(i) *If*

$$\{b_\alpha^{-1} N_\alpha^{r \vee 1}\} \text{ is uniformly integrable}, \tag{9.1}$$

then

$$\{b_\alpha^{-1} |S_{N_\alpha}|^r\} \text{ is uniformly integrable}. \tag{9.2}$$

(ii) *Let $r \geq 1$ and suppose, in addition, that $EX = 0$. If*

$$\{b_\alpha^{-1} N_\alpha^{r/2 \vee 1}\} \text{ is uniformly integrable}, \tag{9.3}$$

then

$$\{b_\alpha^{-1} |S_{N_\alpha}|^r\} \text{ is uniformly integrable}. \tag{9.4}$$

If, in particular, we have $I = [t_0, \infty)$ for some $t_0 > 0$ and let $b_t = t^r$, $b_t = t$, and $b_t = t^{r/2}$ (where we use t instead of α) we obtain the following results related to the classical strong law, the Marcinkiewicz strong law, and the central limit theorem, respectively.

COROLLARY 9.1. *Let $r \geq 1$ and suppose that $E|X|^r < \infty$. If*

$$\left\{\left(\frac{N(t)}{t}\right)^r, t \geq t_0\right\} \text{ is uniformly integrable},$$

then

$$\left\{\left|\frac{S_{N(t)}}{t}\right|^r, t \geq t_0\right\} \text{ is uniformly integrable}.$$

COROLLARY 9.2. *Let $0 < r \leq 2$. Suppose that $E|X|^r < \infty$ and that $EX = 0$ when $r \geq 1$. If*

$$\left\{\left(\frac{N(t)}{t}\right), t \geq t_0\right\} \text{ is uniformly integrable},$$

then

$$\left\{\left|\frac{S_{N(t)}}{t^{1/r}}\right|^r, t \geq t_0\right\} \text{ is uniformly integrable}.$$

COROLLARY 9.3. *Let $r \geq 2$. Suppose that $E|X|^r < \infty$ and that $EX = 0$. If*

$$\left\{\left(\frac{N(t)}{t}\right)^{r/2}, t \geq t_0\right\} \text{ is uniformly integrable},$$

then

$$\left\{\left|\frac{S_{N(t)}}{\sqrt{t}}\right|^r, t \geq t_0\right\} \text{ is uniformly integrable}.$$

Corollaries 9.1–9.3 are due to Lai (1975), Chang and Hsiung (1979), and Yu (1979), respectively. Theorem 9.1 can be proved by the same methods. For proofs of the corollaries and some applications see also Gut (1986).

We now turn our attention to the converse results, corresponding to Theorems 2.1 and 2.2.

THEOREM 9.2. *Let $r \geq 1$, suppose that $P(X \geq 0) = 1$, and $P(X > 0) > 0$. If*

$$\left\{ b_\alpha^{-1} S_{N_\alpha}^r \right\} \text{ is uniformly integrable,} \tag{9.5}$$

then

$$\left\{ b_\alpha^{-1} N_\alpha^r \right\} \text{ is uniformly integrable.} \tag{9.6}$$

THEOREM 9.3. *Let $r \geq 1$ and suppose that $E|X|^r < \infty$ and $EX \neq 0$. If*

$$\left\{ b_\alpha^{-1} |S_{N_\alpha}|^r \right\} \text{ is uniformly integrable,} \tag{9.7}$$

then

$$\left\{ b_\alpha^{-1} N_\alpha^r \right\} \text{ is uniformly integrable.} \tag{9.8}$$

In particular, for $EX \neq 0$, it follows that the converse to Corollary 9.1 holds. Example 2.2 shows that no converse is possible when $EX = 0$.

It follows from Theorem 2.1 that the assumptions in Theorem 9.2 imply that $EX^r < \infty$. Theorem 9.2 thus follows from Theorem 9.3. The proof of Theorem 9.3 is given in Sections 10 and 11; as before we treat the cases $r = 1$ and $r > 1$ separately.

10. Proof of Theorem 9.3 for $r = 1$.

The theorem is a uniform version of Theorem 2.2 and we will use the same idea as in that proof, making all assertions uniform in α. However, here we will work with the weak law of large numbers, and begin by stating a uniformization of it, the proof of which will be given at the end of this section.

LEMMA 10.1. *Let $\{Z_\alpha\}$ be a family of random variables and let $\{a_\alpha\}$ be a bounded set of real numbers. Let $\{Z_{\alpha,k}\}_{k=1}^\infty$ be independent copies of Z_α.*

(a) *If $0 < p < 1$, the following are equivalent:*
  - (i) $E|1/n\Sigma_1^n Z_{\alpha,k} - a_\alpha|^p \to 0$, *uniformly in α, as $n \to \infty$.*
  - (ii) $1/n\Sigma_1^n Z_{\alpha,k} \to_p a_\alpha$, *uniformly in α, as $n \to \infty$.*
  - (iiia) $tP(|Z_\alpha| > t) \to 0$, *uniformly in α, as $t \to \infty$ and*
  - (iiib) $E(Z_\alpha I\{|Z_\alpha| \leq t\}) \to a_\alpha$, *uniformly in α, as $t \to \infty$.*

(b) *If $\{Z_\alpha\}$, furthermore, is uniformly integrable, then (i)–(iii) hold with $a_\alpha = EZ_\alpha$.*

(c) *If $Z_\alpha \geq 0$ a.s. for all α and (one of) (i)–(iii) hold(s), then $\{Z_\alpha\}$ is uniformly integrable.*

For the case of a single random variable (a) reduces to the weak law of large numbers; see Feller (1966), Section VII.7. We also refer to Esseen and Janson (1984) for some other generalizations.

PROOF OF THEOREM 9.3 WHEN $r = 1$. Set $\mu = EX$. Construct, for every α, independent copies $\{N_{\alpha,k}\}_{k=1}^{\infty}$ of N_α as in Section 4 and let $M_{\alpha,n} = \sum_1^n N_{\alpha,k}$. Thus, for a fixed α, $M_{\alpha,n}$ is an increasing sequence of stopping times and $\{S_{M_{\alpha,n}}\}_{n=1}^{\infty}$ is a sequence of partial sums of independent random variables distributed as S_{N_α}.

Set $Z_\alpha = b_\alpha^{-1} S_{N_\alpha}$. Since, by assumption, $\{Z_\alpha\}$ is uniformly integrable, Lemma 10.1(b) implies that

$$\frac{1}{n} b_\alpha^{-1} S_{M\alpha,n} \to_p b_\alpha^{-1} E S_{N_\alpha}, \quad \text{uniformly in } \alpha, \text{ as } n \to \infty. \tag{10.1}$$

On the other hand, $M_{\alpha,n} \geq n$, whence

$$P\big(|S_{M_{\alpha,n}}/M_{\alpha,n} - \mu| > \varepsilon\big) \leq P\big(|S_m/m - \mu| > \varepsilon \text{ for some } m \geq n\big).$$

The right-hand side is independent of α and tends to 0 for every ε by the strong law of large numbers. Hence

$$\frac{1}{M_{\alpha,n}} S_{M_{\alpha,n}} \to_p \mu, \quad \text{uniformly in } \alpha \text{ as } n \to \infty. \tag{10.2}$$

Since $\mu \neq 0$ and $\{b_\alpha^{-1} E S_{N_\alpha}\}$ is bounded, (10.1) and (10.2) yield, by division,

$$\frac{1}{n} b_\alpha^{-1} M_{\alpha,n} \to_p b_\alpha^{-1} \mu^{-1} E S_{N_\alpha}, \quad \text{uniformly in } \alpha, \text{ as } n \to \infty. \tag{10.3}$$

An application of Lemma 10.1(c) with $Z_\alpha = b_\alpha^{-1} N_\alpha$ concludes the proof. □

PROOF OF LEMMA 10.1(a). (iii) ⇒ (i). Truncate Z_α, and similarly $Z_{\alpha,k}$, by defining

$$Z_\alpha^t = Z_\alpha I\{Z_\alpha \leq t\}, \qquad t > 0.$$

We note that if (iiia) holds, then $E|Z_\alpha^t|^2 = o(t)$ and $E|Z_\alpha - Z_\alpha^t|^p = o(t^{p-1})$ as $t \to \infty$, uniformly in α. Hence, by taking $t = n$, we obtain

$$\begin{aligned} E\left|\sum_1^n (Z_{\alpha,k}^n - EZ_\alpha^n)\right|^p &\leq \left(E\left|\sum_1^n (Z_{\alpha,k}^n - EZ_\alpha^n)\right|^2\right)^{p/2} \\ &= \big(nE(Z_\alpha^n - EZ_\alpha^n)^2\big)^{p/2} \leq \big(nE(Z_\alpha^n)^2\big)^{p/2} = o(n^p), \end{aligned}$$

and

$$E\left|\sum_1^n (Z_{\alpha,k} - Z_{\alpha,k}^n)\right|^p \leq nE|Z_\alpha - Z_\alpha^n|^p = o(n^p), \quad \text{uniformly in } \alpha.$$

The c_r-inequality now yields

$$E\left|\sum_1^n Z_{\alpha,k} - nEZ_\alpha^n\right|^p = o(n^p) \quad \text{as } n \to \infty, \text{ uniformly in } \alpha.$$

We have thus shown that

$$(10.4)\qquad \text{(iiia)} \Rightarrow E\left|\frac{1}{n}\sum_{1}^{n} Z_{\alpha,k} - EZ_\alpha^n\right|^p \to 0 \quad \text{uniformly in } \alpha, \text{ as } n \to \infty.$$

Since (iiib) may be written $EZ_\alpha^t \to a_\alpha$, uniformly in α, as $t \to \infty$, it is now clear that (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Use Markov's inequality.

(ii) $\Rightarrow$ (iii). Assume first that the variables Z_α are symmetric. Then, by Feller (1966), formula (V.5.11),

$$\exp(-nP(|Z_\alpha| > n)) \geq 1 - 2P\left(\left|\sum_{1}^{n} Z_{\alpha,k}\right| > n\right) \to 1, \text{ uniformly in } \alpha, \text{ as } n \to \infty.$$

Consequently, $nP(|Z_\alpha| \geq n) \to 0$, uniformly in α, as $n \to \infty$. In general, we symmetrize by letting $\tilde{Z}_\alpha = Z_\alpha - Z'_\alpha$, where Z'_α is an independent copy of Z_α, and obtain $nP(|\tilde{Z}_\alpha| \geq n) \to 0$, uniformly in α, as $n \to \infty$.

Let $A = 1 + \sup|a_\alpha|$. For some integer m and every α we have

$$e^{-1} > P\left(\frac{1}{m}\sum_{1}^{m} Z_{\alpha,k} - a_\alpha > 1\right) \geq P\left(\frac{1}{m}\sum_{1}^{m} Z_{\alpha,k} > A\right) \geq P(Z_\alpha > A)^m.$$

Thus $P(Z_\alpha > A) < e^{-1/m}$ and $P(Z_\alpha \leq A) > 1 - e^{-1/m}$ for all α. Since $tP(Z_\alpha > t)P(Z_\alpha \leq A) \leq tP(\tilde{Z}_\alpha > t - A)$, it follows that $tP(Z_\alpha > t) \to 0$, uniformly in α, as $t \to \infty$.

By using the same arguments for the negative tails we obtain (iiia). By (10.4) we know that

$$\frac{1}{n}\sum_{k=1}^{n} Z_{\alpha,k} - EZ_\alpha^n \to_p 0, \quad \text{uniformly in } \alpha, \text{ as } n \to \infty,$$

which, together with (ii), shows that

$$EZ_\alpha^n \to a_\alpha, \quad \text{uniformly in } \alpha, \text{ as } n \to \infty,$$

which, in view of (iiia), yields (iiib). □

PROOF OF LEMMA 10.1(b). In view of (a) if suffices to prove (iii). (iiia) follows, because $tP(|Z_\alpha| > t) \leq E|Z_\alpha|I\{|Z_\alpha| > t\} \to 0$, uniformly in α, as $t \to \infty$ and (iiib) with $a_\alpha = EZ_\alpha$ follows, because

$$|EZ_\alpha I\{|Z_\alpha| \leq t\} - EZ_\alpha| = |EZ_\alpha I\{|Z_\alpha| > t\}| \leq E|Z_\alpha|I\{|Z_\alpha| > t\}. \qquad \square$$

PROOF OF LEMMA 10.1(c). By (iiib) and monotone convergence it follows that $EZ_\alpha = a_\alpha$ and thus that $EZ_\alpha I\{Z_\alpha > t\} \to 0$, uniformly in α, as $n \to \infty$, which proves the desired uniform integrability. □

11. Proof of Theorem 9.3 for $r > 1$. We repeat the induction argument in Section 5, using Theorem 9.1 instead of the moment inequalities. We thus assume that the theorem is true for $1 \leq r \leq 2^{k-1}$ and let $2^{k-1} < r \leq 2^k$. (The induction is started by the case $r = 1$ established in the preceding section.)

Let us first establish that $\inf_\alpha b_\alpha$ is strictly positive. By assumption and Liapounov's inequality we have

$$E|b_\alpha^{-1/r} S_{N_\alpha}| \le \text{constant}, \quad \text{uniformly in } \alpha.$$

By Wald's lemma we further obtain (note that $N_\alpha \ge 1$)

$$E|S_{N_\alpha}| \ge |ES_{N_\alpha}| = |\mu| EN_\alpha \ge |\mu|,$$

from which it follows that

$$b_\alpha^{-1/r} \le \text{constant}.$$

Next we observe that

$$\left\{ b_\alpha^{-1} |S_{N_\alpha}|^{r/2 \vee 1} \right\} \text{ is uniformly integrable.}$$

This is due to the fact that $b_\alpha^{-1}|S_{N_\alpha}|^{r/2 \vee 1} \le \max\{b_\alpha^{-1}, b_\alpha^{-1}|S_{N_\alpha}|^r\}$. By the induction hypothesis it now follows that

$$\left\{ b_\alpha^{-1} N_\alpha^{r/2 \vee 1} \right\} \text{ is uniformly integrable.}$$

We can thus apply Theorem 9.1(ii) to the sequence $\{X_n - \mu\}$ and conclude that $\{b_\alpha^{-1}|S_{N_\alpha} - N_\alpha \mu|^r\}$ is uniformly integrable. The triangle inequality (5.5) completes the proof. □

Finally, suppose that $EX = 0$. Recall from Section 2, Example 2.2 that the situation here is completely different. By expanding that example a little we shall see that Theorem 9.3 does not hold in this case.

EXAMPLE 11.1. Consider the coin-tossing example from Section 2; that is, let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. random variables such that $P\{X_n = 1\} = P\{X_n = -1\} = \frac{1}{2}$. Define

$$N(t) = \min\{n;\, S_n \ge [t]\} = \min\{n;\, S_n = [t]\} \qquad (t \ge 0).$$

Clearly, $S_{N(t)} = [t]$, and so

$$(0 \le) \frac{S_{N(t)}}{t} = \frac{[t]}{t} \le 1, \quad \text{for all } t > 0,$$

that is, $\{(S_{N(t)}/t)^r\}$ is uniformly integrable for all $r > 0$. On the other hand, we know from random walk theory (see, e.g., Example 2.2) that $E(N(0))^r = +\infty$ for all $r \ge 1$ $(r \ge \frac{1}{2})$ and, since $N(t) \ge N(0)$, $\{(N(t)/t)^r\}$ cannot be uniformly integrable for any $r \ge 1$.

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