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ANSWERS TO SOME QUESTIONS ABOUT INCREMENTS OF A WIENER PROCESS

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Let $W(t)$, $0 \le t < \infty$, be a Wiener process. This paper proves that

$$\limsup_{T\to\infty} \sup_{0<t\le T} \frac{|W(T) - W(T-t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} = 1, \quad \text{a.s.},$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} \frac{|W(T) - W(s-t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} = 1, \quad \text{a.s.}$$

These results give an affirmative answer to the questions posed by Hanson and Russo without additional assumptions.

1. Introduction. The study of convergence properties of increments of a Wiener process attracts the attention of many probabilists. Various important results about this problem are summarized in the book by Csörgő and Révész [3], including some general conclusions obtained in [2] by these authors. In order to investigate the limiting behavior of lag sums (properly normed), Hanson and Russo [4] studied some forms of increments of a Wiener process, resulting in the following Theorem A, which in some ways improves and generalizes the results of [2]:

THEOREM A ([4]). *Let $W(t)$, $0 \le t < \infty$, be a standardized Wiener process. Suppose that $0 < a_T \le T$ for $T > 0$, and*

$$a_T T^\alpha \to \infty \quad \text{as } T \to \infty \quad \text{for each } \alpha > 0. \tag{1}$$

Then one has

$$\limsup_{T\to\infty} \sup_{a_T \le t\le T} \frac{|W(T) - W(T-t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} = 1, \quad a.s. \tag{2}$$

Here and in the sequel we shall define $\log t = \log(\max(t,1))$, $\log\log t = \log\log(\max(t,e))$, for $t \ge 0$. ε stands for a positive number given arbitrarily, and C will be understood as a positive constant independent of n, which can take different values on each appearance.

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In connection with this theorem, Hanson and Russo proposed several questions: (i) Under what conditions on a_T do we get

$$\limsup_{T\to\infty} \frac{|W(T) - W(T - a_T)|}{\{2a_T(\log(T/a_T) + \log\log a_T)\}^{1/2}} \geq 1, \quad \text{a.s.?}$$

I.e., can the Csörgő–Révész condition be relaxed? (ii) Can assumption (1) be weakened or eliminated? (iii) In (2) can $\sup_{a_T \leq t \leq T}$ be replaced by $\sup_{0 < t \leq T}$? (iv) If the answer to the question (iii) is "no," can the denominator in (2) be changed so as to allow this replacement, and if so, how? (v) Under what conditions can we replace lim sup by lim in the theorems of Section 3 in [4]? When we cannot replace lim sup by lim, what is the lim inf?

Soon they answered the first question themselves in [5]. In order to answer questions (ii)–(v), Chen Guijing and Kong Fanchao, and, independently, Lin Zhengyan, investigated the remaining questions, resulting in almost the same results. The present article, which combines and refines our original manuscripts, is a joint work.

2. On questions (ii)–(iv). We succeeded in completely solving the above questions. Also some further facts were obtained.

THEOREM 1. *Let $W(t)$, $0 \leq t < \infty$, be a standardized Wiener process. We have*

$$\limsup_{T\to\infty} \sup_{0<t\leq T} \frac{|W(T) - W(T - t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} = 1, \quad a.s., \tag{3}$$

$$\lim_{T\to\infty} \sup_{0<t\leq T} \sup_{t\leq s\leq T} \frac{|W(s) - W(s - t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} = 1, \quad a.s., \tag{4}$$

$$\limsup_{T\to\infty} \sup_{0<t\leq T} \sup_{0\leq s\leq t} \frac{|W(T) - W(T - s)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} = 1, \quad a.s. \tag{4'}$$

PROOF. We shall proceed step by step.

(A) Using the law of the iterated logarithm, it follows that

$$\text{the left-hand side of (3)} \geq \limsup_{T\to\infty} \frac{|W(T)|}{(2t\log\log T)^{1/2}} = 1, \quad \text{a.s.,} \tag{5}$$

(B) In order to prove that

$$\text{the left-hand side of (4)} \leq 1, \quad \text{a.s.,} \tag{6}$$

we take real numbers $\Theta > 1$ and $Q > 1$ such that

$$1 < 2\frac{(1+\varepsilon)^2}{(2+\varepsilon)\Theta} \triangleq 1 + 2\varepsilon'.$$

For $n = 1,2,\ldots$ and $k = \ldots, -2, -1, 0, 1, 2, \ldots, k_n$, denote $T_n = Q^n$, $t_k = \Theta^k$, where $k \triangleq [(n+1)\log Q/\log \Theta] + 1$. Here and in the sequel $[a]$ stands for the greatest integer $\le a$. Write $\beta = 1/\varepsilon'$ and $k_\Theta = [1/\log \Theta]$, $k_n' = [\log(T_{n+1}/(\log T_n)^\beta)/\log \Theta]$.

When $T \in [T_n, T_{n+1}]$, we have

$$\sup_{0<t\le T} \sup_{t\le s\le T} \frac{|W(s) - W(s-t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}}$$

$$\le \sup_{-\infty<k\le k_n - 1} \sup_{t_k\le t\le t_{k+1}} \sup_{t\le s\le T_{n+1}} \frac{|W(s) - W(s-t)|}{\{2t_k(\log(T_n/t_{k+1}) + \log\log t_k)\}^{1/2}} \tag{7}$$

$$\triangleq \sup_{-\infty<k\le k_{n-1}} A_{nk}.$$

An inspection of the proof of the Csörgő–Révész lemmas (see Lemmas 1.1.1 and 1.2.1 in [3]) convinces one that for any $0 < T, v$, $0 < h \le T$, we have

$$P\left\{\sup_{0\le s', s\le T, 0\le s-s'\le h} h^{-1/2}|W(s) - W(s')| \ge v\right\} \le \frac{CT}{h}\exp\left\{\frac{-v^2}{2+\varepsilon}\right\}, \tag{8}$$

where C is a positive constant depending only on ε. Using this inequality, for $-\infty < k \le k_\Theta$ we have

$$\begin{aligned} P(A_{nk} \ge 1+\varepsilon) &\le P\left\{\sup_{0\le s-t, s\le T_{n+1}, 0<t\le t_{k+1}} t_{k+1}^{-1/2}|W(s) - W(s-t)| \right.\\ &\qquad\qquad \left. \ge (1+\varepsilon)\left(2\frac{t_k}{t_{k+1}}\log\frac{T_n}{t_{k+1}}\right)^{-1/2}\right\} \\ &\le C\frac{T_{n+1}}{t_{k+1}}\exp\left\{-2\frac{(1+\varepsilon)^2}{(2+\varepsilon)\Theta}\log\frac{T_n}{t_{k+1}}\right\} \\ &= C\frac{T_{n+1}}{t_{k+1}}\left(\frac{t_{k+1}}{T_n}\right)^{1+2\varepsilon'} = C\left(\frac{t_{k+1}}{T_n}\right)^{2\varepsilon'} = C\left(\frac{\Theta^{k+1}}{Q^n}\right)^{2\varepsilon'}. \end{aligned} \tag{9}$$

Hence it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{-\infty<k\le k_\Theta} P(A_{nk} \ge 1+\varepsilon) \\ &\quad \le C\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{\Theta^k Q^n}\right)^{2\varepsilon'} + C\sum_{n=1}^{\infty}(k_\Theta + 1)\frac{(\Theta e)^{2\varepsilon'}}{Q^{n(2\varepsilon')}} < \infty. \end{aligned} \tag{10}$$

For the case $k_\Theta < k \le k_n - 1$, using inequality (8) again, we have

$$P(A_{nk} \ge 1+\varepsilon) P\left\{ \sup_{0\le s-t,\, s\le T_{n+1},\, 0\le t\le t_{k+1}} t_{k+1}^{-1/2}|W(s)-W(s-t)| \right.$$

$$\left. \ge (1+\varepsilon)\frac{2t_k}{t_{k+1}}\left(\log\frac{T_n}{t_{k+1}} + \log\log t_k\right)^{1/2}\right\}$$

$$\le C\frac{T_{n+1}}{t_{k+1}}\exp\left\{-2\frac{(1+\varepsilon)^2}{(2+\varepsilon)\Theta}\log\frac{T_n\log t_k}{t_{k+1}}\right\} \tag{11}$$

$$= C\frac{T_{n+1}}{t_{k+1}}\left(\frac{t_{k+1}}{T_n\log t_k}\right)^{1+2\varepsilon'} = C\left(\frac{t_{k+1}}{T_n}\right)^{2\varepsilon'}\left(\frac{1}{\log t_k}\right)^{1+2\varepsilon'}.$$

Note that when $k_\Theta < k \le k_n'$, we have

$$(t_{k+1})^{2\varepsilon'} \le \Theta^{(k_n'+1)2\varepsilon'} \le \left(\frac{\Theta T_{n+1}}{(\log T_n)^\beta}\right)^{2\varepsilon'} = \frac{\Theta^{2\varepsilon'}T_{n+1}^{2\varepsilon'}}{(\log T_n)^2},$$

and from (11) it follows that

$$\sum_{n=1}^{\infty}\sum_{k=k_\Theta+1}^{k_n'} P(A_{nk}\ge 1+\varepsilon) \le C\sum_{n=1}^{\infty}\frac{\Theta^{2\varepsilon'}Q^{2\varepsilon'}}{(\log T_n)^2}\sum_{k=k_\Theta+1}^{k_n'}\frac{1}{(\log t_k)^{1+2\varepsilon'}} \tag{12}$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^{1+2\varepsilon'}} < \infty.$$

For the case $k_n' < k \le k_n - 1$, we have

$$T_n^{1/2} \le t_{k+1} \le \Theta T_{n+1},$$

$$k_n - k_n' \le \frac{\log Q^{n+1}}{\log\Theta} + 1 - \log\frac{Q^{n+1}}{(\log Q^n)^\beta}\Big/\log\Theta + 1,$$

$$\frac{\beta}{\log\Theta}\log\log Q^n + 2 \triangleq k_n''.$$

Using (11) again one sees that

$$\sum_{n=1}^{\infty}\sum_{k=k_n'+1}^{k_n-1} P(A_{nk}\ge 1+\varepsilon) \le C\sum_{n=1}^{\infty}\sum_{k=k_n'+1}^{k_n-1}\left(\frac{\Theta T_{n+1}}{T_n}\right)^{2\varepsilon'}\left(\frac{1}{\log T_n^{1/2}}\right)^{1+2\varepsilon'}$$

$$\le C\sum_{n=1}^{\infty}k_n''(\Theta Q)^2\left(\frac{1}{\log Q^{n/2}}\right)^{1+2\varepsilon'} \tag{13}$$

$$\le C\sum_{n=1}^{\infty}\log\log Q^n\left(\frac{1}{\log Q^{n/2}}\right)^{1+2\varepsilon'}$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon'}} < \infty.$$

Finally, merging (7), (10), (12), and (13) together, we get

$$\sum_{n=1}^{\infty} P\Big(\sup_{-\infty < k \le k_n - 1} A_{nk} \ge 1 + \varepsilon \Big),$$

$$\sum_{n=1}^{\infty} \sum_{-\infty < k \le k_n - 1} P(A_{nk} \ge 1 + \varepsilon) < \infty,$$

and (6) follows by the Borel–Cantelli lemma.

(C) From (5) and (6), we have

$$1 \le \text{the l.h.s. of (3)} \le \text{the l.h.s. of (6)} \le 1, \quad \text{a.s.;}$$

hence (3) holds true.

(D) In order to finish the proof of (4), it is enough to prove that

$$\liminf_{T \to \infty} \sup_{0 < t \le T} \sup_{t \le s \le T} \frac{|W(s) - W(s-t)|}{\{2t(\log(T/t) + \log\log t)\}^{1/2}} \ge 1, \quad \text{a.s.} \tag{14}$$

Let

$$B_n = \sup_{1 \le s \le n} \frac{|W(s) - W(s-1)|}{(2\log n)^{1/2}}.$$

Using the well known probability inequality

$$\begin{aligned} \frac{1}{(2\pi)^{1/2}} \left(\frac{1}{x} - \frac{1}{x^3} \right) \exp(-x^2/2) &\le P(W(1) \ge x) \\ &\le \frac{1}{(2\pi)^{1/2} x} \exp(-x^2/2), \qquad \text{for } x > 0, \end{aligned} \tag{15}$$

it follows that

$$\begin{aligned} \sum_{n=2}^{\infty} P(B_n \le 1 - \varepsilon) &\le \sum_{n=2}^{\infty} P\left\{ \max_{1 \le i \le n} \frac{|W(i) - W(i-1)|}{(2\log n)^{1/2}} \le 1 - \varepsilon \right\} \\ &= \sum_{n=2}^{\infty} \left\{ 1 - P\big(|W(1)| > (1-\varepsilon)(2\log n)^{1/2} \big) \right\}^n \\ &\le \sum_{n=2}^{\infty} \left\{ 1 - \frac{C}{(\log n)^{1/2}} \left(\frac{1}{n} \right)^{(1-\varepsilon)^2} \right\}^n \\ &\le \sum_{n=2}^{\infty} \exp\left\{ -C \frac{n}{(\log n)^{1/2}} \left(\frac{1}{n} \right)^{(1-\varepsilon)^2} \right\} < \infty, \end{aligned}$$

so we have $\liminf_{n \to \infty} B_n \ge 1$, a.s. from the Borel–Cantelli lemma. Notice that when $n \le T \le n+1$, one has

$$\sup_{0<t\le T}\ \sup_{t\le s\le T} \frac{|W(s)-W(s-t)|}{\{2t(\log(T/t)+\log\log t)\}^{1/2}} \ge \sup_{1\le s\le T}\left\{\frac{|W(s)-W(s-1)|}{(2\log T)^{1/2}}\right\} \ge B_n\left(\frac{\log n}{\log(n+1)}\right)^{1/2};$$

therefore, conclusion (14) is proved.

(E) Noting that the left-hand side of (3) $\le$ the left-hand side of (4′) $\le$ the left-hand side of (4), we see that (4′) is true from (3) and (4). The proof of Theorem 1 is now complete. □

Theorem 1 above can be reformulated in a general form (see Theorem 1′ below), which implies many important results about lim sup sup properties of increments of a Wiener process. Examples are the law of the iterated logarithm for Brownian motion, the Csörgő–Révész theorem (see Theorem 1.2.1 in [3]), and the Hanson–Russo theorem (see Theorem A above).

THEOREM 1′. *Let $W(t)$, $0 \le t < \infty$, be a standardized Wiener process. Then we have*

$$\limsup_{T\to\infty}\ \sup_{a_T<t\le T} \frac{|W(T)-W(T-t)|}{\{2t(\log(T/t)+\log\log t)\}^{1/2}} = 1, \quad a.s., \tag{16}$$

$$\limsup_{T\to\infty}\ \sup_{a_T<t\le T}\ \sup_{t\le s\le T} \frac{|W(s)-W(s-t)|}{\{2t(\log(T/t)+\log\log t)\}^{1/2}} = 1, \quad a.s. \tag{17}$$

for a_T such that $0 \le a_T \le T$, $T > 0$, and

$$\limsup_{T\to\infty}\ \sup_{a_T\le t\le T} \frac{|W(T)-W(T-t)|}{\{2t(\log(T/t)+\log\log t)\}^{1/2}} = 1, \quad a.s., \tag{16′}$$

$$\limsup_{T\to\infty}\ \sup_{a_T\le t\le T}\ \sup_{t\le s\le T} \frac{|W(s)-W(s-t)|}{\{2t(\log(T/t)+\log\log t)\}^{1/2}} = 1, \quad a.s. \tag{17′}$$

for any a_T such that $0 < a_T \le T$, $T > 0$.

As a by-product of solving the question (i), Hanson and Russo obtained further results about increments of a Wiener process (see [5]). Using these results and Theorem 1′ above, we obtain easily the following Corollaries 1–4.

First some notation. Let ω be any point in the probability space on which $W(\cdot)$ is defined. We use the following notation:

$L(\omega)$ is the set of limit points (as $T \to \infty$) of

$$\frac{W(T;\omega) - W(T-a_T;\omega)}{\{2a_T(\log(T/a_T) + \log\log T)\}^{1/2}};$$

and, if $\varepsilon > 0$ and $0 \le c_k < b_k < \infty$ for all k, $K(\omega)$ is the set of limit points (as $k \to \infty$) of

$$\frac{W(b_k,\cdot) - W(c_k,\cdot)}{\{2(b_k - c_k)(\log(b_k/(b_k-c_k)) + \log\log b_k)\}^{1/2}},$$

$$c_k^\varepsilon = \max(c_k, \varepsilon b_k),$$

$$S_\varepsilon = \bigcup_{k=1}^{\infty} (c_k^\varepsilon, b_k], \quad \text{and} \quad S'_\varepsilon = S_\varepsilon \cap [e,\infty).$$

COROLLARY 1. *Suppose that* $0 \le c_k < b_k < \infty$ *for* $k = 1,2,\dots$ *and there exists* $\varepsilon_0 \in (0,1)$ *such that*

(18) $$\liminf_{T\to\infty} \mu([0,T] \cap S_{\varepsilon_0})/T > 0,$$

(19) $$\int_{S'_{\varepsilon_0}} (1/t\log t)\,dt = \infty \quad \textit{or}$$

(20) $$\int_{S'_{\varepsilon_0}} \left(1/t(\log t)^2\right) dt = \infty \quad \textit{for all } \gamma < 1,$$

where $\mu(\cdot)$ *stands for Lebesgue measure on* $[0,\infty)$. *Then*

(21) $$P(\omega; K(\omega) = [-1,1]) = 1.$$

COROLLARY 2. *Suppose that* a_T *is measurable and* $0 < a_T \le T$ *for all* $T > 0$. *Then*

(22) $$P(\omega; L(\omega) = [-1,1]) = 1.$$

COROLLARY 3. *If* a_T *is measurable and* $0 < a_T \le T$ *for all* $T > 0$, *then*

(23) $$\limsup_{T\to\infty} \frac{W(T) - W(T-a_T)}{\{2a_T(\log(T/a_T) + \log\log T)\}^{1/2}} = 1, \quad a.s.$$

COROLLARY 4. *Suppose that* c_T *is measurable*, $0 \le c_T < b_T$ *for all* T, b_T *is continuous, and* $b_T \to \infty$ *as* $T \to \infty$. *Then*

(24) $$\limsup_{T\to\infty} \frac{W(b_T) - W(c_T)}{\{2(b_T - c_T)(\log(b_T/(b_T-c_T)) + \log\log b_T\}^{1/2}} = 1, \quad a.s.$$

3. On question (v). For this question we give a preliminary result about Theorem 3.2B in [4]. Csörgő and Révész have proved that if a_T satisfies (a) $0 < a_T \le T$ for $T > 0$, (b) a_T is nondecreasing, and (c) a_T/T is nonincreasing, then

$$\limsup_{T\to\infty} \frac{|W(T) - W(T-a_T)|}{\{2a_T(\log(T/a_T) + \log\log T)\}^{1/2}} = 1, \quad \text{a.s.} \tag{25}$$

If, in addition, (d) $\lim_{T\to\infty}\log(T/a_T)/\log\log T = \infty$, then the lim sup in (25) may be replaced by lim. Further,

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|W(t+s) - W(t)|}{\{2a_T(\log(T/a_T) + \log\log T)\}^{1/2}} = 1, \quad \text{a.s.}$$

can also be strengthened to

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|W(t+s) - W(t)|}{\{2a_T(\log(T/a_T) + \log\log T)\}^{1/2}} = 1, \quad \text{a.s.}$$

These results give the motivation for us to obtain the following theorem.

Theorem 2. *Let $W(t)$, $0 \le t < \infty$, be a standardized Wiener process. Under conditions*
(b′) $a_T \to \infty$ *continuously as* $T \to \infty$,
(d′) $\lim_{T\to\infty} \log(T/a_T)/\log\log a_T = \infty$,
we have

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|W(t+a_T) - W(t)|}{\{2a_T(\log((t+a_T)/a_T) + \log\log a_T)\}^{1/2}} = 1, \quad a.s., \tag{26}$$

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \frac{|W(t+s) - W(t)|}{\{2a_T(\log((t+a_T)/a_T) + \log\log a_T)\}^{1/2}} = 1, \quad a.s. \tag{27}$$

Proof. It is enough to prove that

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|W(t+a_T) - W(t)|}{\{2a_T(\log((t+a_T)/a_T) + \log\log a_T)\}^{1/2}} \ge 1, \quad \text{a.s.} \tag{28}$$

Let $a'_T = \sup_{0\le t\le T}\{a_t\}$, which is nondecreasing. By conditions (b′) and (d′), we have

$$\lim_{T\to\infty} \log(T/a'_T)/\log\log a'_T = \infty, \tag{29}$$

and for T large enough, there exists T' such that $0 \le T' \le T$, $a'_{T'} = a_T$. Then

$$\sup_{0\le t\le T-a_T} \frac{|W(t+a_T) - W(t)|}{\{2a_T(\log((t+a_T)/a_T) + \log\log a_T)\}^{1/2}}$$
$$\ge \sup_{0\le t\le T'-a'_{T'}} \frac{|W(t+a'_{T'}) - W(t)|}{\{2a'_{T'}(\log((t+a'_T)/a'_{T'}) + \log\log a'_{T'})\}^{1/2}},$$

hence

$$
\begin{aligned}
&\liminf_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|W(t+a_T)-W(t)|}{\{2a_T(\log((t+a_T)/a_T)+\log\log a_T)\}^{1/2}}\\
&\quad\ge \liminf_{T\to\infty}\sup_{0\le t\le T/2}\frac{|W(t+a'_T)-W(t)|}{\{2a'_T(\log((t+a'_T)/a'_T)+\log\log a'_T)\}^{1/2}}\\
&\quad\triangleq \liminf_{T\to\infty} A(T),
\end{aligned}
$$

because $T/2 \le T - a'_T$ for T large enough. Using inequality (15), for sufficiently large T we have

$$
\begin{aligned}
&P(A(T)\le 1-\varepsilon)\\
&\quad\le P\left\{\max_{0\le j\le[T/2a'_T]-1}\frac{|W((j+1)a'_T)-W(ja'_T)|}{\{2a'_T(\log(j+1)+\log\log a'_T)\}^{1/2}}\le 1-\varepsilon\right\}\\
&\quad\le \prod_{j=0}^{T/2a'_T-1} P\left\{|W(1)|\le(1-\varepsilon)(2(\log(j+1)+\log\log a'_T))^{1/2}\right\}\\
&\quad\le \prod_{j=0}^{T/2a'_T-1}\left\{1-C\left(\frac{1}{(j+1)\log a'_T}\right)^{1-\varepsilon}\right\} \qquad (30)\\
&\quad\le 2\exp\left\{-C\sum_{j=1}^{[T/2a'_T]}\left(\frac{1}{j\log a'_T}\right)^{1-\varepsilon}\right\}\\
&\quad\le 2\exp\left\{\frac{-C}{(\log a'_T)^{1-\varepsilon}}\left(\frac{a'_T}{T}\right)^{\varepsilon}\right\}\\
&\quad\le 2\exp\left\{-C(\log a'_T)^2\right\}\le 2(a'_T)^{-2},
\end{aligned}
$$

where we make use of the inequality $T/a'_T \ge (\log a'_T)^{2/\varepsilon}$ for all large T, which is a consequence of (29). Define T_k by $a'_T = k$. We have from (30)

$$\liminf_{k\to\infty} A(T_k) \ge 1, \quad \text{a.s.}$$

For each T large enough, there exists k such that $T_{k-1} < T \le T_k$. Then, by an argument similar to those used in the proof of Theorem 3.2A in [4], it is easy to see that

$$
\begin{aligned}
&\limsup_{k\to\infty} B(T_k)\\
&\quad\triangleq \limsup_{k\to\infty}\sup_{0\le t\le T_k}\sup_{0\le s\le 1}\frac{|W(t+s)-W(t)|}{\{2a'_{T_k}(\log(t+a'_{T_k})/a'_{T_k}+\log\log a'_{T_k})\}^{1/2}}\\
&\quad= 0, \quad \text{a.s.}
\end{aligned}
$$

Thus

$$\liminf_{T\to\infty} A(T) \ge \liminf_{k} A(T_{k-1})$$

$$\times \inf_{0\le t\le T_k/2} \frac{\left\{2a'_{T_{k-1}}\left(\log\left((t+a'_{T_{k-1}})/a'_{T_{k-1}}\right)+\log\log a'_T\right)\right\}^{1/2}}{\left\{2a'_{T_k}\left(\log\left((t+a'_{T_k})/a'_{T_k}\right)+\log\log a'_{T_k}\right)\right\}^{1/2}}$$

$$-\limsup_{k\to\infty} B(T_k) \ge 1, \quad \text{a.s.,}$$

and this also completes the proof of the theorem. □

REMARK. Similarly to [1] and [7], we can study lim inf via changing the normalizing factor under a condition weaker than (d′).

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