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A SPECTRAL CRITERION FOR THE FINITENESS OR INFINITENESS OF STOPPED FEYNMAN–KAC FUNCTIONALS OF DIFFUSION PROCESSES[1]

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Consider the Feynman–Kac functional

$$u(q, D; x) = E_x \exp\left(\int_0^{\tau_D} q(x(s))\, ds\right),$$

where D is a bounded open region in R^d, τ_D is the first exit time from D, $q \in C(\bar{D})$, and $x(s)$ is a diffusion process on R^d with generator L. We give a criterion for the finiteness or infiniteness of $u(q, D; x)$ in terms of the top of the spectrum of the Schrodinger operator $L_{q,D}$, an extension of $L + q$ acting on smooth functions which vanish on ∂D. As we also have a variational formula for the top of the spectrum, we thus obtain a criterion explicitly in terms of a variational formula.

1. Let $L = \frac{1}{2}\nabla \cdot a\nabla + b\nabla$ where a is a $d \times d$ matrix with terms $a_{ij} \in C^1(R^d)$ and b is a d vector with components $b_i \in C(R^d)$. Assume a is strictly elliptic on R^d, i.e.,

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \geq c|\lambda|^2$$

for some $c > 0$ independent of $x \in R^d$. Let $D \subset R^d$ be an open, connected set and put $\tau_D = \inf\{t \geq 0: x(t) \notin D\}$. A number of recent papers have studied the consequences of the Feynman–Kac functional $u(q, D; x) = E_x\exp(\int_0^{\tau_D} q(x(s))\, ds)$ being finite in the case that $x(s)$ is Brownian motion or Brownian motion conditioned to exit D at a certain point, with various assumptions on q and D. (See for example [1], [2], [6], [12].)

The finiteness of the above expression is connected with the existence of strictly positive solutions, v, in D of $Lv + qv = 0$. For example, in [1] Chung and Rao show that if D has finite Lebesgue measure, $u(q, D; x) \not\equiv \infty$, ∂D is regular, and q is uniformly Holder continuous on compact subsets of D, then for every bounded $f \in C(\bar{D})$, there exists a unique solution of $\frac{1}{2}\Delta v + qv = 0$, $v|_{\partial D} = f$; in fact $v(x) = E_x(\exp(\int_0^{\tau_D} q(x(s))\, ds) f(x(\tau_D)))$. It was also shown in [1] that $u(q, D; x) \not\equiv \infty$ is equivalent to $\int_0^\infty E_x(\exp(\int_0^t q(x(s))\, ds), \tau_D > t)\, dt < \infty$, if q is bounded, $x(t)$ is Brownian motion, and D is arbitrary. With the exception of [12], none of these papers gives an explicit criterion for the finiteness of $u(q, D; x)$. In [12], which is principally concerned with unbounded potentials, a finiteness criterion is given in the Brownian motion case when D is a ball. This

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result depends heavily on the explicit representation of the Green's function which is available in this case.

In this paper, we first consider a bounded D with a C^2-boundary and $q \in C(\bar{D})$. We make use of the large deviation results of Donsker and Varadhan to elaborate on the finiteness criterion of Chung and Rao and obtain a criterion for finiteness or infiniteness of a Feynman–Kac functional of a diffusion with generator L in terms of the spectrum of the related Schrodinger operator, an extension, $L_{q,D}$, of the operator $L + q$ acting on smooth functions which vanish at ∂D. In fact $L_{q,D}$ is the generator of the semigroup $T_t^{q,D}$ defined by $T_t^{q,D}f(x) = E_x(\exp(\int_0^t q(x(s))\,ds)f(x(t)), \tau_D > t)$ acting on $C(\bar{D})$. $T_t^{q,D}$ leaves invariant $C_0(\bar{D})$, the continuous functions on $\bar{D}$ which vanish at ∂D. Specifically, we shall show that

$$\sup \operatorname{re}(\operatorname{spec}(L_{q,D})) < 0 \quad \text{implies} \quad u(q, D; x) < \infty \quad \text{for all } x \in \bar{D}$$

and

$$\sup \operatorname{re}(\operatorname{spec}(L_{q,D})) > 0 \quad \text{implies} \quad u(q, D; x) = \infty \quad \text{for all } x \in D.$$

Since an explicit variational formula exists for

$$\lambda_{q,D} \equiv \sup \operatorname{re}(\operatorname{spec}(L_{q,D})),$$

we actually obtain a criterion for finiteness or infiniteness explicitly, in terms of a variational formula. In the case that a_{ij}, b_i, $q \in C^2(D)$ and $q \geq 0$, we show that $\lambda_{q,D} = 0$ also implies $u(q, D; x) = \infty$ for all $x \in D$.

Next we consider domains D which are the complement of the closure of the domains considered above, and we assume that q is bounded and continuous on $\bar{D}$. In this case $\lambda_{q,D}$ is not necessarily given by the variational formula. Let $l_{q,D}$ denote the number obtained from the variational formula [see Equation (11)]. As noted above, $l_{q,D} = \lambda_{q,D}$ in the compact case and our criterion in that case can be written as:

(i) $\quad l_{q,D} < 0$ implies $u(q, D; x) < \infty$ for all $x \in \bar{D}$,

(ii) $\quad l_{q,D} > 0$ implies $u(q, D; x) = \infty$ for all $x \in D$.

In the present case where D is not compact, we present a counterexample to (i) and show that (ii) still holds. If one imposes what amounts to a strong positive recurrence condition on the process, then in fact (i) holds. In Section 2, we present the appropriate forms of the large deviation results that we shall need. In Section 3 we treat the compact case and in Section 4 the noncompact case.

2. Let $\Omega = C([0, \infty), R^d)$, the space of continuous R^d-valued paths from $[0, \infty)$. Let $\omega = x(t)$ be a diffusion process on $(\Omega, \mathscr{F}, P_x)$ with generator L as defined in Section 1. (Actually, more precisely, it will be an extension of L which we shall persist in calling L.) For $B \subset R^d$, let $\mathscr{L}_t(\omega, B) = 1/t\int_0^t \chi_{(B)}(x(s))\,ds$. Then $\mathscr{L}_t(\omega, \cdot) \in \mathscr{P}(R^d)$, the space of probability measures on R^d under weak convergence; it is the occupation measure up to time t for the path $\omega = x(\cdot)$. Define I: $\mathscr{P}(R^d) \to R^+$ by $I(\mu) = -\inf_{u \in \mathscr{D}^+} \int_{R^d} Lu/u\,d\mu$, where $\mathscr{D}^+ = \{u \in \mathscr{D}: \inf_{x \in R^d} u(x) > 0\}$ and $\mathscr{D}$ is the domain of the generator L of the process. That I maps into R^+ can be seen by letting $u = \text{const}$ in the variational formula above which defines I. It is also easy to see that I is lower semicontinuous on $\mathscr{P}(R^d)$. The basic large deviation results of Donsker and Varadhan [3], [4] are:

For open $U \subset \mathscr{P}(R^d)$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x(\mathscr{L}_t(\omega, \cdot) \in U) \geq -\inf_{\mu\in U} I(\mu), \quad \text{for all } x \in R^d. \tag{1}$$

For compact $C \subset \mathscr{P}(R^d)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x(\mathscr{L}_t(\omega, \cdot) \in C) \leq -\inf_{\mu\in C} I(\mu), \quad \text{for all } x \in R^d. \tag{2}$$

Now, for any set $G \subset R^d$, define $\mathscr{M}(G) = \{\mu \in \mathscr{P}(R^d)\colon \operatorname{supp}\mu \subset G\}$. On Ω, for open D, we have $\{\mathscr{L}_t(\omega, \cdot) \in \mathscr{M}(D)\} = \{\tau_D > t\}$ and up to a set of P_x-measure zero, $\{\mathscr{L}_t(\omega, \cdot) \in \mathscr{M}(\overline{D})\} = \{\tau_D > t\}$. Now assume D is also bounded. Since $\mathscr{M}(\overline{D})$ is compact in $\mathscr{P}(R^d)$, the upper bound, (2) above, gives

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x(\tau_D > t) \leq -\inf_{\operatorname{supp}\mu\subset\overline{D}} I(\mu), \quad \text{for all } x \in \overline{D}. \tag{3}$$

In fact then, it is not hard to show [11] that for $q \in C(\overline{D})$,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} \log E_x\left(\exp\left(\int_0^t q(x(s))\,ds\right), \tau_D > \tau\right) \\ &\quad\leq \sup_{\operatorname{supp}\mu\subset\overline{D}} \left[\int_{R^d} q\,d\mu - I(\mu)\right] \equiv l_{q,D} \quad \text{for all } x \in \overline{D}.\end{aligned} \tag{4}$$

We would like to obtain a lower bound corresponding to the upper bound in (4). However, we cannot use (1) directly. The problem is that, regardless of what G is, sets of the form $\mathscr{M}(G)$ are never open in $\mathscr{P}(R^d)$. However, Theorem 8.1 in [4] shows that for G open, $\mu \in \mathscr{M}(G)$, and N any neighborhood of μ,

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x(\mathscr{L}_t(\omega, \cdot) \in N \cap \mathscr{M}(G)) \geq -I(\mu), \quad \text{for all } x \in D. \tag{5}$$

In particular, letting $G = D$, we obtain, similarly to (4), for $q \in C(\overline{D})$,

$$\liminf_{t\to\infty} \frac{1}{t} \log E_x\left(\exp\int_0^t q(x(s))\,ds, \tau_D > t\right) \geq \sup_{\operatorname{supp}\mu\subset D} \left[\int_{R^d} q\,d\mu - I(\mu)\right], \tag{6}$$

for all $x \in D$.

In fact,

$$\sup_{\operatorname{supp}\mu\subset D} \left[\int_{R^d} q\,d\mu - I(\mu)\right] = \sup_{\operatorname{supp}\mu\in\overline{D}} \left[\int_{R^d} q\,d\mu - I(\mu)\right].$$

The problem with proving this is that $I(\mu)$ is only lower semicontinuous. However, if we let $D_t = t\overline{D} = \{tx, x \in \overline{D}\}$ for $t > 0$, and $J_t = \sup_{\operatorname{supp}\mu\subset D_t}[\int_{R^d} q\,d\mu - I(\mu)]$, then J_t is nondecreasing in t and hence has at most a countable number of jumps. But by the smoothness of ∂D and the explicit formula for $I(\mu)$ which we present below, sufficient regularity exists to conclude that if J_t had a jump at some particular t, then it would have a jump at any other t, which is impossible. Thus we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log E_x\left(\exp\left(\int_0^{\tau_D} q(x(s))\,ds\right), \tau_D > t\right) \ge \sup_{\operatorname{supp}\mu\subset\bar{D}}\left[\int_{R^d} q\,d\mu - I(\mu)\right] = l_{q,D}, \quad \text{for all } x\in D. \tag{7}$$

Consider the semigroup $T_t^{q,D}$ on $C(\bar{D})$ which was defined in the introduction. As mentioned there, $T_t^{q,D}$ leaves invariant $C_0(\bar{D})$ and is generated by $L_{q,D}$ which is an extension of $L+q$ acting on smooth functions which vanish on ∂D. Let

$$\lambda_{q,D} \equiv \limsup_{t\to\infty}\frac{1}{t}\log\|T_t^{q,D}\| = \limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in\bar{D}}(T_t 1)(x).$$

Donsker and Varadhan [5] show that $\lambda_{q,D}\in\operatorname{spec}(L_{q,D})$ and $\lambda_{q,D} = \sup\operatorname{re}(\operatorname{spec}(L_{q,D}))$, and furthermore that

$$\sup\operatorname{re}\big(\operatorname{spec}(L_{q,D})\big) = l_{q,D}. \tag{8}$$

It is also of interest to note that if we consider $T_t^{q,D,H}$ defined like $T_t^{q,D}$ but acting on $L^2(R^d)$, and call its generator $L_{q,D,H}$, then in fact since D is compact, it can be shown that

$$\lambda_{q,D} = \lambda_{q,D,H}, \tag{9}$$

where

$$\lambda_{q,D,H} = \sup\operatorname{re}\big(\operatorname{spec}(L_{q,D,H})\big).$$

We utilize (4), (7), and (8) to prove our results in the compact case. In the unbounded case, (4) and (8) do not generally hold. This will be discussed further in Section 4.

Finally, we give the formula for the I function [10]. For μ with compact support,

$$I(\mu) = \sup_{h\in C^2(R^d)}\left(\int_{R^d}\left(\frac{\nabla g}{g} - a^{-1}b\right)a\left(\frac{\nabla g}{g} - a^{-1}b\right)g^2\,dx - \int_{R^d}(\nabla h - a^{-1}b)a(\nabla h - a^{-1}b)g^2\,dx\right) \tag{10}$$

if $d\mu/dx = g^2$ and $\int_{R^d}|\nabla g|^2\,dx < \infty$, and $I(\mu)=\infty$, otherwise. (By $\int_{R^d}|\nabla g|^2\,dx < \infty$ we mean that g has one generalized $L^2(R^d)$ derivative.) This formula presumably holds even if μ does not have compact support. For our purposes, we will *define* $I(\mu)$ by (10) even when μ does not have compact support.

3. From (10) we have

$$l_{q,D} = \sup_{\substack{g:\ \int_D g^2\,dx=1\\ g=0 \text{ on } \partial D}}\ \inf_{h\in C^2(D)}\left[\int_D qg^2\,dx - \int_D\left(\frac{\nabla g}{g} - a^{-1}b\right)a\left(\frac{\nabla g}{g} - a^{-1}b\right)g^2\,dx + \int_D(\nabla h - a^{-1}b)a(\nabla h - a^{-1}b)g^2\,dx\right]. \tag{11}$$

We now prove

THEOREM 1. *Let D be open and bounded with a C^2-boundary ∂D and let $L_{q,D}$ be as defined above.*

(a) *If* $l_{q,D} = \sup \operatorname{re}(\operatorname{spec}(L_{q,D})) < 0$, *then* $u(q, D; x) < \infty$ *for all* $x \in \overline{D}$.

(b) *If* $l_{q,D} = \sup \operatorname{re}(\operatorname{spec}(L_{q,D})) > 0$, *then* $u(q, D; x) = \infty$ *for all* $x \in D$.

(c) *If* $q \geq 0$ *and the coefficients* a_{ij}, b_i, *and also* q *are in* $C^2(\overline{D})$, *then* $l_{q,D} = \sup \operatorname{re}(\operatorname{spec}(L_{q,D})) = 0$ *implies that* $u(q, D; x) = \infty$ *for all* $x \in D$.

PROOF. First assume $\lambda_{q,D} = l_{q,D} < 0$. Then we have for $x \in \overline{D}$,

$$\begin{aligned}
u(q, D; x) &= E_x \exp\left(\int_0^{\tau_D} q(x(s))\, ds\right) \\
&= \sum_{n=0}^{\infty} E_x\left(\exp\left(\int_0^{\tau_D} q(x(s))\, ds\right), n \leq \tau_D < n+1\right) \\
&\leq e^{\|q\|} \sum_{n=0}^{\infty} E_x\left(\exp\left(\int_0^{n} q(x(s))\, ds\right), n \leq \tau_D < n+1\right) \\
&\leq e^{\|q\|} \sum_{n=0}^{\infty} E_x\left(\exp\left(\int_0^{n} q(x(s))\, ds\right)\right), \tau_D \geq n).
\end{aligned}$$

But for each $x \in \overline{D}$, (4) implies that there exists a constant c_x such that

$$E_x\left(\exp\left(\int_0^n q(x(s))\, ds\right), \tau_D \geq n\right) \leq c_x e^{(n\lambda_{q,D}/2)}, \qquad n = 1, 2, \dots .$$

Hence

$$\sum_{n=0}^{\infty} E_x\left(\exp\left(\int_0^n q(x(s))\, ds\right), \tau_D \geq n\right) \leq c_x \sum_{n=0}^{\infty} e^{(n\lambda_{q,D}/2)} < \infty$$

and thus $u(q, D; x) < \infty$. Now assume $\lambda_{q,D} = l_{q,D} > 0$. Then we have for all $x \in D$,

$$\begin{aligned}
u(q, D; x) &= E_x \exp\left(\int_0^{\tau_D} q(x(s))\, ds\right) \geq E_x\left(\exp\left(\int_0^{\tau_D} q(x(s))\, ds\right), \tau_D > t\right) \\
&= E_x\left(E_x\left(\exp\left(\int_0^{\tau_D} q(x(s))\, ds\right)\middle|\, \tau_D > t\right), \tau_D > t\right).
\end{aligned}$$

But

$$\begin{aligned}
&E_x\left(\exp\left(\int_0^{\tau_D} q(x(s))\, ds\right)\middle|\, \tau_D > t\right) \\
&\quad = \exp\left(\int_0^t q(x(s))\, ds\right) E_x\left(\exp\left(\int_t^{\tau_D} q(x(s))\, ds\right)\middle|\, \tau_D > t\right) \\
&\quad = \exp\left(\int_0^t q(x(s))\, ds\right) E_{x(t)} \exp\left(\int_0^{\tau_D} q(x(s))\, ds\right) \\
&\quad = \exp\left(\int_0^t q(x(s))\, ds\right) u(q, D; x(t)).
\end{aligned}$$

So

$$u(q, D; x) \geq E_x\left(u(q, D; x(t))\exp\left(\int_0^t q(x(s))\,ds\right), \tau_D > t\right)$$

$$\geq \inf_{y \in \overline{D}} u(q, D; y) E_x\left(\exp\left(\int_0^t q(x(s))\,ds\right), \tau_D > t\right).$$

This inequality holds for all $t > 0$. By (7)

$$\liminf_{t \to \infty} \frac{1}{t} \log E_x\left(\exp\left(\int_0^t q(x(s))\,ds\right), \tau_D > \tau\right) = \infty.$$

Thus we will be done with part (b) once we show that $\inf_{y \in \overline{D}} u(q, D; y) > 0$. But since the operator L is strictly elliptic and $\overline{D}$ is compact, it is clear that there exists a $t_0 > 0$ and an $\varepsilon > 0$ such that

$$P_x(\tau_D < t_0) > \varepsilon, \quad \text{for all } x \in \overline{D}.$$

Since q is bounded, we thus have

$$u(q, D, x) \geq e^{-\|q\| t_0} \varepsilon, \quad \text{for all } x \in \overline{D}.$$

Now we prove part (c). Since a_{ij}, b_i, and q are in $C^2(\overline{D})$ and ∂D is a C^2-boundary, the semigroup $T_t^{q, D}$ leaves $C^2(\overline{D})$ invariant. (For a probabilistic proof of this in the case $q \equiv 0$ and $D = R^d$, see [7]. A modification of this method works in the present situation.)

Consider $T_t^{q, D}$ restricted to $C^2(\overline{D})$. $T_t^{q, D}$ leaves invariant the cone of positive functions in $C^2(\overline{D})$. As $L_{q, D}$ has a compact resolvent, the Krein–Rutman theory of positive operators [9] provides for the existence of an eigenvalue at $\lambda_{q, D} = \sup \operatorname{re}(\operatorname{spec}(L_{q, D}))$ and corresponding nonnegative eigenfunction ϕ which is positive on D and vanishes on ∂D. Because we have restricted the semigroup to $C^2(\overline{D})$, ϕ is automatically in $C^2(\overline{D})$. Applying Itô's formula to ϕ gives

$$(12) \quad E_x\left(\exp\left(\int_0^{\tau_D \wedge t} q(x(s))\,ds\right)\phi(x(\tau_D \wedge t))\right) = \phi(x) \quad \text{for } x \in \overline{D} \text{ and } t \geq 0.$$

Now $\exp(\int_0^{\tau_D \wedge t} q(x(s))\,ds)\phi(x(\tau_D \wedge t))$ converges pointwise to zero as $t \to \infty$ and since $q \geq 0$, it is dominated by $\|\phi\|\exp(\int_0^{\tau_D} q(x(s))\,ds)$. Thus if $E_x \exp(\int_0^{\tau_D} q(x(s))\,ds)$ were finite, the left-hand side of (12) would converge to zero as $t \to \infty$ by the dominated convergence theorem. This would give $\phi(x) = 0$, which is a contradiction. This proves part (c). The problem with proving this for general q is that we cannot dominate $\exp(\int_0^{\tau_D \wedge t} q(x(s))\,ds)$. It is not uniformly integrable, for in fact we can show that if $u(q, D; x) < \infty$, then

$$u(q, D; x) < \liminf_{t \to \infty} E_x \exp\left(\int_0^{\tau_D \wedge t} q(x(s))\,ds\right). \qquad \square$$

4. Let D satisfy D^c compact. As mentioned in Section 2, Equations (4) and (8) (and (9)) do not hold in general in this case. The reason that (4) does not hold is that the upper bound (2) is only valid in general for compact sets C, not all closed sets. In fact it is easy to provide examples in which $\inf_{\mu \in \mathscr{P}(R^d)} I(\mu) > 0$. In this case if we let $C = R^d$, it is clear that (2) cannot hold. To provide a

counterexample to part (a) of Theorem 1 (that is, $l_{q,D} < 0$ implies $u(q, D; x) < \infty$ for all $x \in \overline{D}$), we consider $L = \frac{1}{2}d^2/dx^2 + b(d/dx)$ for $b > 0$ a constant. In this case, the I function reduces to $I(\mu) = \frac{1}{2}\int_{-\infty}^{\infty}(g')^2\,dx + b^2/2$ if $d\mu/dx = g^2$ and $g' \in L^2(R)$. So $\inf_{\mu \in \mathscr{P}(R)} I(\mu) = b^2/2$. Let $D = (0, \infty)$ and pick $q = c$ for $0 < c < b^2/2$. Then $l_{q,D} = c - b/2 < 0$ and $u(q, D; x) = E_x^b e^{c\tau_D}$, where $\tau_D = \inf\{t > 0\colon x(t) = 0\}$ and E_x^b is the expectation with respect to the process. This is obviously infinite; indeed $P_x^b(\tau_D = \infty) > 0$. Note that in fact if we make b larger, then $l_{q,D}$ becomes smaller, but τ_D for the new process is even bigger than it was for the original one, in the sense that $P^{b_1}(\tau_d > a) > P^b(\tau_D > a)$ for $b_1 > b$ and $a > 0$. This turns the relationship in Theorem 1 between $l_{q,D}$ and the size of $u(q, D; x)$ upside down. The reason is that in the present case, $\inf_{\mu \in \mathscr{P}(R)} I(\mu) = b^2/2$ is not giving the asymptotic rate of decay of the probability that the process stays away from zero, but rather, the asymptotic rate of decay of the probability that the process stays away from infinity; it is essentially the rate of escape to infinity of the process. Thus, the larger b is, the larger is the rate of escape to infinity and the bigger is τ_D. Donsker and Varadhan show that the upper bound (2) does hold under a certain condition which requires that the process be positive recurrent in a strong sense [4].

CONDITION A. There exists a function $V(x)$ such that $\{x \in R^d\colon V(x) \geq a\}$ is compact for each $a > -\infty$ and there exists a sequence $u_n \in \mathscr{D}$, the domain of the generator L, such that

(i) $u_n \geq 1$ for all $n, x \in R^d$;
(ii) for each compact $W \subset R^d$, $\sup_{x \in W} \sup_n u_n(x) < \infty$;
(iii) for each $x \in R^d$, $\lim_{n \to \infty}(Lu_n/u_n)(x) = V(x)$;
(iv) for some $N < \infty$, $\sup_{n,x}(Lu_n/u_n)(x) \leq N$.

Part (b) of Theorem 1 will still hold for unbounded D. We cannot show part (c) because the Krein–Rutman theory no longer applies; there is no compact resolvent. We now prove

THEOREM 2. (a) *If* $l_{q,D} < 0$ *and Condition* A *holds, then*

$$u(q, D; x) < \infty \text{ for all } x \in \overline{D};$$

(b) *If* $l_{q,D} > 0$, *then* $u(q, D; x) = \infty$, *for all* $x \in D$.

PROOF. If Condition A holds, then the upper bound (2) holds for all closed sets and thus (4) holds. Now part (a) can be proved just as its counterpart in Section 3 was proved. Now consider part (b). Let $D_n = \{x \in D\colon |x| < n\}$. Since $l_{q,D} > 0$, one can show that in fact $l_{q,D_n} > 0$ for sufficiently large n. Consider any $x \in D$. Pick n so that $|x| < n$ and $l_{q,D_n} > 0$. The calculation in Section 3 gave us

$$u(q, D; x) \geq E_x\left(u(q, D; x(t))\exp\left(\int_0^t q(x(s))\,ds\right), \tau_D > t\right).$$

At this point, in Section 3 we took $u(q, D; x(t))$ out of the expectation. Here we add an additional step and write

$$E_x\left(u(q, D; x(t))\exp\left(\int_0^t q(x(s))\,ds\right), \tau_D > t\right)$$
$$\geq E_x\left(u(q, D; x(t))\exp\left(\int_0^t q(x(s))\,ds\right), \tau_{D_n} > t\right)$$
$$\geq \inf_{|y|\leq n} u(q, D; y)E_x\left(\exp\left(\int_0^t q(x(s))\,ds\right), \tau_{D_n} > t\right).$$

The argument in Section 3 shows that $\inf_{|y|\leq n} u(q, D; y) > 0$, and since $l_{q, D_n} > 0$, equation (7) applied to D_n shows that

$$\liminf_{t\to\infty} \frac{1}{t}\log E_x\left(\exp\left(\int_0^t q(t(s))\,ds\right), \tau_{D_n} > t\right) = \infty.$$

Hence $u(q, D; x) = \infty$ for all $x \in D$. □

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