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A LIMIT THEOREM FOR THE POSITION OF A TAGGED PARTICLE IN A SIMPLE EXCLUSION PROCESS

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We prove that the position of a tagged particle in a stationary simple exclusion process satisfies a law of large numbers. For this purpose, we show the extremality of an invariant measure for the process "seen" from the tagged particle, and we use the ergodicity properties of the initial process.

1. Introduction. On the set of sites $S = \mathbb{Z}^d$, we consider the evolution of infinitely many indistinguishable particles, according to a simple exclusion process, which is described as follows: Let $p(x, y)$ be a translation invariant probability transition function on S; each site has an exponential clock with parameter one, all the clocks being mutually independent. There is at most one particle per site. When the clock at site x rings, the particle at x (if there is one) chooses a site y with probability $p(x, y)$, moves to y if the site y is vacant and remains at x otherwise. More formally, we can define a Markov process (η_t) on $\{0,1\}^S$. For every site x, $\eta_t(x)$ equals 1 if x is occupied and 0 otherwise. For $\rho \in [0,1]$ the Bernoulli product measure ν_ρ with marginals $\nu_\rho\{\eta(x) = 1\} = \rho$ is an invariant measure for this process (see [7], Chapter VIII).

Here, we are interested in the asymptotic behavior of a tagged particle which is initially at site 0, when the other particles of the underlying process have the initial distribution ν_ρ. Assuming that $p(x, y)$ has a finite first moment, Spitzer has computed EX_t (where X_t is the position at time t of the tagged particle) and proved the existence of an almost sure limit for X_t/t ([9]—see also [7], Chapter VIII, Section 4). Following Spitzer, Liggett has conjectured that this limit was constant, and therefore equal to $(1-\rho)\sum_{x \in S} xp(0,x)$ ([7], Chapter VIII, Section 7).

In dimension one, in the nearest-neighbour asymmetric case, Kipnis has proved this result ([5]), using the fact that in this particular case, there exists a correspondence between the simple exclusion process and the zero range process. In this paper, we prove:

THEOREM 1. *Let (η_t) be a simple exclusion process on $\mathbb{Z}^d$, such that at time 0 there is a particle at the origin (the tagged particle), and, at all sites different from 0, the particles are placed according to the Bernoulli distribution ν_ρ. Assume that the transition function $p(x, y)$ is translation invariant, has a finite first moment and moreover is not nearest neighbour in the case of dimension one. Denote by X_t the position at time t of the tagged particle. Then X_t/t converges almost surely to $(1-\rho)\sum_{x \in S} xp(0,x)$ when t goes to infinity.*

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REMARK. A natural interpretation of the preceding formula is that, due to the interaction, in the mean a fraction ρ of the jumps is suppressed.

We proceed as follows: To get the limit of X_t/t, we introduce two auxiliary Markov processes, the (X_t, η_t) process and the exclusion process seen from the tagged particle $(\tau_{X_t}\eta_t)$. [The shift operator τ_x is defined by $(\tau_x\eta)(z) = \eta(x+z)$ for all x and z.] This second process is clearly the image of the first under the mapping $(X_., \eta_.) \to (\tau_{X_.}\eta_.)$ (see [4] and [7], Chapter VIII, Section 4 for more details on this correspondence and on the construction of these processes). Then, we show the extremality of an invariant measure exhibited by Spitzer ([9]) for the $(\tau_{X_t}\eta_t)$-process (Proposition 3). For this, we first prove (in Proposition 2) that the Bernoulli measures are extremal invariant for the simple exclusion process in the asymmetric case (Liggett has obtained all the extremal invariant measures in the symmetric case—[7], Chapter VIII).

The next problem would be to obtain a central limit theorem for X_t. This question has been completely solved in the symmetric case, by Arratia [1] [in dimension $d = 1$, when $p(0,1) = p(0,-1) = \frac{1}{2}$, he proved that the correct renormalisation was $t^{1/4}$] and by Kipnis and Varadhan [6] [in dimension $d \geqq 2$, and in dimension $d = 1$, when the support of $p(0,\cdot)$ contains more than two points]. In the asymmetric case, in dimension $d = 1$, there are proofs only when $p(0,1) = 1$ (Kesten, see [9]) and when $p(0,1) = 1 - p(0,-1)$ (Kipnis [5]). The problem remains open in all the other cases.

2. The asymptotic behavior of the tagged particle.

We first study the simple exclusion process.

PROPOSITION 2. *The Bernoulli measures ν_ρ are extremal invariant for the simple exclusion process in the asymmetric case.*

PROOF. Let P_t denote the semigroup of the process, and Ω its infinitesimal generator. For every function f on $\{0,1\}^S$ which depends on finitely many coordinates, Ω is given by

$$\Omega f(\eta) = \sum_{x, y \in S} \eta(x)[1-\eta(y)]\,p(x,y)[f(\eta^{xy}) - f(\eta)],$$

where

$$\eta^{xy}(z) = \begin{cases} \eta(z) & \text{if } z \neq x \text{ and } z \neq y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x. \end{cases}$$

If P_t^* is the adjoint of P_t in $\mathbb{L}^2(\nu_\rho)$, it is equivalent to say that the Bernoulli measure ν_ρ is extremal invariant for the simple exclusion process, or that there exists no event A, with $0 < \nu_\rho(A) < 1$, such that $P_t^* 1_A = 1_A$ in $\mathbb{L}^2(\nu_\rho)$ ([8]). So let A be such that $P_t^* 1_A = 1_A$ in $\mathbb{L}^2(\nu_\rho)$. We have that

$$\langle P_t P_t^* 1_A, 1_A\rangle = \langle P_t^* 1_A, P_t^* 1_A\rangle = \langle 1_A, 1_A\rangle.$$

On the other hand, we have

$$|\langle P_tP_t^*1_A, 1_A\rangle| \leqq \|1_A\|_2\|P_tP_t^*1_A\|_2 \leqq \langle 1_A, 1_A\rangle,$$

because P_t and P_t^* are contraction operators. So $\|P_tP_t^*1_A\|_2 = \|1_A\|_2$, and therefore $P_tP_t^*1_A = 1_A$ in $\mathbb{L}^2(\nu_\rho)$.

We now apply:

Theorem (Chernoff [2]). *Let e^{tA} and e^{tB} be contraction semigroups on a Banach space X. Suppose $C = \overline{A + B}$ is the generator of a contraction semigroup. Then $(e^{tA/n}e^{tB/n})^n$ converges to e^{tC} in the strong operator topology.*

Notice that the generator Ω^* of P_t^* is given by

$$\Omega^* f(\eta) = \sum_{x, y\in S} \eta(x)[1 - \eta(y)]\,p(y, x)[f(\eta^{xy}) - f(\eta)],$$

where f depends on finitely many coordinates, since we have that for every bounded function g in $\mathbb{L}^2(\nu_\rho)$,

$$\int\left[\frac{d}{dt}P_t^*f\right]_{t=0} g\,d\nu_\rho = \frac{d}{dt}\left[\int fP_tg\,d\nu_\rho\right]_{t=0} = \int f\Omega g\,d\nu_\rho = \int g\Omega^* f\,d\nu_\rho.$$

The operator $\Omega + \Omega^*$ is the generator of a symmetric simple exclusion process, whose speed rate is

$$c(x, y, \eta) = \eta(x)[1 - \eta(y)][p(x, y) + p(y, x)].$$

The semigroup P_t can be extended as a Markov semigroup on $\mathbb{L}^2(\nu_\rho)$, whose generator Ω_{ν_ρ} is the closure of Ω. Moreover, the adjoint of Ω_{ν_ρ} is the closure of Ω^* (see [7], Chapter IV, Proposition 4.1). So, $e^{t(\Omega+\Omega^*)}$ is a contraction semigroup on $\mathbb{L}^2(\nu_\rho)$, and $(e^{t\Omega/n}e^{t\Omega^*/n})^n$ converges to $e^{t(\Omega+\Omega^*)}$ in the strong operator topology of $\mathbb{L}^2(\nu_\rho)$. The equality $P_tP_t^*1_A = 1_A$ in $\mathbb{L}^2(\nu_\rho)$ for all t implies that $(P_{t/n}P_{t/n}^*)^n1_A = 1_A$ in $\mathbb{L}^2(\nu_\rho)$, so $e^{t(\Omega+\Omega^*)}1_A = 1_A$ in $\mathbb{L}^2(\nu_\rho)$.

Since the measure ν_ρ is extremal invariant for the symmetric simple exclusion process, we necessarily have that $\nu_\rho(A)$ equals 0 or 1. This proves that the Bernoulli measures are extremal invariant in the asymmetric case too. □

For $(\tau_{X_t}\eta_t)$, the tagged particle process, Spitzer ([9]) has proved that the measure $\nu_\rho(\cdot|\eta(0) = 1)$ is invariant (see also [7], Chapter VIII, Section 4). Proposition 2 enables us to show

Proposition 3. *Assume that $p(x, y)$ is translation invariant, and not nearest neighbour in dimension one. Then the measure $\lambda_\rho = \nu_\rho(\cdot|\eta(0) = 1) = (\eta(0)/\rho)\nu_\rho$ is extremal invariant for the simple exclusion process seen from the tagged particle.*

Proof. We already know the invariance of λ_ρ and we argue by contradiction to show its extremality. If λ_ρ is not extremal, there is a set of configurations

E invariant under the semigroup $\tilde{P}_t$ of $(\tau_{X_t}\eta_t)$ and such that $0 < \lambda_\rho(E) < 1$. Then, for the set $E_1 = E \cap \{\eta(0) = 1\}$ we have $0 < \nu_\rho(E_1) < \rho$, and for ν_ρ a.s. configuration η of E_1, $\tau_{X_t}\eta_t$ is almost surely in E_1. Denote by F the set of all translates of configurations in E_1. If η belongs to F, there is a point a in $\mathbb{Z}^d$ such that $\zeta = \tau_a\eta$ is in E_1, therefore $\tau_{X'_t}\zeta_t = \tau_{(a+X'_t)}\eta_t$ is almost surely in E_1 (where X'_t is the position at time t of the particle initially on a). It follows that the set F is closed for the evolution of the (η_t)-process. Then as the measure ν_ρ is extremal invariant (by Proposition 2), $\nu_\rho(F)$ equals 0 or 1. But $\nu_\rho(F) = 0$ implies that $\nu_\rho(E_1) = 0$, which contradicts the assumptions. So $\nu_\rho(F) = 1$. Similarly, letting $E_2 = E^c \cap \{\eta(0) = 1\}$ and $G = \bigcup_{x \in S}\tau_{-x}E_2$, we have $\nu_\rho(G) = 1$.

We now use Lemma 4, the proof of which will be given later.

LEMMA 4. *In the setup of Proposition* 3, *for almost every configuration* ξ, *there exists a subset of sites* $N = \{a, b, c, a_1, \dots, a_n, b_1, \dots, b_l, c_1, \dots, c_k\}$ *such that*:

- $\tau_a\xi \in E_1$ *and* $\tau_b\xi \in E_2$;
- $\xi(c) = 0 = \xi(a_1) = \cdots = \xi(a_n)$;
- $a_1, \dots, a_n$ *are all different from* c, $b_1, \dots, b_l$ *from* a *and* $c_1, \dots, c_k$ *from* b;
- $p(a, a_1)p(a_1, a_2) \cdots p(a_n, b) > 0$, $p(b, b_1)p(b_1, b_2) \cdots p(b_l, c) > 0$, $p(a, c_1)p(c_1, c_2) \cdots p(c_k, c) > 0$.

This means that there is a path from a to b avoiding c through empty sites, from a to c avoiding b, and from b to c avoiding a. Therefore, let ξ be a fixed configuration, and N the subset of $\mathbb{Z}^d$ given in the lemma. Let T be a fixed real number, and H the subset of the trajectories of the (η_t)-process for which, before time T, no particle can leave or enter N.

Using the basic coupling ([7], Chapter II) based on independent Poisson processes of rates $p(x, y)$ for all $x, y \in S$, we have that H has a nonvanishing probability, since the absence of jumps between N and N^c is expressed by the absence of points in $[0, T]$ for the corresponding Poisson processes, and since the set N is finite. We also have that the (η_t)-process has before time T the same law as the product of two independent exclusion processes $(\eta_{1,t})$ and $(\eta_{2,t})$ taking place, respectively, on N and on N^c.

Since the set N is finite, we can explicitly construct the evolution of the first process between times 0 and T, if the initial configuration is $\xi = (\xi_1, \xi_2)$. Using two different trajectories, we obtain at time T the configurations $\xi_T = (\xi_{1,T}; \xi_{2,T})$ and $\xi'_T = (\xi'_{1,T}; \xi_{2,T})$. We assume for this purpose that each clock of N rings at most once before T.

CONSTRUCTION OF $\xi_{1,T}$. Between times 0 and T, first we get the site b vacant and the site c occupied: The particles go through the intermediate sites $b_0 = b, b_1, \dots, b_l, b_{l+1} = c$ in such a way that at time T, except for b and c, the occupied sites are the same as at time 0. More precisely, let $i_0 = 0$, and for $k \geqq 1$, if $i_{k-1} < l + 1$, let $i_k = \inf\{i_{k-1} < j \leqq l + 1;\ \xi(b_j) = 0\}$. The particles go successively from b_{i_k-1} to b_{i_k}, from b_{i_k-2} to $b_{i_k-1}, \dots$ and from $b_{i_{k-1}}$ to $b_{i_{k-1}+1}$ for all $0 \leqq i_k \leqq l+1$.

Then the particle on site a goes to b through the sites $a_1, \dots, a_n$ (it is possible to perform the jumps in this order, because we have assumed $a_1, \dots, a_n$ vacant). All the other particles in N do not move.

Initially, $\tau_a \xi$, the configuration seen from a, was an element of E_1, which is closed for the evolution of the semigroup $\tilde{P}_t$. The particle being at a at time 0 is at b at time T, so $\tau_b \xi_T$ is an element of E_1.

CONSTRUCTION OF $\xi'_{1,T}$. Between times 0 and T, we get the site a vacant and the site c occupied: The particles go through the intermediate sites $c_1, \dots, c_k$ in such a way that at time T, the occupied sites are the same as at time 0 (except for a and c). The other particles in N do not move before time T.

Since $\tau_b \xi$ was an element of E_2, and since the particle which was on b does not move during $[0, T]$, the configuration $\tau_b \xi'_T$ is still an element of E_2.

Finally, the configurations ξ_T and ξ'_T are identical, and this gives a contradiction since $\tau_b \xi_T = \tau_b \xi'_T$ cannot belong at the same time to E_1 and E_2. The assumption $\nu_\rho(F) = 1 = \nu_\rho(G)$ leads to a contradiction, so we have $\lambda_\rho(F) = 0$ or $\lambda_\rho(G) = 0$, thus $\lambda_\rho(E) = 0$ or $\lambda_\rho(E) = 1$. The measure λ_ρ is extremal invariant. □

PROOF OF LEMMA 4. We can assume without loss of generality that $p(x, y)$ is irreducible.

We first study the case of the dimension one. Since $p(x, y)$ is not nearest neighbour, there is a path from 0 to 1. So there are two integers k and q such that $p(0, k) > 0$, $p(0, q) > 0$, k and q relatively prime and ($q = 1$ and $k > 1$) or ($k > 0$, $q < 0$ and $|k| > |q|$).

Let ξ be a fixed configuration. Since $\nu_\rho(F) = \nu_\rho(G) = 1$, there exist two points u and v in $\mathbb{Z}^d$ such that $\tau_u \xi \in E_1$, $\tau_v \xi \in E_2$ and $u \leq v$. Among the intermediate sites on the path from u to v, there are necessarily $n+2$ consecutive sites $a, a_1, \dots, a_n, b$ such that $\xi(a_1) = \cdots = \xi(a_n) = 0$, $\tau_a \xi \in E_1$ and $\tau_b \xi \in E_2$. Let c belong to the infinite set $C_1 = \{x \in \mathbb{Z};\ x > b,\ x > a,\ x > a_1, \dots, x > a_n$ and if $x - a = \alpha k + \beta q$ with $\alpha, \beta \in \mathbb{N}$, then $\alpha > 0\}$ such that $\xi(c) = 0$. We can write the numbers $c - a$, $b - a$, $c - b$ as linear combinations of k and q, since k and q are relatively prime. To go through a path from b to c which avoids a, a particle performs the transitions of step k, and then those of step q. To move from a to c avoiding b, a particle performs first the transitions of step k if $b - a$ is a multiple of q, and otherwise those of step q first.

We now turn to the case of the dimension $d \geqq 2$. Because of the irreducibility of $p(x, y)$, we can assume without loss of generality that $p(0, (1, \dots, 0)) > 0$, $p(0, (0, 1, 0, \dots, 0)) > 0, \dots, p(0, (0, \dots, 0, 1)) > 0$. On $\mathbb{Z}^d$, we define the partial order relation: $x \leqq y$ if it is true coordinatewise. Then, if $x \leqq y$, there is a path from x to y.

From now on, the proof follows verbatim the case $d = 1$, except that it can happen that neither $u \leqq v$ nor $v \leqq u$. We then pick an element w of the infinite set $C = \{x \in \mathbb{Z}^d;\ x \geqq u \text{ and } x \geqq v\}$ such that $\xi(w) = 1$. Suppose that $\tau_w \xi \in E_2$. We just replace u and v by u and w. Later on, the point c belongs to $C_2 = \{x \in \mathbb{Z}^d;\ x \geqq b \geqq a \text{ and } a, b, x \text{ not on a "horizontal" line}\}$. It is clear that there exists a path from b to c avoiding a since $a \leqq b \leqq c$, and that there exists a path from a to c avoiding b since a, b, c are not on a "horizontal" line (so the path can possibly move around b). □

We can now proceed to the

PROOF OF THEOREM 1. For every z belonging to S, denote by N_t^z the number of jumps of size z of the tagged particle before time t; notice that $e_i \cdot X_t = \sum_{z \in S} e_i \cdot z N_t^z$, where $(e_i, 1 \leqq i \leqq d)$ is a basis of $\mathbb{Z}^d$. Since $(\tau_{X_t}\eta_t)$ is a Markov process, the quantities

$$C_t^z = N_t^z - \int_0^t p(0, z) 1_{\{(\tau_{X_s}\eta_s)(z) = 0\}}\, ds$$

are mean zero martingales without common jumps, so they are orthogonal. (All the properties of pure jump martingales that we use in this paper are reviewed in [3], pages 242–245). Moreover, the condition $\sum_{z \in S} \|z\| p(0, z) < \infty$ implies that

$$\sum_{z \in S} e_i \cdot z C_t^z = e_i \cdot X_t - \sum_{z \in S} e_i \cdot z p(0, z) \int_0^t 1_{\{(\tau_{X_s}\eta_s)(z) = 0\}}\, ds$$

and $\sum_{z \in S} |e_i \cdot z| C_t^z$ are also mean zero martingales. Hence we have

$$\begin{aligned} \mathbb{E}(e_i \cdot X_1) &= \mathbb{E}\Big(\sum_{z \in S} e_i \cdot z p(0, z) \int_0^1 1_{\{(\tau_{X_s}\eta_s)(z) = 0\}}\, ds \Big) \\ &= (1 - \rho) \sum_{z \in S} e_i \cdot z p(0, z) \end{aligned} \tag{1}$$

and

$$\mathbb{E}\Big(\sup_{0 \leqq s \leqq t} |e_i \cdot X_s| \Big) \leqq \mathbb{E}\Big(\sum_{z \in S} |e_i \cdot z| N_t^z \Big) = (1 - \rho) t \sum_{z \in S} |e_i \cdot z| p(0, z). \tag{2}$$

[Remember the initial distribution of the $(\tau_{X_t}\eta_t)$ process is the invariant distribution λ_ρ.]

Now, to obtain the limit of $e_i \cdot X_t / t$, we write for every $s \geqq 0$, $e_i \cdot X_{s+1} = e_i \cdot X_s + (e_i \cdot X_1) \circ \theta_s$ [where the shift operator θ_t is defined on the trajectories by $\theta_t((X_\cdot, \eta_\cdot)) = (X_{t+\cdot}, \eta_{t+\cdot})$] and then

$$\begin{aligned} \int_0^t (e_i \cdot X_1) \circ \theta_s\, ds &= \int_0^t (e_i \cdot X_{s+1} - e_i \cdot X_s)\, ds \\ &= e_i \cdot X_t + \int_0^1 (e_i \cdot X_s) \circ \theta_t\, ds - \int_0^1 e_i \cdot X_s\, ds, \end{aligned}$$

because for $t \leqq u$, $e_i \cdot X_u = e_i \cdot X_t + (e_i \cdot X_{u-t}) \circ \theta_t$.

First, since $\sup_{0 \leqq s \leqq 1} |e_i \cdot X_s|$ is almost surely finite by (2), $(1/t)\int_0^1 e_i \cdot X_s\, ds$ converges a.s. to zero when t goes to infinity.

Then, we write

$$\left| \frac{1}{t} \int_0^1 (e_i \cdot X_s) \circ \theta_t \, ds \right| \leqq \frac{1}{[t]} \left(\int_0^2 \sum_{z \in S} |e_i \cdot z| N_s^z \, ds \right) \circ \theta_{[t]}$$

(where $[t]$ is the integer part of t), and we use (2), the stationarity of the $(\tau_{X_t}\eta_t)$ process, and a Borel–Cantelli argument to conclude that $(1/t)\int_0^1 (e_i \cdot X_s) \circ \theta_t \, ds$ converges a.s. to zero when t goes to infinity.

Finally, the ergodicity of λ_ρ (this measure is extremal invariant by Proposition 3, and therefore ergodic—see [8]) implies the almost sure convergence of $(1/t)\int_0^t (e_i \cdot X_1) \circ \theta_t \, ds$ to $\mathbb{E}(e_i \cdot X_1) = (1 - \rho)\sum_{z \in S} e_i \cdot z p(0, z)$.

So, for every $1 \leqq i \leqq d$, $(e_i \cdot X_t)/t$ converges a.s. to $(1 - \rho)\sum_{z \in S} e_i \cdot z p(0, z)$ when t goes to infinity. □

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