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ON A COMBINATORIAL CONJECTURE CONCERNING DISJOINT OCCURRENCES OF EVENTS

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Recently van den Berg and Kesten have obtained a correlation-like inequality for Bernoulli sequences. This inequality, which goes in the opposite direction of the FKG inequality, states that the probability that two monotone (i.e., increasing or decreasing) events "occur disjointly" is smaller than the product of the individual probabilities. They conjecture that the monotonicity condition is immaterial, i.e., that the inequality holds for all events. In the present paper we try to make clear the intuitive meaning of the conjecture and prove some nontrivial special cases, one of which, a pure correlation inequality, is an extension of Harris' FKG inequality.

1. Introduction. In van den Berg and Kesten (1985) (hereinafter called [2]) a conjecture is stated which has the intuitive interpretation given by the following example.

(1.1) EXAMPLE. Suppose two children make a list of their wishes for Christmas. The first child is satisfied if he gets at least one of the combinations in the following list:

(1) a green teddy-bear and a blue car;
(2) a red teddy-bear;
(3) a blue car and a blue football.

The second child has the following list:

(1′) a blue teddy-bear and a blue car;
(2′) a red teddy-bear;
(3′) a red football;
(4′) a blue football.

Now suppose Santa Claus takes two boxes and puts in each of them a teddy-bear, a football, and a car. However, he does not consider the colors and chooses the toys randomly from large sacks, each sack containing one type of toy in several colors. We assume that this happens in such a way that the six colors in the two boxes may be considered as independent random variables, and that the contents of the two boxes are stochastically identical (i.e., the color of the football in the

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first box has the same distribution as that of the football in the second box, etc.). Consider the following two options:

(a) Santa Claus gives only one box to the two children and they must try to share the contents of this box in such a way that both are satisfied, i.e., get at least one of the combinations in their respective lists. It is easy to check that this is only possible if the box contains at least one of the following compositions of combinations of the first and second list:

$1 \times 3'$: a green teddy-bear, a blue car, and a red football;
$1 \times 4'$: a green teddy-bear, a blue car, and a blue football;
$2 \times 3'$: a red teddy-bear and a red football;
$2 \times 4'$: a red teddy-bear and a blue football;
$3 \times 2'$: a blue car, a blue football, and a red teddy-bear.

(b) This option is as follows: Both children receive a box but they are not allowed to exchange toys. In this case the box given to the first child must contain at least one of the combinations $1, 2, 3$ and the box given to the second child must contain at least one of the combinations $1', 2', 3', 4'$. (Note that these events are independent.)

When for each type of toy the probability distribution of its colors is known, one can calculate, for both options, the probability that both children are satisfied. The conjecture in [2] is equivalent to saying that this probability for the first option is not larger than for the second option and that this holds for arbitrary numbers of different toys and possible colors, for all probability distributions of the colors, and for any pair of lists of wishes.

The investigations which led to the conjecture in [2] were motivated by the following percolation problem:

(1.2) EXAMPLE. Let each bond b of a locally finite graph G, independently of the other bonds be *open* with probability p_b and *closed* with probability $1 - p_b$. A path from s to s' is a sequence $s = s_1, b_1, s_2, b_2, \ldots, b_{n-1}, s_n = s'$, where $s_1, s_2, \ldots, s_n$ are sites of G and each b_i is a bond of G connecting s_i and s_{i+1}, $i = 1, \ldots, n-1$. An *open path* is a path of which all bonds are open. Two paths are *disjoint* if they have no bonds in common.

Let V_1, V_2, W_1, and W_2 be sets of sites of G. Further, let A be the event that there exists an open path from a site of V_1 to a site of V_2, and B the corresponding event with respect to W_1 and W_2. It follows from a result of Harris (1960) that A and B are positively correlated, i.e., $P(A \cap B) \geq P(A)P(B)$. (We come back to Harris' result in Section 4.) Now the problem is whether, on the other hand, the probability that there exist two *disjoint* open paths, one of which goes from a site of V_1 to a site of V_2 and the other from a site of W_1 to a site of W_2, is *at most* $P(A)P(B)$. In [2] it is shown that this is indeed the case. However, the following related problem is unsolved: Consider again the above graph G. This time the bonds are not randomly open or closed, but they have a random direction. More precisely, if b is a bond with endpoints s_1, s_2 then it has, independently of the other bonds, probability $p_b(s_1, s_2)$ to be directed from s_1 to s_2 and probability $p_b(s_2, s_1) = 1 - p_b(s_1, s_2)$ to be directed from s_2 to s_1. A *directed path* from s to s' is a sequence as shown above, with the additional property that each b_i is directed from s_i to s_{i+1}, $i = 1, \dots, n-1$.

The problem, analogous to the one for the open-closed case, is now whether the probability that there exist two disjoint directed paths one of which goes from a site of V_1 to a site of V_2 and the other from a site of W_1 to a site of W_2 is, again, smaller than or equal to the product of the individual probabilities.

These two problems (the solved open-closed problem and the unsolved random-direction problem) represent special cases of the conjecture.

In Section 2 we give a formal description of the conjecture after introducing the necessary definitions and notation. We also present an attractive special case which, as shown in Section 3, turns out to be equivalent to the full conjecture. In Section 3 we also show some other equivalent forms of the conjecture, try to make clear the relation to the examples in Section 1, introduce additional definitions and notation, and give some general results concerning the conjecture.

In Section 4 we state our main result, Theorem 4.2, which consists of four nontrivial, proved, special cases of the conjecture. The first is an extension of the special case proved in [2], and also contains Harris' inequality mentioned in Example 2. In Section 4 we further give some corollaries and examples.

The proofs of the four cases of Theorem 4.2 are rather long and, except for the first two cases whose proofs are related, completely different. Therefore, they are given in three different Sections, 5, 6 and 7.

2. Formal statement of the conjecture. Let $\Omega = S_1 \times S_2 \times \cdots \times S_n$ with $S_1, S_2, \dots, S_n$ finite subsets of $\mathbb{N}$. *Realizations* (i.e., elements of Ω) are denoted by $\omega = (\omega_1, \dots, \omega_n)$. The *support* of an *event* (a subset of Ω) is defined as the set of all indices on which it depends. More precisely, if $A \subset \Omega$ then

(2.1) $\operatorname{supp}(A) := \{i | 1 \le i \le n, \exists \omega, \omega' \in \Omega, \forall j \ne i\ \omega_j = \omega'_j;\ \omega \in A, \omega' \notin A\}.$

Two events A and B are said to be *perpendicular* to each other, denoted by $A \perp B$, if $\operatorname{supp}(A) \cap \operatorname{supp}(B) = \varnothing$. For $\omega \in \Omega$ and $K \subset \{1, \dots, n\}$ we define the *cylinder*

$$[\omega]_K := \{\omega' | \omega' \in \Omega,\ \omega'_i = \omega_i, \text{for all } i \in K\}. \tag{2.2}$$

(2.3) REMARKS.

(i) Note that our definition of a cylinder is more restrictive than the usual one.

(ii) Though $[\omega]_K$ depends on Ω we omit this parameter.

If $A, B \subset \Omega$ we say that ω is a *disjoint realization* of A and B if ω is an element of both A and B but "for disjoint reasons." Formally, the set $A \square B$ of disjoint realizations of A and B is defined as:

$$\begin{aligned} A \square B := \{\omega \in \Omega | \exists K, L \subset \{1, \dots, n\},\ K \cap L = \varnothing, \\ [\omega]_K \subset A, \text{and } [\omega]_L \subset B\}. \end{aligned} \tag{2.4}$$

(2.5) REMARK. Note that we again omit the parameter Ω.

Our subject, the conjecture stated in van den Berg and Kesten (1985), is the following

(2.6) CONJECTURE. *Let* $n \in \mathbb{N} \setminus \{0\}$. *Let* S_i *be a finite subset of* $\mathbb{N}$ *and* μ_i *a probability measure on* S_i, $i = 1, \dots, n$. *Further, define* $\Omega = S_1 \times S_2 \times \cdots \times S_n$ *and* $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$. *Then*

$$\forall A, B \subset \Omega, \qquad \mu(A \square B) \leq \mu(A)\mu(B). \tag{2.7}$$

The special case in which, for each i, $S_i = \{0,1\}$ and $\mu_i(0) = \mu_i(1) = \frac{1}{2}$ gives

(2.8) CONJECTURE. *If* $\Omega = \{0,1\}^n$ $(n \in \mathbb{N} \setminus \{0\})$, *then*

$$\forall A, B \subset \Omega, \qquad |A \square B| 2^n \leq |A|\,|B|, \tag{2.9}$$

where $|\cdot|$ *denotes cardinality.*

It will be shown in Section 3 that the above special case is equivalent to the full conjecture (2.6).

3. General results concerning the $\square$-operation. Several results in this section, especially in the beginning, are almost trivial. However, they may help to get familiar with the $\square$-operation and make it possible to shorten the proofs of the more interesting results.

We start by stating some properties of the $\square$-operation leading to equivalent definitions of $A \square B$ and, subsequently, to the equivalents of (2.7).

Next we show the connection with the examples in Section 1. We also prove, as announced in Section 2, that the special case (2.8) implies the full (2.6). We do this with the help of a more general principle which will be used throughout Sections 5–7 and is therefore presented as a separate lemma (3.4). Finally, we prove another useful result (Lemma 3.12) and introduce some additional notation and definitions. It is easily seen that the $\square$-operation has the following properties and we omit the proof:

(3.1) LEMMA.

(i) $A \square B \subset A \cap B$.
(ii) *If* $A \perp B$, *then* $A \square B = A \cap B$.
(iii) $A \square B = B \square A$.
(iv) $(A_1 \cup A_2) \square B \supset ((A_1 \square B) \cup (A_2 \square B))$.

Using these properties, several definitions of $A \square B$, equivalent to the one in Section 2, can be given. First we define the following. By "a cylinder of A" we mean a cylinder contained in A. Further, a set C is called a *maximal cylinder* of A if C is a cylinder of A and there is no cylinder C' of A with $C \subsetneq C'$.

(3.2) LEMMA.

(i) $A \square B = \bigcup\{C \cap C' | C$ *is a cylinder of* A, C' *is a cylinder of* B, *and* $C \perp C'\}$.

(ii) $A \square B = \bigcup\{C \cap C' | C$ *is a maximal cylinder of* A, C' *is a maximal cylinder of* B, *and* $C \perp C'\}$.

(iii) $A \square B = \bigcup\{A' \cap B' | A' \subset A,\ B' \subset B$, *and* $A' \perp B'\}$.

PROOF. (i) Follows immediately from definition (2.4) and the definition of "$\perp$."

(ii) It is clear that the r.h.s. of (i) does not change if we restrict ourselves to maximal cylinders.

(iii) By (i) it is obvious that the l.h.s. of (iii) is contained in the r.h.s. We prove the other direction as follows: By Lemma 3.1(iii, iv), $A \square B \supset \bigcup\{A' \square B' | A' \subset A,\ B' \subset B\}$ which contains, of course, $\bigcup\{A' \square B' | A' \subset A,\ B' \subset B,\ A' \perp B'\}$ which, by Lemma 3.1(ii) is equal to the r.h.s. of 3.2(iii). □

Using the above lemma we get several equivalents of conjecture (2.6):

(3.3) LEMMA. *The following statements* (i, ii, iii, iv) *are equivalent to* (2.7):

(i) $\mu(\bigcup_{1 \le i \le m}(C_i \cap C_i')) \le \mu(\bigcup_{1 \le i \le m} C_i)\mu(\bigcup_{1 \le i \le m} C_i')$, *where* $m \in \mathbb{N} \setminus \{0\}$, $C_i, C_i' \subset \Omega$ *are cylinders*, $C_i \perp C_i'$, $i = 1, \dots, m$.

(ii) $\mu(\bigcup_{1 \le i \le m}(A_i \cap B_i)) \le \mu(\bigcup_{1 \le i \le m} A_i)\mu(\bigcup_{1 \le i \le m} B_i)$, *where* $m \in \mathbb{N} \setminus \{0\}$, $A_i, B_i \subset \Omega$, $A_i \perp B_i$, $i = 1, \dots, m$.

(iii) $\mu(\bigcup\{C_i \cap C_j' | i \in I,\ j \in I',\ C_i \perp C_j'\}) \le \mu(\bigcup_{i \in I} C_i)\mu(\bigcup_{i \in I'} C_i')$, *where* I, I' *are finite index sets, and* C_i, $i \in I$ *and* C_j', $j \in I'$ *are cylinders in* Ω.

(iv) $\mu(\bigcup\{A_i \cap B_j | i \in I,\ j \in I',\ A_i \perp B_j\}) \le \mu(\bigcup_{i \in I} A_i)\mu(\bigcup_{i \in I'} B_i)$, *where* I, I' *are finite index sets, and* A_i, $i \in I$ *and* B_j, $j \in I'$ *are subsets of* Ω.

PROOF. (2.7) ⇔ (i): (2.7) implies (i) by taking $A = \bigcup_{1 \le i \le m} C_i$, $B = \bigcup_{1 \le i \le m} C_i'$, and noting that, by (3.2(i)), the l.h.s. of (i) is contained in $A \square B$. Conversely, (i) implies (2.7) by taking, for $(C_i, C_i')_{1 \le i \le m}$ all possible pairs (C, C') with C a cylinder of A, C' a cylinder of B, and $C \perp C'$, and again using 3.2(i).

(2.7) ⇔ (ii): As the above proof; this time use 3.2(iii) instead of 3.2(i).
(2.7) ⇔ (iii): Analogous to the first case.
(2.7) ⇔ (iv): Analogous to the second case. □

We shall now briefly discuss the Examples 1.1 and 1.2 in light of the above definitions and results. As to Example 1.1, let, if there are n different types of toys, $S_1, S_2, \dots, S_n$ represent the sets of possible colors, and take $\Omega = S_1 \times S_2 \times \cdots \times S_n$.

The combinations $1, 2, 3, \dots, k$ on the first list and $1', 2', 3', \dots, l'$ on the second list correspond to cylinders $C_1, C_2, C_3, \dots, C_k$ and $C_1', C_2', C_3', \dots, C_l'$, respectively. Further, the set of compositions $1 \times 3', 1 \times 4', \dots$ corresponds exactly to $\{C_i \cap C_j' | 1 \le i \le k,\ 1 \le j \le l,\ C_i \perp C_j'\}$ and now, noting Lemma 3.3(iii), it is clear that the example, in its general setting (i.e., arbitrary number of different toys, etc.) is indeed an interpretation of the conjecture.

As to Example 1.2, assume that G is a finite graph (otherwise we can use obvious limit arguments). Now let $\Omega = \{0,1\}^{|E|}$, where E is the set of bonds of G and take, for $\omega \in \Omega$, $\omega_i = 1$ or 0 according as the bond b_i is open or closed (or, in the random-direction case, according as the direction of b_i). Then the events {there exists an open (directed) path from a site of V_1 to a site of V_2} and {there exists an open (directed) path from a site of W_1 to a site of W_2} can be considered as sets $A, B \subset \Omega$. It is easy to check that the event {there exist two disjoint open (directed) paths one of which goes from a site of V_1 to a site of V_2 and the other from a site of W_1 to a site of W_2} corresponds to $A \square B$, which clarifies the connection with the conjecture.

Lemma 3.3 yields rather trivial equivalents of conjecture (2.6). More interesting is the equivalence of this conjecture to conjecture (2.8). This equivalence will be proved by using the following lemma which is also useful in many other applications.

(3.4) LEMMA. *Let, for $1 \le i \le n$, $1 \le j \le m$, S_i and T_j be finite subsets of* $\mathbb{N}$, *and μ_i and ν_j probability measures on S_i and T_j, respectively. Further, let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ and $\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_m$ be the respective product measures on Ω $(= S_1 \times S_2 \times \cdots \times S_n)$ and Ω' $(= T_1 \times T_2 \times \cdots \times T_m)$. Finally, let A and B be subsets of Ω, and $f\colon \Omega' \to \Omega$ a map with the following properties* (i) *and* ((ii) *or* (ii′)):

(i) $\mu(\omega) = \nu(f^{-1}(\omega))$ *for all* $\omega \in \Omega$.

(ii) *If C_1 and C_2 are cylinders contained in Ω, and $C_1 \perp C_2$, then* $f^{-1}(C_1) \perp f^{-1}(C_2)$.

(ii′) *If C_1 and C_2 are maximal cylinders of A and B, respectively, and $C_1 \perp C_2$, then $f^{-1}(C_1) \perp f^{-1}(C_2)$.*

Then

$$\nu(A' \square B') \le \nu(A')\nu(B') \quad \textit{implies} \quad \mu(A \square B) \le \mu(A)\mu(B),$$

where $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$.

REMARKS.

(a) Note that we do not in (ii) and (ii′) require that $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are cylinders.

(b) Note that (ii′) is weaker than (ii) so that the latter is superfluous, since we require (i) and ((ii) or (ii′)). However, we also state (ii) because in many cases treated in this article, this stronger condition does hold.

PROOF. By Lemma 3.2(ii) we have $f^{-1}(A \square B) = f^{-1}(\bigcup\{C_1 \cap C_2 | C_1$ is a maximal cylinder of A, C_2 is a maximal cylinder of B, $C_1 \perp C_2\})$ which, of course, equals $\bigcup\{f^{-1}(C_1) \cap f^{-1}(C_2) | C_1$ is a maximal cylinder of A, C_2 is a maximal cylinder of B, $C_1 \perp C_2\}$ which, by property (ii′) of f, is contained in $\bigcup\{A'_1 \cap B'_1 | A'_1 \subset A', B'_1 \subset B', A'_1 \perp B'_1\}$ which, by Lemma 3.2(iii), is equal to

$A'\square B'$. Hence

$$\mu(A\square B) = \nu\big(f^{-1}(A\square B)\big) \le \nu(A'\square B') \le \nu(A')\nu(B') = \mu(A)\mu(B). \qquad \square$$

(3.5) LEMMA. *The conjectures* (2.6) *and* (2.8) *are equivalent.*

PROOF. We only have to prove that if conjecture (2.8) is true then conjecture (2.6) is also true, since the other direction is trivial. So suppose conjecture (2.8) is true. Let $\Omega = S_1 \times S_2 \times \cdots \times S_n$ with $S_i = \{s_{i1}, s_{i2}, \dots, s_{i,k_i}\}$, $i = 1, \dots, n$. Further let, for $1 \le i \le n$, μ_i be a probability measure on S_i, and $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$. Define $\rho_{i,j} = \mu_i(s_{i,j}) = \mu\{\omega_i = s_{i,j}\}$. Since we have a finite system, it is clear that, for each $A \subset \Omega$, $\mu(A)$ is a continuous function of $(\rho_{i,j})$ $1 \le i \le n$, $1 \le j \le k_i$. Using this and the fact that every $\rho_{i,j}$ can be approximated to arbitrary precision by numbers of the form $l2^{-M}$, $l, M \in \mathbb{N}$, it is clearly sufficient to consider the case that there exist integers M and $C_{i,j}$, $1 \le i \le n$, $1 \le j \le k_i$, such that $\rho_{i,j} = C_{i,j}2^{-M}$. So assume that the $\rho_{i,j}$'s are indeed of this form. Now consider, for each $i = 1, 2, \dots, n$, the set $\{0,1\}^M$, and order the elements of this set, e.g., lexicographically: $(0,0,\dots,0)$, $(1,0,\dots,0)$, $(0,1,\dots,0)$, $(1,1,0,\dots,0)$, etc. (we refer to this ordering in Section 5). Define the map f_i: $\{0,1\}^M \to S_i$ as follows: The first $C_{i,1}$ elements (with respect to the above ordering) are all mapped to s_{i1}, the next $C_{i,2}$ elements to $s_{i,2}$, etc. (Note that $\sum_j C_{i,j} = 2^M$ so that f_i is defined on all of $\{0,1\}^M$.) Now apply Lemma (3.4) with Ω and μ as above and $\Omega' = \{0,1\}^{nM}$ (i.e., $T_i = \{0,1\}$, $1 \le i \le nM$), ν the uniform distribution on Ω', and f: $\Omega' \to \Omega$ as defined by

$$\begin{aligned} &f\big(\omega'_1, \dots, \omega'_M, \omega'_{M+1}, \dots, \omega'_{2M}, \dots, \dots, \omega'_{(n-1)M+1}, \dots, \omega'_{nM}\big) \\ &\quad = \big(f_1(\omega'_1, \dots, \omega'_M), f_2(\omega'_{M+1}, \dots, \omega'_{2M}), \dots, \dots, f_n(\omega'_{(n-1)M+1}, \dots, \omega'_{nM})\big). \qquad \square \end{aligned}$$

We finish this section with some additional notation and with two lemmas which are useful in the proofs of the results in Section 4.

(3.6) NOTATION. Let n, m be positive integers. Denote, for $k_i \in \mathbb{N}$, $\omega^i \in \mathbb{N}^{k_i}$, $A_i \subset \mathbb{N}^{k_i}$, $i = 1, \dots, m$:

$$(3.7) \qquad (\omega^1, \dots, \omega^m) := \big(\omega^1_1, \dots, \omega^1_{k_1}, \dots, \omega^m_1, \dots, \omega^m_{k_m}\big).$$

$$(3.8) \qquad [A_1, \dots, A_m] := A_1 \times A_2 \times \cdots \times A_m.$$

Further, for $S_1, \dots, S_n \subset \mathbb{N}$ finite, $\Omega = \prod_{i=1}^n S_i$, $l, r \ge 0$, $l + r \le n$, $A \subset \prod_{i=1}^l S_i$, $B \subset \prod_{i=l+r+1}^n S_i$:

$$(3.9) \qquad [A, *^r, B] := \left[A, \prod_{i=l+1}^{l+r} S_i, B\right].$$

REMARKS.

(a) Of course, when we use the notation of (3.9), the S_i, $l + 1 \le i \le l + r$, are assumed to be known.

(b) If no confusion is possible, we omit the commas in (3.7), (3.8), and (3.9) and the "r" in (3.9).

(c) If an A_i in (3.8) consists of one element ω, we write "ω" instead of "$\{\omega\}$."

(d) The notation (3.9) can be extended in an obvious way to more "$*$"'s.

(3.10) LEMMA. *Let, for $i = 1,\dots,n$, S_i be a finite subset of $\mathbb{N}$ and μ_i a probability measure on S_i. Let $\Omega = \prod_{i=1}^n S_i$ and $\mu = \prod_{i=1}^n \mu_i$. Define, for π a permutation of $\{1,\dots,n\}$, $\omega = (\omega_1,\dots,\omega_n) \in \Omega$, and $D \subset \Omega$:*

$$\pi(\omega) := (\omega_{\pi(1)},\dots,\omega_{\pi(n)}) \quad \text{and} \quad \pi(D) = \{\pi(\omega) | \omega \in D\}.$$

Further, let $S_i' = S_{\pi^{-1}(i)}$ and $\mu_i' = \mu_{\pi^{-1}(i)}$, $i = 1,\dots,n$, $\Omega' = \prod_{i=1}^n S_i'$, and $\mu' = \prod_{i=1}^n \mu_i'$. Then, for all $A, B \subset \Omega$:

$$\text{(3.11)} \qquad \begin{gathered} \mu'(\pi(A)) = \mu(A), \qquad \mu'(\pi(B)) = \mu(B), \\ \mu'(\pi(A) \square \pi(B)) = \mu(A \square B). \end{gathered}$$

PROOF. The proof is straightforward. □

(3.12) LEMMA. *Let $A, B \subset \prod_{i=1}^n S_i$, $K_A \subset \{1,\dots,n\} \setminus \operatorname{supp}(B)$, $K_B \subset \{1,\dots,n\} \setminus \operatorname{supp}(A)$, $K_{AB} = K_A \cup K_B$, $K = \{1,\dots,n\} \setminus K_{AB}$. By the preceding lemma we may assume for our purpose that, for certain $r, s, t \geq 0$, $K = \{1,\dots,r\}$, $K_A = \{r+1,\dots,r+s\}$, $K_B = \{r+s+1,\dots,r+s+t = n\}$ and $K_{AB} = \{r+1,\dots,n\}$. Define, for $\omega' \in \prod_{i \in K_{AB}} S_i$, and $D \subset \Omega$,*

$$D(\omega') := \left\{\omega \in \prod_{i \in K} S_i \middle| (\omega\, \omega') \in D\right\}.$$

Let μ_i be a probability measure on S_i $i = 1,2,\dots,n$, $\mu = \prod_{i=1}^n \mu_i$ and $\tilde{\mu} = \prod_{i \in K} \mu_i$. If

$$\text{(3.13)} \qquad \forall\, \omega' \in \prod_{i \in K_{AB}} S_i\; \tilde{\mu}(A(\omega') \square B(\omega')) \leq \tilde{\mu}(A(\omega'))\tilde{\mu}(B(\omega')),$$

then

$$\text{(3.14)} \qquad \mu(A \square B) \leq \mu(A)\mu(B).$$

PROOF. Define, in addition to the above, for $\omega_A \in \prod_{i \in K_A} S_i$ and $\omega_B \in \prod_{i \in K_B} S_i$

$$\hat{A}(\omega_A) := \left\{\tilde{\omega} \in \prod_{i \in K} S_i \middle| [\tilde{\omega}\omega_A *] \subset A\right\}$$

and

$$\check{B}(\omega_B) := \left\{\tilde{\omega} \in \prod_{i \in K} S_i \middle| [\tilde{\omega} * \omega_B] \subset B\right\}.$$

It is easily seen that if $\omega' = (\omega_A \omega_B) \in \prod_{i \in K_{AB}} S_i$ then $\hat{A}(\omega_A) = A(\omega')$, $\check{B}(\omega_B) = B(\omega')$, and $(A \square B)(\omega') = A(\omega') \square B(\omega') = \hat{A}(\omega_A) \square \check{B}(\omega_B)$.

Let $\mu_A = \prod_{i \in K_A} \mu_i$, $\mu_B = \prod_{i \in K_B} \mu_i$, and $\mu_{AB} = \mu_A \times \mu_B$. If the condition in (3.13) holds, then

$$\begin{aligned}
\mu(A \square B) &= \sum_{\omega'} \mu([(A \square B)(\omega')\omega']) = \sum_{\omega'} \mu_{AB}(\omega')\tilde{\mu}(A(\omega') \square B(\omega')) \\
&\le \sum_{\omega'} \mu_{AB}(\omega')\tilde{\mu}(A(\omega'))\tilde{\mu}(B(\omega')) \\
&= \sum_{\omega_A}\sum_{\omega_B} \mu_A(\omega_A)\mu_B(\omega_B)\tilde{\mu}(\hat{A}(\omega_A))\tilde{\mu}(\check{B}(\omega_B)) \\
&= \sum_{\omega_A} \mu_A(\omega_A)\tilde{\mu}(\hat{A}(\omega_A)) \sum_{\omega_B} \mu_B(\omega_B)\tilde{\mu}(\check{B}(\omega_B)) \\
&= \sum_{\omega_A} \mu([\hat{A}(\omega_A)\omega_A *]) \sum_{\omega_B} \mu([\check{B}(\omega_B) * \omega_B]) = \mu(A)\mu(B),
\end{aligned}$$

where ω' is summed over $\prod_{i \in K_{AB}} S_i$, ω_A over $\prod_{i \in K_A} S_i$ and ω_B over $\prod_{i \in K_B} S_i$. □

4. Statement of the main results. We state in Theorem 4.2 four special cases of conjecture (2.6) which are proved in Sections 5–7. The theorem is followed by a short discussion of each of the cases.

(4.1) Remark. In addition to the cases of Theorem 4.2 we can also prove (2.6) in the case where the maximal cylinders of A or of B are mutually disjoint. The proof is straightforward. [Use Lemma 3.2(ii) and the fact that each set is the union of its maximal cylinders.] Further, we have a (rather complicated) proof for the case in which $\Omega = \{0,1\}^n$, μ is the uniform distribution on Ω, and A or B has at most 3 maximal cylinders.

For the first case of Theorem 4.2 we need some definitions. Let, as usual, $S_1, \dots, S_n$ be finite subsets of $\mathbb{N}$ and $\Omega = \prod_{i=1}^n S_i$. If $\omega, \omega' \in \Omega$ then $\omega \ge \omega'$ means $\omega_i \ge \omega_i'$, $i = 1, \dots, n$. A set $A \subset \Omega$ is called *increasing* or *positive* if $\omega' \in A$ whenever $\omega' \in \Omega$, $\omega' \ge \omega$, and $\omega \in A$. Analogously, A is decreasing or negative if $\omega' \in A$ whenever $\omega' \in \Omega$, $\omega' \le \omega$, and $\omega \in A$. The events A and B in the open-closed case of 1.2 are examples of increasing events. However, the corresponding events in the random-direction case can be represented neither as increasing nor as decreasing events.

(4.2) Theorem. *Let, for $1 \le i \le n$, S_i be a finite subset of $\mathbb{N}$ and μ_i a probability measure on S_i. Let $\Omega = S_1 \times \cdots \times S_n$, $\mu = \mu_1 \times \cdots \times \mu_n$, and $A, B \subset \Omega$. In each of the following cases we have*

$$\mu(A \square B) \le \mu(A)\mu(B). \tag{4.3}$$

(a) *A and B are both an intersection of an increasing and a decreasing event.*

(b) *$\Omega = \{0,1\}^n$ and A, B are both permutation invariant [i.e., if the coordinates of an element of $A(B)$ are permuted, the result is again an element of $A(B)$].*

(c) *There are cylinders* C_i, $i \in I$, *and* C'_j, $j \in J$, *such that* $A = \bigcup_{i \in I} C_i$, $B = \bigcup_{j \in J} C'_j$ *and for all* $i \in I$, $j \in J$: $C_i \perp C'_j$ *or* $C_i \cap C'_j = \varnothing$.

(d) *There are cylinders* C_i, $i \in I$, *such that* $A = \bigcup_{i \in I} C_i$, C_i *is a maximal cylinder of* A, $i \in I$, *and for all* i, $j \in I$: $C_i \perp C_j$ *or* $\mathrm{supp}(C_i) \equiv \mathrm{supp}(C_j)$.

PROOF. The cases (a) and (b) are proved in Section 5, the case (c) in Section 6 and the case (d) in Section 7. □

(4.4) DISCUSSION OF THEOREM 4.2.

(a) Note that this result includes the case that A and B are both increasing and the case that A is increasing and B decreasing. The first has been proved, for $\Omega = \{0,1\}^n$ by van den Berg and Kesten (1985), who obtained it as a special case of a result concerning so-called NBU measures. They had several other (unpublished) more direct proofs. One of these is closely related to the proof of the clutter theorem. [This theorem is treated by McDiarmid (1980, 1981) and has been applied before by Hammersley (1961).] We use a refinement of it (which we call the splitting method) to prove (a) and (b).

If A is increasing and B decreasing and $\Omega = \{0,1\}^n$, then $A \square B = A \cap B$ and we get, by (a), $\mu(A \cap B) \le \mu(A)\mu(B)$. This is equivalent to a correlation inequality of Harris (1960). In fact, Harris' inequality says that increasing events are positively correlated to each other but, since the complement of an increasing event is decreasing, this is the same as saying that an increasing and a decreasing event are always negatively correlated to each other. See also the discussion of (c) which also contains Harris' inequality as a special case.

(b) In spite of serious attempts we have not been able to prove the permutation-invariant case more generally, i.e., for $\Omega = \{0,1,\dots,k\}^n$, $k \ge 2$. That result would have the interesting consequence that the multinomial distribution is SNBU [see van den Berg and Kesten (1985) for a discussion of SNBU].

(c) Note that, in this case, $A \square B = A \cap B$ so that we have a correlation inequality. As a special case we have $\Omega = \{0,1\}^n$, A increasing and B decreasing [because the maximal cylinders of an increasing event are always of the form $[\omega^1]_K$ and those of a decreasing event of the form $[\omega^0]_K$ where ω^1 is the element $(1,1,\dots,1)$ and ω^0 the element $(0,0,\dots,0)$]; this reduces again to Harris' inequality which was also obtained as a special case of (a). Harris' inequality has been extended by Fortuin, Kasteleyn, and Ginibre (1971) to a larger class of probability measures on $\{0,1\}^n$. The FKG inequality in turn is contained in a rather general theory developed by Ahlswede and Daykin (1979). However, apart from some common special cases, like Harris' inequality, there does not seem to be a relationship between conjecture (2.6) and the results of Ahlswede and Daykin.

Another example of (c) is the following:

(4.5) EXAMPLE. Define, for positive integers l, m,

$$B_{l,m} := \{(x,y) \in \mathbb{Z}^2 \mid 0 \le x \le l, 0 \le y \le m\}.$$

The *boundary* of $B_{l,m}$ is defined as

$$\delta(B_{l,m}) := \{(x,y) \in B_{l,m} | x = 0 \text{ or } x = l \text{ or } y = 0 \text{ or } y = m\},$$

and the *interior* of $B_{l,m}$ as

$$\operatorname{int}(B_{l,m}) := B_{l,m} \setminus \delta(B_{l,m}).$$

The sets $B_{l,m}$, $l, m \in \mathbb{N} \setminus \{0\}$, and their images under translations $\mathbb{Z}^2 \to \mathbb{Z}^2$ are called *boxes*. Now suppose that each site $s \in \mathbb{Z}^2$ is, independent of the other sites, black with probability p_s and white with probability $1 - p_s$. A box is called *black* (*white*) if its boundary is black (white) and its interior is white (black). Let V be a finite region in $\mathbb{Z}^2$, e.g., for a certain positive integer r,

$$V = \{(x,y) \in \mathbb{Z}^2 | |x|, |y| \le r\}.$$

Further, let A be the event {there exists a black box in V} and B the event {there exists a white box in V}. It is not difficult to check that this falls under case (c) in our theorem, so we get $P(A \cap B) \le P(A)P(B)$. Note that Harris' inequality cannot be applied here because neither A nor B is increasing or decreasing.

(d) This case has the following interesting consequence:

(4.6) COROLLARY. *Let $x_1, x_2, \ldots, x_n$ be independent random variables with values in $\mathbb{R}$ (or another set, it turns out that the set is immaterial). Let, for $1 \le i \le n$, A_i, B_i, and C_i be Borel sets. Then*

$$\begin{aligned} &P\{\text{there are three different } i, j, k \text{ such that} \\ &\qquad x_i \in A_i,\ x_j \in B_j, \text{ and } x_k \in C_k\} \\ &\le P\{\exists i\, x_i \in A_i\} P\{\exists i\, x_i \in B_i\} P\{\exists i\, x_i \in C_i\}. \end{aligned} \tag{4.7}$$

REMARKS.

(i) The result can be extended to four types of sets or more (i.e., A_i''s, B_i''s, C_i''s, D_i''s, etc.). We show how (4.7) follows from our theorem; the above-mentioned extension can be proved by induction on the number of different types of sets.

(ii) For the case with two types of sets, and for the case in which, for each i, the sets with index i are mutually disjoint, there is a more direct proof. However, if there are no additional conditions, we do not know a proof of (4.7) which is more direct than that of the full Theorem 2.1(d) of which it is a corollary.

PROOF OF THE COROLLARY. First we remark that the l.h.s. of (4.7) is completely determined by the probabilities

$$\begin{aligned} &P_i(q,r,s) := P\{x_i \in A(q) \cap B(r) \cap C(s)\}, \\ &\qquad\qquad q, r, s \in \{0,1\}, 1 \le i \le n, \end{aligned} \tag{4.8}$$

where, for a set V, $V(0)$ denotes V and $V(1)$ denotes V^c. Therefore, it is sufficient to prove the corollary for the case in which each x_i can only have a finite number of values. Hence it is equivalent to the following:

(4.9) *Let, for $i = 1, 2, \ldots, n$, S_i be a finite subset of $\mathbb{N}$, μ_i a probability measure on S_i, and A_i, B_i, C_i subsets of S_i. Further, let $\Omega = S_1 \times \cdots \times S_n$ and $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$. Then*

$$\mu\{(\omega_1, \ldots, \omega_n) | \text{ there exist different } i, j, k \text{ such that } \omega_i \in A_i, \omega_j \in B_j, \text{ and } \omega_k \in C_k\} \leq \mu\{(\omega_1, \ldots, \omega_n) | \exists i \ \omega_i \in A_i\} \times \mu\{(\omega_1, \ldots, \omega_n) | \exists j \ \omega_j \in B_j\} \mu\{(\omega_1, \ldots, \omega_n) | \exists k \ \omega_k \in C_k\}. \tag{4.10}$$

PROOF OF (4.9). Define $A := \{(\omega_1, \ldots, \omega_n) | \exists i \ \omega_i \in A_i\}$, and define B and C analogously. It is easy to see that the event of the l.h.s. of (4.10) is a subset of $A \square (B \square C)$, and that A, B, and C have the property mentioned in case (d) of Theorem 4.6. Now the result follows by applying this theorem twice. □

5. Proof of Theorem 4.2, cases (a) and (b). In case (b) we have $\Omega = \{0, 1\}^n$, and for (a) it is sufficient, by virtue of Lemma 3.4 [take f as in the proof of Lemma 3.5, noting that $f^{-1}(A)$ is increasing (decreasing) if A is increasing (decreasing)] to restrict ourselves to the binary case. The proofs are based on the splitting method mentioned in Section 4, which we shall explain here. First some definitions [mind the notation (3.6)–(3.9), which will be used frequently]. In the following we always assume $n > 1$.

(5.1) DEFINITION. If $A \subset \{0, 1\}^n$, then

$$A^1 = \left\{\omega | \omega \in \{0, 1\}^{n-1}, (\omega 1) \in A\right\},$$

$$A^0 = \left\{\omega | \omega \in \{0, 1\}^{n-1}, (\omega\, 0) \in A\right\},$$

$$A^{01} = A^0 \cap A^1 = \left\{\omega | \omega \in \{0, 1\}^{n-1}, [\omega\, *] \subset A\right\}.$$

Clearly, for $V \subset \{0, 1\}^{n-1}$

$$V \subset A^1 \Leftrightarrow [V 1] \subset A,$$

$$V \subset A^0 \Leftrightarrow [V 0] \subset A,$$

$$V \subset A^{01} \Leftrightarrow [V *] \subset A.$$

The following observations are frequently used.

(5.2) OBSERVATIONS.

(i) *If* $\tilde{\omega} \in \{0, 1\}^{n-1}$, $A \subset \{0, 1\}^n$, $\omega_n \in \{0, 1\}$, $\omega = (\tilde{\omega} \omega_n)$, *and* $K \subset \{1, \ldots, n-1\}$, *then*: $[\omega]_K \subset A \Leftrightarrow [\tilde{\omega}]_K \subset A^{01}$.

(ii) $A^i \cap B^i = (A \cap B)^i$, *where i denotes* 0, 1, *or* 01.

(iii) $A = [A^1 1] \cup [A^0\, 0]$.

(5.3) LEMMA. *For* $A, B \subset \{0,1\}^n$

(i) $(A\square B)^1 = (A^1\square B^{01}) \cup (B^1\square A^{01})$;
(ii) $(A\square B)^0 = (A^0\square B^{01}) \cup (B^0\square A^{01})$;
(iii) $(A\square B)^{01} = (A^1\square B^{01} \cap B^0\square A^{01}) \cup (A^1\square B^{01} \cap A^0\square B^{01}) \cup (B^1\square A^{01} \cap A^0\square B^{01}) \cup (B^1\square A^{01} \cap B^0\square A^{01})$.

PROOF. (i) We prove that the l.h.s. is contained in the r.h.s. The reverse can be proved analogously.

$$\omega = (\omega_1,\ldots,\omega_{n-1}) \in (A\square B)^1 \Rightarrow (\omega 1) \in A\square B$$
$$\Rightarrow \exists K, L \subset \{1,\ldots,n\}\ K\cap L = \varnothing, \qquad [\omega 1]_K \subset A, \qquad [\omega 1]_L \subset B.$$

For K, L as above, at least one of these sets does not contain n. Suppose $n \notin L$. We show that this implies $\omega \in A^1\square B^{01}$. [Analogously (note the symmetry), $n \notin K$ implies $\omega \in B^1\square A^{01}$.] $n \notin L \Rightarrow L \subset \{1,\ldots,n-1\} \Rightarrow [\omega]_L \subset B^{01}$ [see 5.2(i)]. Further, $[\omega 1]_K \subset A \Rightarrow [\omega]_{K'} \subset A^1$, where $K' = K\setminus\{n\}$. Hence, because $L\cap K' = \varnothing$, $\omega \in A^1\square B^{01}$.

(ii) Analogous to (i) (by 0–1 symmetry).

(iii) Follows from (i), (ii), and Definition 5.1. □

(5.4) DEFINITION. For the pair A, $B \subset \{0,1\}^n$ we define the pair A^*, ${}^*B \subset \{0,1\}^{n+1}$ by

$$A^* = \left\{(\omega_1,\ldots,\omega_{n+1}) \in \{0,1\}^{n+1} \middle| (\omega_1,\ldots,\omega_n) \in A\right\},$$
$${}^*B = \left\{(\omega_1,\ldots,\omega_{n+1}) \in \{0,1\}^{n+1} \middle| (\omega_1,\ldots,\omega_{n-1},\omega_{n+1}) \in B\right\}.$$

Further, if $\mu = \mu_1 \times \cdots \times \mu_n$, where each μ_i is a probability measure on $\{0,1\}$, then $\overset{*}{\mu}$ is the probability measure $\mu_1 \times \cdots \times \mu_n \times \mu_n$ on $\{0,1\}^{n+1}$. Note that $A^* = [A\,*]$ and ${}^*B = \pi([B\,*])$ (where π is the map which exchanges the last two coordinates). Also,

$$\text{(5.5)} \qquad \text{(i)}\ \overset{*}{\mu}(A^*) = \mu(A); \qquad \text{(ii)}\ \overset{*}{\mu}({}^*B) = \mu(B).$$

Roughly speaking, $A*$ and $*B$ are obtained by "making A and B independent in the last coordinate by splitting this coordinate." Analogous operations can be defined for the coordinates $1,\ldots,n-1$. Intuitively, one would expect that, after applying one of these split operations, the probability of $A\square B$ always increases. This would imply that conjecture (2.6) is true, because then, after successively "splitting" all coordinates $1,\ldots,n$ we would have, for the "new" A and B (denoted by $A^{\cdot}$, ${}^{\cdot}B$),

$$A^{\cdot} \perp {}^{\cdot}B, \quad \text{hence } \mu(A\square B) \le \dot{\mu}(A^{\cdot}\square {}^{\cdot}B) = \dot{\mu}(A^{\cdot})\dot{\mu}({}^{\cdot}B) = \mu(A)\mu(B).$$

However, Ahlswede gave a counterexample and since then we have observed that it goes wrong very often. It appears that the probability of $A\square B$ does increase if $(A\square B)^{01} = A^{01}\square B^{01}$, which (as we shall show) holds for case (a) of our theorem. We shall prove that a weaker condition is also sufficient, which we use to prove case (b) of the theorem.

(5.6) Lemma. *Let* $A, B \subset \{0,1\}^n$. *Then*

$$A^*\Box^*B = [A^0\Box B^0\,0\,0] \cup [A^1\Box B^0\,1\,0] \cup [A^0\Box B^1\,0\,1] \cup [A^1\Box B^1\,1\,1].$$

Proof. Let $r, s \in \{0,1\}$; we show that for all $\omega \in \{0,1\}^{n-1}$, $(\omega\, r s) \in A^*\Box^*B \Leftrightarrow \omega \in A\Box B^s$. $(\omega\, r\, s) \in A^*\Box^*B \Leftrightarrow \exists K, L \subset (1,\dots, n+1)$ s.t. $K \cap L = \varnothing$, $[\omega\, r s]_K \subset A^*$, $[\omega\, r s]_L \subset {}^*B$. However, because $(n+1) \notin \mathrm{supp}(A^*)$ and $n \notin \mathrm{supp}(^*B)$, the last statement is equivalent to $\exists K' \subset \{1,\dots, n-1\}$, $L' \subset \{1,\dots, n-1\}$ s.t. $K' \cap L' = \varnothing$, $[\omega]_{K'} \subset A^r$, and $[\omega]_{L'} \subset B^s \Leftrightarrow \omega \in A^r\Box B^s$. □

(5.7) Lemma. *Let* $A, B \subset \{0,1\}^n$.

(i) *If*

$$(A\Box B)^{01} = (A^1\Box B^{01} \cap A^0\Box B^{01}) \cup (B^1\Box A^{01} \cap B^0\Box A^{01}),$$

then $\overset{*}{\mu}(A^*\Box^*B) \geq \mu(A\Box B)$.

(ii) *If* $(A\Box B)^{01} = A^{01}\Box B^{01}$, *then* $\overset{*}{\mu}(A^*\Box^*B) \geq \mu(A\Box B)$.

Proof. See Lemma 5.3(iii). We only have to prove (i) because (ii) is weaker. Suppose the condition in (i) holds. Let $\tilde{\omega} = (\omega_1, \dots, \omega_{n-1})$ be given. We show that the conditional probability of $A^*\Box^*B$ is always at least the conditional probability of $A\Box B$. We have four cases (a, b, c, d). First let $p = \mu(\{\omega | \omega_n = 1\})$.

(a) $\omega \notin ((A\Box B)^1 \cup (A\Box B)^0)$. Then the conditional probability of $A\Box B$ is 0.

(b) $\tilde{\omega} \in (A\Box B)^1 \setminus (A\Box B)^0$. Then $(\tilde{\omega}, \omega_n) \in A\Box B$ iff $\omega_n = 1$, which has probability p. On the other hand, by Lemma 5.3, $\tilde{\omega} \in A^1\Box B^{01}$ or $\tilde{\omega} \in B^1\Box A^{01}$. If $\tilde{\omega} \in A^1\Box B^{01}$ then it is sufficient, in order to have $(\tilde{\omega}\omega_n\omega_{n+1}) \in A^*\Box^*B$, that $\omega_n = 1$, which has probability p. Analogously, if $\tilde{\omega} \in B^1\Box A^{01}$ it is sufficient that $\omega_{n+1} = 1$, which also has probability p.

(c) $\tilde{\omega} \in (A\Box B)^0 \setminus (A\Box B)^1$. This case is analogous to case (b).

(d) $\tilde{\omega} \in (A\Box B)^1 \cap (A\Box B)^0$.

Hence $\tilde{\omega} \in (A\Box B)^{01}$. So the conditional probability of $A\Box B$ equals 1. We have to show that also the conditional probability of $A^*\Box^*B$ equals 1. By the condition in (i),

$$\tilde{\omega} \in A^1\Box B^{01} \cap A^0\Box B^{01} \quad \text{or} \quad \tilde{\omega} \in B^1\Box A^{01} \cap B^0\Box A^{01}.$$

Suppose the first holds (by 0–1 symmetry the reasoning is analogous if the latter holds).

$$\tilde{\omega} \in A^1\Box B^{01} \Rightarrow \text{(see Lemma 5.6)}[\tilde{\omega}\,1\,*] \subset A^*\Box^*B.$$

$$\tilde{\omega} \in A^0\Box B^{01} \Rightarrow [\tilde{\omega}\,0\,*] \subset A^*\Box^*B.$$

Hence $[\tilde{\omega}\, *\, *] \subset A^*\Box^*B$, so that, indeed, the conditional probability of $A^*\Box^*B$ equals 1. □

(5.8) Proof of Theorem 4.2, case (a). We show that, if $n \geq 2$, $A, B \subset \{0,1\}^n$ and A, B have the property mentioned in case (a), then the condition in

Lemma 5.7(ii) holds. This is sufficient because A^* and $*B$ also fall under case (a), so we can successively split all coordinates. (More formally, the proof can be completed by induction on the number of "unsplit" coordinates.)

So, let $A, B \subset \{0,1\}^n$, $A = D \cap E$, $B = F \cap G$, where D and F are increasing, and E and G decreasing subsets of $\{0,1\}^n$. Suppose $\omega \in (A \Box B)^{01}$. We shall show that this implies $\omega \in A^{01} \Box B^{01}$, so that $(A \Box B)^{01} \subset A^{01} \Box B^{01}$. The reverse inclusion is trivial. $\omega \in (A \Box B)^{01} \Rightarrow (\omega\, 0) \in A \Box B$ and $(\omega\, 1) \in A \Box B$. $(\omega\, 0) \in A \Box B \Rightarrow \exists K, L \subset \{1, \dots, n-1\}$ s.t. $K \cap L = \varnothing$, $\omega \equiv 1$ on $K \cup L$, and $[\omega\, 0]_K \subset D$, $[\omega\, 0]_L \subset F$. $(\omega\, 1) \in A \Box B \Rightarrow \exists K', L' \subset \{1, \dots, n-1\}$ s.t. $K' \cap L' = \varnothing$, $\omega \equiv 0$ on $K' \cup L'$, $[\omega 1]_{K'} \subset E$, $[\omega 1]_{L'} \subset G$. Fix such K, L, K', L'. We have:

$$\omega \equiv 1 \text{ on } K \cup L \text{ and } \omega \equiv 0 \text{ on } K' \cup L' \Rightarrow (K \cup L) \cap (K' \cup L') = \varnothing.$$

Hence K, L, K', and L' are mutually disjoint subsets of $\{1, \dots, n-1\}$ and it is easily seen that $[\omega]_{K \cup K'} \subset A^{01}$ and $[\omega]_{L \cup L'} \subset B^{01}$. □

(5.9) Proof of Theorem 4.2, case (b). We show that if $A, B \subset \{0,1\}^n$ are permutation invariant, then condition (i) in Lemma 5.7 holds. The proof can then be completed by induction on n [using Lemma 3.12, noting that $\{n, n+1\} \notin \operatorname{supp}(A^*) \cap \operatorname{supp}(*B)$ and that, in the notation of the lemma, for all $i, j \in \{0,1\}$ the pair $A^*(i,j)$, $*B(i,j)$, also falls under Theorem 4.2, case (b)]. Suppose

$$\omega \in (A^1 \Box B^{01} \cap B^0 \Box A^{01}) \setminus (B^1 \Box A^{01} \cup A^0 \Box B^{01}). \tag{5.10}$$

$\omega \in A^1 \Box B^{01} \Rightarrow \exists K, L \subset \{1, \dots, n-1\}$ s.t. $[\omega]_K \subset A^1$, $[\omega]_L \subset B^{01}$, and $K \cap L = \varnothing$. Fix such K, L. We have $[[\omega]_K 1] \subset A$, $[[\omega]_L *] \subset B$. Assume $\exists i \in L\ \omega_i = 1$. Fix such an i. Define $K' = K \cup \{i\}$, $L' = L \setminus \{i\}$. Obviously $K' \cap L' = \varnothing$. By the permutation invariance of A and B we have $[[\omega]_{K'} *] \subset A$ and $[[\omega]_{L'} 1] \subset B$, so that $[\omega]_{K'} \subset A^{01}$ and $[\omega]_{L'} \subset B^1$; hence $\omega \in B^1 \Box A^{01}$. This is in contradiction to (5.10). Hence the assumption "$\exists i \in L\ \omega_i = 1$" is false, so that $\omega \equiv 0$ on L. Analogously, because ω is in $B^0 \Box A^{01}$ but not in $A^0 \Box B^{01}$, we get that for certain disjoint $\tilde{K}, \tilde{L} \subset \{1, \dots, n-1\}$: $[\omega]_{\tilde{K}} \subset A^{01}$, $[\omega]_{\tilde{L}} \subset B^0$, and $\omega \equiv 1$ on $\tilde{K}$. Hence, since $\omega \equiv 0$ on L and 1 on $\tilde{K}$, L and $\tilde{K}$ are disjoint.

Summarizing, $[\omega]_L \subset B^{01}$, $[\omega]_{\tilde{K}} \subset A^{01}$, and $L \cap \tilde{K} = \varnothing$, i.e., $\omega \in A^{01} \Box B^{01}$. However, this is in contradiction to (5.10). Therefore, we may conclude that the r.h.s. of (5.10) equals $\varnothing$, hence

$$(A^1 \Box B^{01}) \cap (B^0 \Box A^{01}) \subset (B^1 \Box A^{01}) \cup (A^0 \Box B^{01}). \tag{5.11}$$

Of course, we may replace the r.h.s. of (5.11) by its intersection with the l.h.s., which is

$$\begin{gathered}((B^1 \Box A^{01}) \cap (A^1 \Box B^{01}) \cap (B^0 \Box A^{01})) \\ \cup ((A^0 \Box B^{01}) \cap (A^1 \Box B^{01}) \cap (B^0 \Box A^{01})),\end{gathered}$$

which is contained in the r.h.s. of the condition (i) in Lemma (5.7). Analogously, by 0–1 symmetry, we can prove also that $A^0 \Box B^{01} \cap B^1 \Box A^{01}$ is contained in the r.h.s. of 5.7(i). The required result now follows from Lemma 5.3(iii). □

6. Proof of Theorem 4.2, case (c). We first state some definitions and lemmas. A *representation* of an event A is a set $\{[\omega^i]_{K_i} | i \in I\}$ of cylinders s.t. $\bigcup_{i \in I} [\omega^i]_{K_i} = A$.

(6.1) DEFINITION. Let $\mathscr{A} = \{[\omega^i]_{K_i} | i \in I\}$ and $\mathscr{B} = \{[\omega^j]_{K_j} | j \in J\}$ be sets of cylinders. The pair $\mathscr{A}$, $\mathscr{B}$ is called semi-disjoint if

$$\forall\, i \in I,\ j \in J\colon K_i \cap L_j = \varnothing \quad \text{or} \quad [\omega^i]_{K_i} \cap [\omega^j]_{L_j} = \varnothing .$$

A pair of events A, B is called semi-disjoint if there exists a semi-disjoint pair of representations of A and B.

REMARK. Note that these pairs of events form exactly case (c) in the theorem.

The set of maximal cylinders of an event A is denoted by $\mathrm{MR}(A)$. Clearly $\mathrm{MR}(A)$ is a representation of A.

(6.2) LEMMA. *Let A and B be events and let $\mathscr{A}$ be a representation of A. If the property $(*)$ (see below) holds for all ω and for all K then the pair $\mathscr{A}$, $\mathrm{MR}(B)$ is semi-disjoint.*

$$(*) \qquad ([\omega]_K \in \mathscr{A}, \omega \in B) \to [\omega]_{K^c} \subset B.$$

PROOF. Let $[\gamma]_L \in \mathrm{MR}(B)$, $[\eta]_K \in \mathscr{A}$, and $[\eta]_K \cap [\gamma]_L \neq \varnothing$. Then for a suitable σ this intersection can be written as $[\sigma]_{K \cup L}$. (Note that, for such σ, $[\eta]_K = [\sigma]_K$, $[\gamma]_L = [\sigma]_L$.) It is not difficult to see that $[\sigma]_{L \setminus K} = \bigcup\{[\omega]_{K^c} | \omega \in [\sigma]_{K \cup L}\}$ which, by $(*)$, is contained in B. Hence $[\sigma]_{L \setminus K}$ is a cylinder of B. But then, $[\gamma]_{L \setminus K}$ is also a cylinder of B. This is possible only if $L \setminus K = L$. Hence $K \cap L = \varnothing$. □

(6.3) LEMMA. *The pair of sets D, E is semi-disjoint if and only if the pair $\mathrm{MR}(D), \mathrm{MR}(E)$ is semi-disjoint.*

PROOF. The "if-part" is trivial so we only have to prove the other direction: If the pair D, E is semi-disjoint, then by definition there is a semi-disjoint pair $\mathscr{D}$, $\mathscr{E}$ of representations of D and E. It is not difficult to show that if we take $A = D$, $\mathscr{A} = \mathscr{D}$, and $B = E$, then $(*)$ in Lemma 6.2 holds for all ω and K. Hence, by that lemma, the pair $\mathscr{D}, \mathrm{MR}(E)$ is semi-disjoint. Applying Lemma 6.2 once more [this time we take $A = E$, $\mathscr{A} = \mathrm{MR}(E)$ and $B = D$] gives the result. □

(6.4) LEMMA. *Let $\Omega = \prod_{i=1}^n \{1, \dots, k_i\}$ and let $\mu = \prod_{i=1}^n \mu_i$, where μ_i is the uniform distribution on $\{1, \dots, k_i\}$ $(i = 1, \dots, n)$. (Hence μ is the uniform distribution on Ω.) If A, B is a semi-disjoint pair of subsets of Ω, then*

$$\mu(A \square B) \leq \mu(A)\mu(B).$$

Remark. By applying Lemma 3.4 (in the same way as we did in the proof of Lemma 3.5), Lemma 6.4 can be extended to case (c) of Theorem 4.2. (It would even be sufficient to give a proof for the case in which each $k_i = 2$; however, our proof for general k_i is not more complicated.)

Proof of Lemma 6.4. The case $n = 1$ is trivial. We shall prove that if the result holds for $n - 1$ (where $n \geq 2$), then it also holds for n. The proof consists of five parts, (i)–(v). First, let Ω, A, and B be as in the conditions of the lemma. By "making A (B) decreasing (increasing) in the last coordinate" we will obtain $\tilde{A}$ and $\tilde{B}$ for which we prove in parts (i)–(iii) that: $\mu(\tilde{A}) = \mu(A)$, $\mu(\tilde{B}) = \mu(B)$, and $\mu(\tilde{A} \cap \tilde{B}) = \mu(A \cap B)$. Further, we show in part (iv) that the pair $\tilde{A}, \tilde{B}$ is also semi-disjoint, so that $\tilde{A} \cap \tilde{B} = \tilde{A} \square \tilde{B}$. Hence, it is sufficient to prove that $\mu(\tilde{A} \square \tilde{B}) \leq \mu(\tilde{A})\mu(\tilde{B})$. This will be done in part (v) by applying the induction hypothesis to $\Omega' = \prod_{i=1}^{n-1}\{1, \ldots, k_i\}$.

First some definitions:

(6.5) **Definition.**

$$\mathscr{B}^+ = \{C \in \mathrm{MR}(B) | n \in \mathrm{supp}(C)\},$$

$$\mathscr{B}^- = \{C \in \mathrm{MR}(B) | n \notin \mathrm{supp}(C)\},$$

$$B^+ = \cup \mathscr{B}^+, \qquad B^- = \cup \mathscr{B}^-.$$

$\mathscr{A}^+$, $\mathscr{A}^-$, A^+, and A^- are defined analogously.

(6.6) **Definition.** If $V \subset \{1, \ldots, k_n\}$, then $\vec{V} = \{l, l+1, \ldots, k_n\}$, where $l = k_n - |V| + 1$, and $\overleftarrow{V} = \{1, \ldots, |V|\}$.

(6.7) **Definition.**

(a) $\tilde{B}^+ = \bigcup_{\omega' \in \Omega'}\{[\omega' \overrightarrow{V(\omega')}]\}$, where $V(\omega') = \{j | (\omega' j) \in B^+\}$.
(b) $\tilde{A}^+ = \bigcup_{\omega' \in \Omega'}\{[\omega' \overleftarrow{W(\omega')}]\}$, where $W(\omega') = \{j | (\omega' j) \in A^+\}$.
(c) $\tilde{B} = B^- \cup \tilde{B}^+$.
(d) $\tilde{A} = A^- \cup \tilde{A}^+$.

We are now ready to start the real work.

(i) It is easy to see, by conditioning on $\omega_1, \ldots, \omega_{n-1}$, that, for all D for which $\mathrm{supp}(D) \subset \{1, \ldots, n-1\}$

$$\mu(D \cap B^+) = \mu(D \cap \tilde{B}^+) \quad \text{and} \quad \mu(D \cap A^+) = \mu(D \cap \tilde{A}^+).$$

(ii) Using (i), we get

$$\mu(B) = \mu(B^-) + \mu(B^+) - \mu(B^- \cap B^+)$$
$$= \mu(B^-) + \mu(\tilde{B}^+) - \mu(B^- \cap \tilde{B}^+) = \mu(\tilde{B}),$$

and analogously, $\mu(A) = \mu(\tilde{A})$.

(iii) Application of Lemma 6.3 yields $A^+ \cap B^+ = \varnothing$, and by conditioning on $\omega_1, \dots, \omega_{n-1}$, it follows that also $\tilde{A}^+ \cap \tilde{B}^+ = \varnothing$, so that

$$\begin{aligned}\mu(A \cap B) &= \mu((A^- \cup A^+) \cap (B^- \cup B^+)) \\ &= \mu(A^- \cap B^-) + \mu(A^- \cap B^+) + \mu(A^+ \cap B^-) \\ &\quad - \mu(A^- \cap B^- \cap B^+) - \mu(A^- \cap B^- \cap A^+).\end{aligned}$$

Now, by (i), we may, in the last expression, replace B^+ by $\tilde{B}^+$ and A^+ by $\tilde{A}^+$ and then, following the equations backwards, we get $\mu(A \cap B) = \mu(\tilde{A} \cap \tilde{B})$.

(iv) We shall now show that the pair $\tilde{A}, \tilde{B}$ is semi-disjoint. First of all, it is clear that $\mathscr{A}^- \cup \mathrm{MR}(\tilde{A}^+)$ and $\mathscr{B}^- \cup \mathrm{MR}(\tilde{B}^+)$ are representations of $\tilde{A}$ and $\tilde{B}$, respectively. It will turn out that this pair of representations is semi-disjoint. The pair $\mathrm{MR}(\tilde{A}^+), \mathrm{MR}(\tilde{B}^+)$ is obviously semi-disjoint, because, as we saw in (iii), $\tilde{A}^+ \cap \tilde{B}^+ = \varnothing$. By Lemma 6.3, the pair $\mathscr{A}^-, \mathscr{B}^-$ is also semi-disjoint. It remains to show that the pairs $\mathscr{A}^-, \mathrm{MR}(\tilde{B}^+)$ and $\mathscr{B}^-, \mathrm{MR}(\tilde{A}^+)$ are also semi-disjoint and, by symmetry, it is sufficient to treat the first pair. This will be done by using Lemma 6.2.

Suppose $\omega = (\omega_1, \dots, \omega_n) \in \tilde{B}^+$ and $[\omega]_K \in \mathscr{A}^-$. Let $K' = \{1, \dots, n-1\} \setminus K$. Define the map S: $\Omega \times \Omega \to \{1, \dots, k_n\}$ by

$$S(\eta, \gamma) = \{j | [\eta]_K \cap [\gamma]_{K'} \cap [* \; j] \subset B^+\}.$$

REMARK. Note that the intersection of the three cylinders in the above expression contains only one element.

Define $\tilde{S}(\eta, \gamma)$ analogously, replacing B^+ by $\tilde{B}^+$. We know, because the pair $\mathscr{A}^-, \mathscr{B}^+$ is semi-disjoint (Lemma 6.3), that for all $\eta \in \Omega$, $S(\omega, \omega) \subset S(\eta, \omega)$. Further, the definition of $\tilde{B}^+$ implies that

$$\forall\, \eta \in \Omega, \qquad \tilde{S}(\eta, \omega) = \overrightarrow{S(\eta, \omega)}.$$

So we get

$$\begin{aligned} &\forall\, \eta \in \Omega, && \tilde{S}(\omega, \omega) \subset \tilde{S}(\eta, \omega), \text{ and finally, because } \omega_n \in \tilde{S}(\omega, \omega), \\ &\forall\, \eta \in \Omega, && \omega_n \in \tilde{S}(\eta, \omega), \text{ so that} \\ &\forall\, \eta \in \Omega, && [\eta]_K \cap [\omega]_{K^c} \subset \tilde{B}^+, \\ & && \text{hence } [\omega]_{K^c} \subset \tilde{B}^+, \end{aligned}$$

and so, by Lemma 6.2, the pair $\mathscr{A}^-, \mathrm{MR}(\tilde{B}^+)$ is semi-disjoint.

(v) First define, for $D \subset \Omega$ and $1 \le i \le k_n$,

$$D^i = \{\omega' \in \Omega' | (\omega' i) \in D\}.$$

Before we apply the induction hypothesis, we also have to show that the pair $\tilde{A}^i, \tilde{B}^i$ is semi-disjoint ($i = 1, \dots, k_n$). This is easily seen by taking the representations

$$\mathscr{A}^i = \{C^i | C \in \mathrm{MR}(\tilde{A})\}, \qquad \mathscr{B}^i = \{C^i | C \in \mathrm{MR}(\tilde{B})\}.$$

The induction step is as follows. Let μ' denote $\mu|\Omega'$ (i.e., μ' is the uniform distribution on Ω'). Let $p = 1/k_n$. We have

$$\mu(A)\mu(B) - \mu(A \cap B) = \mu(\tilde{A})\mu(\tilde{B}) - \mu(\tilde{A} \cap \tilde{B})$$
$$= p\sum_{i=1}^{k_n} \mu'(\tilde{A}^i) p \sum_{j=1}^{k_n} \mu'(\tilde{B}^j) - p\sum_{i=1}^{k_n} \mu'\big((\tilde{A} \cap \tilde{B})^i\big).$$

The last summation equals

$$p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \mu'(\tilde{A}^i \cap \tilde{B}^i),$$

which, by the induction hypothesis, is at most

$$p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \mu'(\tilde{A}^i)\mu'(\tilde{B}^i).$$

So we have

$$\mu(A)\mu(B) - \mu(A \cap B) \geq p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \left[\mu'(\tilde{A}^i)\mu'(\tilde{B}^j) - \mu'(\tilde{A}^i)\mu'(\tilde{B}^i)\right]$$
$$= p^2 \sum_{i<j} \left[\mu'(\tilde{A}^i) - \mu'(\tilde{A}^j)\right]\left[\mu'(\tilde{B}^j) - \mu'(\tilde{B}^i)\right],$$

which is nonnegative because, for all i, j, $i < j$ implies $\tilde{A}^j \subset \tilde{A}^i$ and $\tilde{B}^i \subset \tilde{B}^j$. □

7. Proof of Theorem 4.2, case (d). Clearly, there are mutually disjoint $K_1, K_2, \ldots \subset \{1, \ldots, n\}$ such that the support of each maximal cylinder of A is one of the K_i's. By Lemma 3.12 [noting that each $A(\omega')$ falls again under case (d) of the theorem] we may assume that $\bigcup_i K_i = \{1, \ldots, n\}$ and by Lemma 3.10 that each K_i consists of consecutive numbers. Further, we can reduce the problem to the case that for suitable r_i, s_i

$$\Omega = \prod_{i=1}^{n} \{1, \ldots, r_i\}, \qquad A = \{\omega \in \Omega | \exists i\ \omega_i \leq s_i\}. \tag{7.1}$$

This can be seen as follows. For each i, let

$$\Omega_i = \left\{\omega' \in \prod_{j \in K_i} S_j \middle| [\ast\ \omega'\ \ast] \text{ is a maximal cylinder of } A\right\};$$

take $r_i = |\prod_{j \in K_i} S_j|$ and $s_i = |\Omega_i|$, i.e., the number of maximal cylinders of A which have support K_i. Obviously, there exists a 1–1 map f_i: $\{1, \ldots, r_i\} \to \prod_{j \in K_i} S_j$, which maps $\{1, \ldots, s_i\}$ onto Ω_i. Let $\nu_i = (\prod_{j \in K_i} \mu_j) \circ f_i$. Now define $f = \prod_i f_i$: $\prod_i \{1, \ldots, r_i\} \to \Omega$, and $\nu = \prod_i \nu_i$. Note that $f^{-1}(A) = \{\omega \in \prod_i \{1, \ldots, r_i\} | \exists j\ \omega_j \leq s_j\}$, and check that the properties (i) and (ii′) in Lemma 3.4 hold. Application of that lemma gives that we may indeed restrict ourselves to case (7.1).

Now we apply Lemma 3.4 once more to reduce 7.1 to the binary case. Let

$$\Omega' = \{0,1\}^n \times \prod_{i=1}^{n} \{1,\dots,s_i\} \times \prod_{i=1}^{n} \{s_i+1,\dots,r_i\}.$$

Define g: $\Omega' \to \Omega$ by

$$\begin{aligned} g(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n) \\ = \big(g_1(x_1,y_1,z_1), g_2(x_2,y_2,z_2),\dots,g_n(x_n,y_n,z_n)\big), \end{aligned}$$

where, for $1 \le i \le n$,

$$g_i(x_i,y_i,z_i) = \begin{cases} y_i, & \text{if } x_i = 0, \\ z_i, & \text{if } x_i = 1. \end{cases}$$

Also define, for $1 \le i \le n$, the probability measures μ'_{ix}, μ'_{iy}, and μ'_{iz} on $\{0,1\}$, $\{1,\dots,s_i\}$ and $\{s_i+1,\dots,r_i\}$, respectively, by

$$\begin{aligned} &\mu'_{ix}(0) = 1 - \mu'_{ix}(1) = \mu\{\omega_i \le s_i\}; \\ &\mu'_{iy}(j) = \mu\{\omega_i = j\}/\mu\{\omega_i \le s_i\}, \qquad j = 1,\dots,s_i; \\ &\mu'_{iz}(j) = \mu\{\omega_i = j\}/\mu\{\omega_i > s_i\}, \qquad j = s_i+1,\dots,r_i. \end{aligned}$$

Finally, define

$$\mu' = \prod_{i=1}^{n} \mu'_{ix} \times \prod_{i=1}^{n} \mu'_{iy} \times \prod_{i=1}^{n} \mu'_{iz}.$$

It is not difficult to see that the properties (i) and (ii) of Lemma 3.4 hold (with $f = g$ and $\nu = \mu'$) and that

$$g^{-1}(A) = \big\{(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n) \in \Omega | \exists i\, x_i = 0\big\}.$$

Application of this lemma and of Lemma 3.12 [note that $g^{-1}(A)$ does not depend on the y_i's and z_i's] reduces the problem to the case that

$$\Omega = \{0,1\}^n, \qquad A = \{\omega \in \Omega | \exists i\, \omega_i = 0\}, \tag{7.2}$$

and for this case we have the following direct proof.

If Ω and A are as in (7.2) then, for arbitrary $B \subset \Omega$,

$$\begin{aligned} &\omega \in A \square B \Leftrightarrow \omega \in B \quad \text{and} \\ &\exists i\ \omega_i = 0, (\omega_1,\dots,\omega_{i-1},1,\omega_{i+1},\dots,\omega_n) \in B. \end{aligned} \tag{7.3}$$

Call an element $\omega \in B$ *maximal* if there is no $\omega' \ne \omega$ in B with $\omega' \ge \omega$. Denote the set of maximal elements of B by $B_{\max}$. Obviously, by (7.3), $A \square B \subset B \setminus B_{\max}$, hence

$$\mu(A \square B) \le \mu(B) - \mu(B_{\max}). \tag{7.4}$$

Further, for each $\omega \in \Omega$

$$\begin{aligned} \mu(\{\omega' | \omega' \le \omega\}) &= \prod_{\omega_i = 0} \mu_i(\omega_i) = \mu(\omega) \Big/ \prod_{\omega_i = 1} \mu_i(\omega_i) \\ &\le \mu(\omega)/\mu(1,\dots,1) = \mu(\omega)/(1 - \mu(A)). \end{aligned} \tag{7.5}$$

Hence

$$
(7.6)\quad \begin{aligned} \mu(B) &= \mu\Big(\bigcup_{\omega \in B_{\max}} \{\omega' | \omega' \in B,\, \omega' \le \omega\}\Big) \le \sum_{\omega \in B_{\max}} \mu(\{\omega' | \omega' \in B,\, \omega' \le \omega\}) \\ &\le \sum_{\omega \in B_{\max}} \mu(\omega)/(1 - \mu(A)) = \mu(B_{\max})/(1 - \mu(A)), \end{aligned}
$$

so that

$$
(7.7)\qquad \mu(B_{\max}) \ge \mu(B)(1 - \mu(A)).
$$

Combining (7.4) and (7.7) we get

$$
(7.8)\qquad \mu(A \square B) \le \mu(B) - \mu(B)(1 - \mu(A)) = \mu(A)\mu(B). \quad \square
$$

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