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LIMITING DISTRIBUTIONS AND REGENERATION TIMES FOR MULTITYPE BRANCHING PROCESSES WITH IMMIGRATION IN A RANDOM ENVIRONMENT

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Sufficient conditions for the existence of a limiting distribution for a multitype branching process with immigration in a random environment, $Z(t)$, are given. In the case when the environment is an independent, identically distributed sequence, sufficient conditions are given which insure that the tail of the distribution of $v = \inf\{t > 0\colon Z(t) = 0\}$ decreases exponentially fast, and an application of this fact to random walk in a random environment is indicated.

1. Introduction and statement of main results. A population consisting of several types of individuals may evolve in many ways. One possibility is described by a multitype branching process with immigration in a random environment (MBPIRE).

To describe such a process, let t be an integer, denoting time, and let V be the nonnegative orthant of Z^d. Elements of V will be written in boldface.

A d-type branching process with immigration in a random environment is the superposition of a sequence of d-type branching processes in a random environment that begin at different times. D-type branching processes in a random environment are discussed by Athreya and Karlin (1971) and by Tanny (1981).

More specifically, let F be the set of $d + 1$ tuples of measures on V with finite mean. The random environment is a stationary, ergodic sequence $e = \{e(t, *)\colon t \in Z\}$ of F-valued random variables. We partition the vector $e(t, *)$ into the first d components, which we denote by $p(t, *)$ and the last coordinate, which we denote by $a(t, *)$. The mth coordinate of $p(t, *)$ gives the offspring distribution at time t for type-m parents. $a(t, *)$ gives the distribution at time t for the immigrants. For each integer k we define $Z(k, t)$ to be the branching process in a random environment which begins at time k. That is to say, conditional on e,

$$P[Z(k, t) = \mathbf{0}|e] = 1, \quad \text{if } t < k, \tag{1}$$

$$P[Z(k, k) = \mathbf{v}|e] = a(k, \mathbf{v}), \tag{2}$$

$$P[Z(k, t) = \mathbf{v}|Z(k, t-1) = \mathbf{b}(m), e] = p(t, \mathbf{v})_m, \quad \text{if } t > k, \tag{3}$$

where $\mathbf{b}(m)$ is the mth standard basis vector in V. In addition to the assumption that conditional on e each of the processes $Z(k, *)$ have independent lines of descent, we also assume that conditional on e, each of the processes $Z(k, *)$ are independent.

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The d-type branching process with immigration in a random environment starting at time 0, $Z(t)$, is the offspring born at time t to the immigrants who arrived between time 0 and $t - 1$ inclusive:

$$Z(0) = \mathbf{0}, \tag{4}$$

$$Z(t) = \sum_{k=0}^{t-1} Z(k, t), \qquad t > 0. \tag{5}$$

Kaplan (1973) investigated multitype branching processes with immigration, in the setting of time-homogeneous offspring distributions. He gave necessary and sufficient conditions for the existence of limiting distributions for these processes.

The existence of limiting distributions for MBPIRE can be addressed as well. Let $I(t) = E[Z(t, t)|e]$, and let $M(t)$ be the $d \times d$ matrix whose mth row is the expected number of offspring born to a type-m parent at time t, conditioned on e: $E[Z(t-1, t)|Z(t-1, t-1) = \mathbf{b}(m), e]$. $\{M(t)\colon t \in Z\}$ is a stationary, ergodic sequence.

Athreya and Karlin (1971) and Tanny (1981) use the maximal Lyapunov exponent of $\{M(t)\colon t \in Z\}$ to give extinction criteria for multitype branching processes in a random environment. For $x \in R^d$ let $|x| = \Sigma_{i=1}^d |x_i|$, and for any $d \times d$ matrix M let $|M|$ be the operator norm of M induced by $|x|$. We have from Section 3,

Theorem 3.3. *Suppose that $Z(t)$ is a MBPIRE with*

(i) $$E(\log^+|I(0)|) < \infty, \qquad E(\log^+|M(1)|) < \infty, \quad \text{and}$$

(ii) $$\lim_{k \to \infty} k^{-1} E(\log|M(1) \cdots M(k)|) < 0.$$

Then $\pi(\mathbf{v}) = \lim_{t \to \infty} P(Z(t) = \mathbf{v})$ exists and defines a probability distribution on V.

If in addition to (i) *and* (ii)

(iii) $$P(p(1, \mathbf{0})_i > 0, \text{ for } i = 1, 2, \ldots, d) > 0,$$

then $\pi(\mathbf{0}) > 0$.

Condition (iii) means that with positive probability no offspring will be produced by any of the individuals present so that the process starts over with the next wave of immigrants.

Kesten, Kozlov and Spitzer (1975) used a single-type branching process with immigration in a random environment, $B(t)$, to give limit laws for a random walk in a random environment, $Y(t)$, satisfying $P(|Y(t) - Y(t-1)| = 1) = 1$ and $P(\lim_{t \to \infty} Y(t) = \infty) = 1$. One main lemma is

(6) *If $v = \min\{t > 0\colon B(t) = 0\}$ then there are positive constants k_1 and k_2 such that $P(v > t) < k_1 \exp(-k_2 t)$.*

Key (1983) showed that if $P(|Y(t) - Y(t-1)| < 1) = 1$ and $P(\lim_{t \to \infty} Y(t) = \infty) = 1$, then Kesten, Kozlov and Spitzer's program can be carried out using a two-type MBPIRE. Since one of the two types never had offspring, (6) was still sufficient. To extend the analysis to the case where $P(-L \le Y(t) - Y(t-1) \le 1) = 1$ and $P(\lim_{t\to\infty} Y(t) = \infty) = 1$ one must prove the analog of (6) for a general $L+1$-type MBPIRE. We have, from Section 4,

THEOREM 4.2. *Suppose that $Z(t)$ is a MBPIRE with random environment $e = \{(p(t,*), a(t,*)): t \ge 0\}$. Let $v = \min\{t > 0: Z(t) = \mathbf{0}\}$. If*

(i) *$E(|I(0)|^q) < \infty$ and $E(|M(1)|^q) < \infty$ for some $q > 0$,*
(ii) *$\lim_{k\to\infty} k^{-1}E(\log|M(1)\cdots M(k)|) < 0$,*
(iii) *$P(p(1,\mathbf{0})_i > 0$ for $i = 1,2,\dots,d) > 0$,*
(iv) *e is an iid sequence,*
(v) *$p(t,*)$ and $a(t,*)$ are independent for each t,*

then there exist positive constants k_1 and k_2 such that

$$P(v > t) < k_1 \exp(-k_2 t).$$

2. Generating functions and an auxiliary process. Since $a(t,*)$ and the coordinates of $p(t,*)$ are measures on V given e, we may define

$$f_t\colon [0,1]^d \to [0,1]^d \quad \text{and} \quad g_t\colon [0,1]^d \to [0,1]$$

to be the (random) generating functions of $p(t,*)$ and $a(t,*)$, respectively. From the discussion of $Z(k,*)$ and $Z(*)$ in Section 1 and using the multinomial notation, $\mathbf{s}^{\mathbf{v}} = s_1^{v_1} \cdots s_d^{v_d}$ for $\mathbf{s} \in [0,1]^d$, we have

$$E\left[\mathbf{s}^{Z(k,k)}|e\right] = g_k(\mathbf{s}), \tag{7}$$

$$E\left[\mathbf{s}^{Z(k,t)}|e\right] g_k \circ f_{k+1} \circ \cdots \circ f_t(\mathbf{s}), \qquad t > k, \tag{8}$$

and from (5) it follows that

$$\begin{aligned} E\left[\mathbf{s}^{Z(t)}|e\right] &= \sum_{k=0}^{t-1} E\left[\mathbf{s}^{Z(k,t)}|e\right] \\ &= \sum_{k=0}^{t-1} g_k \circ f_{k+1} \circ \cdots \circ f_t(\mathbf{s}), \qquad t > 0. \end{aligned} \tag{9}$$

For $t > 0$ we define the auxiliary process $Z'(t)$:

$$Z'(t) = \sum_{k=0}^{t-1} Z(-k,0). \tag{10}$$

Since $Z(k,t) \in V$ we have

$$Z'(t+1)_i \ge Z'(t)_i, \quad \text{for } i = 1,2,\dots,d. \tag{11}$$

(8), (10) and the conditional independence of the $Z(k,*)$ imply that

$$E\left[\mathbf{s}^{Z'(t)}|e\right] = \sum_{k=1}^{t} g_{-k} \circ f_{-k+1} \circ \cdots \circ f_0(\mathbf{s}), \qquad t > 0. \tag{12}$$

LEMMA 2.1. *For each $t > 0$, $Z(t)$ and $Z'(t)$ have the same distribution.*

PROOF. Use (9), (12) and that e is stationary to show that $Z(t)$ and $Z'(t)$ have the same generating function. □

LEMMA 2.2. *Let $\mathbf{v} \in V$. Then $\lim_{t\to\infty} P[Z'(t) = \mathbf{v}|e]$ exists almost surely.*

PROOF. For $\mathbf{x}$ and $\mathbf{v}$ in V let $x \le v$ mean $x_i \le v_i$ for $i = 1, \dots, d$. (11) implies that for any $x \in V$,

$$P[Z'(t+1) \le \mathbf{x}|e] \le P[Z'(t) \le \mathbf{x}|e] \quad \text{almost surely.}$$

Therefore, for any $x \in V$,

$$\lim_{t\to\infty} P[Z'(t) \le \mathbf{x}|e] \quad \text{exists almost surely.}$$

Since $P[Z'(t) = \mathbf{v}|e]$ is determined by a linear combination of terms of the form $P[Z'(t) \le x|e]$, and the number of terms and the coefficients depend only on d,

$$\lim_{t\to\infty} P[Z'(t) = \mathbf{v}|e]$$

exists almost surely. □

3. Moment estimates and products of random matrices. Suppose that $\{A(j)\colon j \in Z\}$ is a stationary, ergodic sequence of $d \times d$ matrix-valued random variables with $E(\log^+|A(0)|) < \infty$. Then it is a simple consequence of Kingman's subadditive ergodic theorem that there exists $a \in [0, \infty)$ such that

$$\lim_{n\to\infty} n^{-1}E\big(\log|A(1)\cdots A(n)|\big) = \lim_{n\to\infty} n^{-1}E\big(\log|A(-n)\cdots A(-1)|\big) = \log(a),$$

$$\lim_{n\to\infty} n^{-1}\log|A(a)\cdots A(n)| = \log(a) \quad \text{a.s.,} \tag{13}$$

$$\lim_{n\to\infty} n^{-1}\log|A(-n)A(-1)| = \log(a) \quad \text{a.s.} \tag{14}$$

$\log(a)$ is called the maximal Lyapunov exponent of the sequence $\{A(j)\colon j \in Z\}$. If $\{A(j)\colon j \in Z\}$ is a constant sequence then a is the spectral radius of $A(0)$.

The chain rule and (8) imply that for $t > k$, and $1 = b(1) + \cdots + b(d)$,

$$\begin{aligned} E[Z(k,t)|e] &= D(g_k \circ f_{k+1} \circ \cdots \circ f_t(\mathbf{s}))|_{\mathbf{s}=\mathbf{1}} \\ &= Dg_k(\mathbf{1}) \cdot Df_{k+1}(\mathbf{1}) \cdots D\mathbf{f}_t(1) \\ &= I(k)M(k+1)\cdots M(t). \end{aligned} \tag{15}$$

LEMMA 3.1. *Suppose that $E(\log^+|I(0)|) < \infty$, $E(\log^+|M(0)|) < \infty$ and $\lim_{n\to\infty} n^{-1}\log|M(-n)\cdots M(0)| = \log m < 0$. Then there exists a random variable $K = K(e) < \infty$ a.s. such that for all $t > 0$, $E[|Z'(t)||e] \le K$.*

PROOF. By (10),

$$\begin{aligned} E\left[|Z'(t)||e\right] &= \sum_{k=1}^{t} E\left[|Z(-k,0)||e\right] \quad \text{(since } |v| \text{ is linear in } v\text{)} \\ &= \sum_{k=1}^{t} |I(-k)M(-k+1)\cdots M(0)| \\ &\le \sum_{k=1}^{\infty} |I(-k)|\cdot|M(-k+1)\cdots M(0)|. \end{aligned}$$

By hypothesis, $\lim_{n\to\infty} n^{-1}\log|M(-n)\cdots M(0)| = \log m < 0$, and by the ergodic theorem $\lim_{n\to\infty} n^{-1}\log|I(-n)| \le 0$. Therefore,

$$\sum_{k=1}^{\infty} |I(-k)|\cdot|M(-k+1)\cdots M(0)|$$

is convergent almost surely by the root test, which proves the lemma. □

LEMMA 3.2. *For each* $\mathbf{v}$ *in* V, $\lim_{t\to\infty} P(Z'(t) = \mathbf{v}) = \pi(\mathbf{v})$ *exists, and under the hypotheses of Lemma* 3.1, $\pi(\mathbf{v})$ *is a probability measure on* V.

PROOF. Using the dominated convergence theorem we have

$$\begin{aligned} \lim_{t\to\infty} P(Z'(t) = \mathbf{v}) &= \lim_{t\to\infty} E(P[Z'(t) = \mathbf{v}|e]) \\ &= E\Big(\lim_{t\to\infty} P[Z'(t) = \mathbf{v}|e]\Big), \end{aligned}$$

so $\pi(v)$ exists by Lemma 2.2.

By Lemma 3.1 and Chung [(1974), Problem 6, page 94], the probability measures $P[Z'(t) = *|e]$, $t > 0$, on V are tight a.s., so

$$\sum_{\mathbf{v}\in V} \lim_{t->\infty} P[Z'(t) = \mathbf{v}|e] = 1 \quad \text{a.s.} \tag{16}$$

Therefore, $\sum_{\mathbf{v}\in V}\pi(\mathbf{v}) = 1$ as well. □

THEOREM 3.3. *Suppose that* $Z(t)$ *is a MBPIRE with*

(i) $E(\log^+|I(0)|) < \infty, \quad E(\log^+|M(1)|) < \infty,$ *and*

(ii) $\lim_{k\to\infty} k^{-1}E(\log|M(1)\cdots M(k)|) < 0.$

Then $\lim_{t\to\infty} P(Z(t) = \mathbf{v}) = \pi(\mathbf{v})$, *a probability distribution on* V.

If in addition to (i) *and* (ii)

(iii) $P(p(1,\mathbf{0})_i > 0 \text{ for } i = 1,2,\ldots,d) > 0,$

then $\pi(\mathbf{0}) > 0$.

PROOF. $P(Z(t) = v) = P(Z'(t) = v)$ by Lemma 2.1, so the first assertion follows from Lemma 3.2.

To show that $\pi(\mathbf{0}) > 0$ we proceed by contradiction. If $\pi(\mathbf{0}) = 0$, then

$$\lim_{t\to\infty} P[Z(t) = \mathbf{0}|e] = 0 \quad \text{a.s.} \tag{17}$$

For each $\mathbf{v}$ in V let

$$\begin{aligned} q(t, \mathbf{v}) &= \sum_{x\in V} p(t+1, \mathbf{0})^{\mathbf{x}+\mathbf{v}} a(t, \mathbf{x}) \\ &= P[Z(t+1) = \mathbf{0}|Z(t) = \mathbf{v}, e], \quad \text{if } t > 0. \end{aligned}$$

Then for each $\mathbf{v}$ in V,

$$P[Z(t+1) = \mathbf{0}|e] > P[Z(t) = \mathbf{v}|e] \cdot q(t, \mathbf{v}) \tag{18}$$

Taken together, (17) and (18) imply that for all v in V,

$$\lim_{t\to\infty} P[Z(t) = \mathbf{v}|e] \cdot q(t, \mathbf{v}) = 0 \quad \text{a.s.} \tag{19}$$

Since e is a stationary sequence, (9) and (12) show that for each $t > 0$ and each v in V, $P[Z(t) = \mathbf{v}|e] \cdot q(t, \mathbf{v})$ and $P[Z'(t) = \mathbf{v}|e] \cdot q(0, \mathbf{v})$ have the same distribution.

Lemma 2.2 shows that $\lim_{t\to\infty} P[Z'(t) = \mathbf{v}|e] \cdot q(0, \mathbf{v})$ exists almost surely for each $\mathbf{v}$ and V, and (19) implies that this limit must be 0. Therefore,

$$\lim_{t\to\infty} P[Z'(t) = \mathbf{v}|e] = 0 \quad \text{on } \{q(0, \mathbf{v}) > 0\}.$$

On the other hand, $\{p(1, \mathbf{0})_i > 0,\ i = 1, 2, \dots, d\} \subset \{q(0, \mathbf{v}) > 0\}$ for each $\mathbf{v}$ in V, so (iii) implies that

$$\sum_{\mathbf{v}\in V} \lim_{t\to\infty} P[Z'(t) = \mathbf{v}|e] = 0$$

on a set of positive measure, which contradicts (16). □

REMARK. Hypothesis (iii) may be replaced by other hypotheses and the second conclusion of the theorem will still hold. For example, suppose that there are only two types, and each type-1 particle only gives birth to type-2 particles, and all of the immigrants are type-2 particles. Then (iii) may be replaced by

$$P(p(1, \mathbf{0})_2 p(2, \mathbf{0})_2 > 0) > 0. \tag{iii$'$}$$

This type of behavior might be observed in a stratified population model.

4. Regeneration times. Kesten, Kozlov and Spitzer (1975) and Key (1983) used branching processes with immigration in a random environment to find limiting distributions for random walk in a random environment. (See Key (1984) for the definition of random walk in a random environment.) If $\{X(t): t \geq 0\}$ denotes a random walk in a random environment satisfying

$$\lim_{t\to\infty} X(t) = \infty \quad \text{a.s.}, \qquad |X(t+1) - X(t)| \leq 1 \quad \text{a.s.},$$

we found limiting distributions for $\{X(t): t \geq 0\}$ by finding limiting distributions for $T(n) = \inf\{t: X(t) = n\}$, as $n \to \infty$.

To find a limiting distribution for $T(n)$ we constructed a branching process with immigration in a random environment with nonrandom immigration, $Z(t)$, and examined the limiting behavior of

$$\sum_{t=0}^{n} f(Z(t)) \tag{20}$$

for certain linear functionals f. $Z(t)$ and f were chosen in such a way that (20) and $T(n)$ had the same limiting distributions.

The key to finding a limiting distribution for (20) is to use the sequence of regeneration times $v(n)$ defined by

$$v(0) = 0, \qquad v(n+1) = \inf\{t > v(n)\colon Z(t) = \mathbf{0}\},$$

to replace (20) with

$$\sum_{t=0}^{v(n)} f(Z(t)) = \sum_{j=1}^{n} \sum_{t=v(j-1)}^{v(j)} f(Z(t)).$$

If the environment for $X(t)$ is an iid sequence, then so is the environment for $Z(t)$, which makes

$$\sum_{t=v(j-1)}^{v(j)} f(Z(t)), \qquad j = 1, 2, \ldots,$$

an iid sequence. One key tool in studying this sequence is the estimate

$$P(v(1) > t) < k_1 \exp(-k_2 t) \tag{21}$$

for some k_1 and $k_2 > 0$. Kesten, Kozlov and Spitzer (1975) obtained this estimate in the case where $Z(t)$ is a single-type branching process.

If the hypothesis $|X(t+1) - X(t)| \leq 1$ a.s. is weakened to $-L \leq X(t+1) - X(t) \leq 1$ a.s., the general argument for finding limiting distributions for $X(t)$ seems to go through, except now $Z(t)$ is multitype. The only part that seems not to be a line by line rewriting of Kesten, Kozlov and Spitzer's argument is the proof of (21).

LEMMA 4.1. *Let $\{A(j)\colon j \geq 1\}$ be an iid sequence of $d \times d$ matrix-valued random variables with*

(i) $$E\big(|A(1)|^q\big) < \infty \quad \text{for some } q \geq 0,$$

(ii) $$\lim_{n\to\infty} n^{-1} E\big(\log|A(1) \cdots A(n)|\big) = \log a < 0.$$

Then for some h in $(0, q]$ there are positive constants C and w with $w < 1$ such that

$$E\big(|A(1) \cdots A(t)|^h\big) \leq C w^t.$$

PROOF. By (ii), there exists an integer k such that

$$c = E\big(\log|A(1) \cdots A(k)|\big) < 0.$$

For this value of k

$$\lim_{h\to 0} E\left(|A(1)\cdots A(k)|^h\right)^{1/h} = \exp(c) < 1.$$

Therefore there exists $h > 0$ such that

$$E\left(|A(1)\cdots A(k)|^h\right) = u < 1.$$

Since the $A(j)$ are independent we have for all $n > 0$

$$E\left(|A(1)\cdots A(n)|^h\right) \leq Cw^n,$$

where $w = u^{1/k}$. Since $w < 1$, the lemma is proved. □

THEOREM 4.2. *Suppose that $Z(t)$ is a MBPIRE with random environment $e = \{(p(t,*), a(t,*)) \colon t \geq 0\}$. Let $v = \min\{t > 0 \colon Z(t) = \mathbf{0}\}$. If*

(i) $E(|I(0)|^q) < \infty$ *and* $E(|M(1)|^q) < \infty$ *for some* $q > 0$,
(ii) $\lim_{k\to\infty} k^{-1}E(\log|M(1)\cdots M(k)|) < 0$.
(iii) $P(p(1,\mathbf{0})_i > 0, i = 1,2,\ldots,d) > 0$,
(iv) *e is an iid sequence,*
(v) *$p(t,*)$ and $a(t,*)$ are independent for each t,*

then there exist positive constants k_1 and k_2 such that

$$P(v > t) < k_1 \exp(-k_2 t).$$

REMARK. The version of this proof for a single-type MBPIRE with nonrandom immigration was given by Kozlov, but never published.

PROOF. It is sufficient to prove that the power series

$$F(z) = \sum_{t=0}^{\infty} P(v(1) > t) z^t$$

has a radius of convergence which exceeds 1. Let $v = v(1)$. Let

$$v(t) = P(v > t),$$
$$h(r,t) = P\left(Z(r,t) \neq \mathbf{0}, \sum_{j=r+1}^{t-1} Z(j,t) = 0\right),$$
$$g(r,t) = P\left(\sum_{j=r}^{t-1} Z(j,t) = \mathbf{0}\right).$$

Then

$$\begin{aligned} v(t) &= P(v > t, Z(t) \neq \mathbf{0}) \\ &= P\left(v > t, \sum_{k=0}^{t-1} Z(k,t) \neq \mathbf{0}\right) \\ &= \sum_{k=0}^{t-2} P\left(v > t, Z(k,t) \neq \mathbf{0}, \sum_{j=k+1}^{t-1} Z(j,t) = \mathbf{0}\right) \\ &\quad + P(v > t, Z(t-1,t) \neq \mathbf{0}). \end{aligned} \tag{22}$$

Next,

$$(23)\quad \begin{aligned} P\left(v > t, Z(k,t) \neq \mathbf{0}, \sum_{j=k+1}^{t-1} Z(j,t) = \mathbf{0}\right) &= P\left(v > k, Z(k,t) \neq \mathbf{0}, \sum_{j=k+1}^{t-1} Z(j,t) = \mathbf{0}\right) \\ &= P(v > k) P\left(Z(k,t) \neq \mathbf{0}, \sum_{j=k+1}^{t-1} Z(j,t) = \mathbf{0}\right) \\ &\qquad \text{(by the independence hypotheses on the environment)} \\ &= v(k) h(k,t). \end{aligned}$$

Since e is a stationary sequence,

$$(24)\qquad g(k,t) = P(Z(t-k) = \mathbf{0}).$$

Let $g(t) = P(Z(t) = \mathbf{0})$. It follows from (24) that

$$(25)\qquad g(k,t) = g(t-k).$$

Next, let $h(t) = g(t) - g(t-1)$. Since $h(k,t) + g(t,t) = g(k+1,t)$, (25) implies that

$$(26)\qquad h(k,t) = h(t-k).$$

Substituting (23) into (22) and then using (26) gives

$$v(t) = \sum_{k=0}^{t-1} v(k) h(t-k).$$

In addition, we have $v(0) = 1$, $h(k) > 0$ for all $k > 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} h(k) &= \lim_{n \to \infty} 1 - g(n) \\ &= \lim_{n \to \infty} 1 - P(Z(t) = \mathbf{0}) \\ &< 1 \quad \text{(by Theorem 3.3)}. \end{aligned}$$

Therefore, $\{v(k)\colon k = 0, 1, 2, \dots\}$ is a renewal sequence, and standard renewal theoretic arguments show that if

$$V(z) = \sum_{t=0}^{\infty} v(t) z^t$$

and

$$H(z) = \sum_{t=1}^{\infty} h(t) z^t,$$

then

$$V(z) = (1 - H(z))^{-1}.$$

Since $H(1) < 1$, $F(z)$ will have a radius of convergence larger than 1 if $H(z)$ has a radius of convergence larger than 1. That the radius of convergence of $H(z)$ is larger than 1 follows from this estimate of $h(t)$, with h, C, and w as in Lemma 4.1:

$$\begin{aligned}
h(t) &= h(0,t)\\
&< P(Z(0,t) \neq \mathbf{0})\\
&= E(P[Z(0,t) \neq \mathbf{0}|e])\\
&= E\left(E\left[|Z(0,t)|^h|e\right]\right)\\
&\qquad \left(\text{since } |v| \text{ is a linear function of } v \text{ and } x^h > 1 \text{ for } x > 1\right)\\
&< E\left(|I(0)|^h|M(1)\cdots M(t)|^h\right)\\
&< CE\left(|I(0)|^h\right)w^t \quad (\text{by Lemma 4.1}).
\end{aligned}$$

Since $w < 1$ the radius of convergence of $H(z)$ is greater than 1. □

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