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A SHARP AND STRICT L^p-INEQUALITY FOR STOCHASTIC INTEGRALS[1]

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A new proof of a sharp L^p-inequality for stochastic integrals is given that makes it possible to show that strict inequality holds in all nontrivial cases.

1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $(\mathscr{F}_t)_{t \geq 0}$ a nondecreasing right-continuous family of sub-σ-fields of $\mathscr{F}$ where $\mathscr{F}_0$ contains all $A \in \mathscr{F}$ with $P(A) = 0$. Suppose that $M = (M_t)_{t \geq 0}$ is a real martingale adapted to $(\mathscr{F}_t)_{t \geq 0}$ such that almost all of the paths of M are right-continuous on $[0, \infty)$ and have left limits on $(0, \infty)$. Let $V = (V_t)_{t \geq 0}$ be a predictable process with values in $[-1, 1]$ and denote by $N = V \cdot M$ the stochastic integral of V with respect to M: N is an adapted right-continuous process with left limits on $(0, \infty)$ such that

$$N_t = \int_{[0,\, t]} V_s \, dM_s \quad \text{a.s.}$$

For background and the basic results that we take for granted here, see [3] and [4].

Let p^* be the maximum of p and q where $1 < p < \infty$ and $1/p + 1/q = 1$. Set $\|M\|_p = \sup_t \|M_t\|_p$. Then [1],

$$\|N\|_p \leq (p^* - 1)\|M\|_p \tag{1}$$

and $p^* - 1$ is the best constant. However, our original proof of (1) has the disadvantage of not preserving the strict inequality of the discrete-time version (Theorem 1.1 of [1]) in the transition, via approximation, to the continuous-time case. Therefore, the following theorem and its proof give additional information and insight.

THEOREM 1. *If* $p \neq 2$ *and* $0 < \|M\|_p < \infty$, *then*

$$\|N\|_p < (p^* - 1)\|M\|_p. \tag{2}$$

For example, if $p \neq 2$ and $\|M\|_p = 1$, then

$$\left\| \int_{[0,\, \infty)} V_t \, dM_t \right\|_p < p^* - 1. \tag{3}$$

Here the integral denotes N_∞, the almost sure pointwise limit of N. It is also the limit in L^p of N, hence the left-hand side of (3) is equal to $\|N\|_p$.

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2. The inequality without strictness. To prepare for the proof of the theorem, we shall give a new proof of (1). Let $1 < p < \infty$ and $\|M\|_p < \infty$. Denote by $Z = (X, Y)$ the stochastic integral with values in $\mathbb{R}^2$ where

$$X = N + M = (V + 1) \cdot M \tag{4}$$

and

$$Y = N - M = (V - 1) \cdot M. \tag{5}$$

Define $v\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$v(x, y) = \left|\frac{x + y}{2}\right|^p - (p^* - 1)^p\left|\frac{x - y}{2}\right|^p.$$

Since $N = (X + Y)/2$ and $M = (X - Y)/2$, we have that

$$Ev(Z_t) = \|N_t\|_p^p - (p^* - 1)^p\|M_t\|_p^p.$$

Consequently, if

$$Ev(Z_t) \leq 0 \tag{6}$$

for all $t \geq 0$, then (1) holds.

Instead of proving (6) directly, we shall prove an analogous inequality for a majorant u of v (see [2]) with the following key property: If $x, y, h, k \in \mathbb{R}$ and $hk \leq 0$, then the mapping

$$s \to u(x + hs, y + ks)$$

is concave on $\mathbb{R}$. The function $u\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the symmetry condition

$$u(x, y) = u(y, x) = u(-x, -y),$$

so it is enough to recall its definition on the set where $|y| < x$: If

$$w(x, y) = \alpha_p x^p\left[1 - \frac{p^*(x - y)}{2x}\right],$$

where $\alpha_p = p[p^*/(p^* - 1)]^{1-p}$, then, for $1 < p \leq 2$,

$$\begin{aligned} u(x, y) &= v(x, y) \quad \text{if } (1 - 2/p^*)x < y < x,\\ &= w(x, y) \quad \text{if } -x < y < (1 - 2/p^*)x. \end{aligned}$$

For $p > 2$,

$$\begin{aligned} u(x, y) &= w(x, y) \quad \text{if } (1 - 2/p^*)x < y < x,\\ &= v(x, y) \quad \text{if } -x < y < (1 - 2/p^*)x. \end{aligned}$$

The final step in the proof of (1) is to show that

$$Eu(Z_t) \leq 0. \tag{7}$$

Although this follows from the discrete case, proved in [2], it may be instructive to give a direct proof here. We shall do this in Section 4 using Itô's formula.

3. Strictness of the inequality.

PROOF OF THEOREM 1. Because $\|M\|_p$ is finite, the almost sure limit M_∞ exists and satisfies $\|M_\infty\|_p = \|M\|_p$. By (1), a similar statement holds for N_∞, hence also for X_∞ and Y_∞. It is clear from the definition of u that $|u(x, y)|$ is majorized by a constant multiple of $|x|^p + |y|^p$, so

$$|u(Z_t)| \le c_p(|X_t|^p + |Y_t|^p) \le c_p[(X^*)^p + (Y^*)^p], \tag{8}$$

where $X^* = \sup_t|X_t|$. Using (7), Doob's L^p-inequality for the maximal function of a martingale, and the dominated convergence theorem, we see that $Z_\infty = (X_\infty, Y_\infty)$ satisfies

$$Eu(Z_\infty) \le 0. \tag{9}$$

(i) Consider the case $p > 2$. Then, in addition to (9), we have that

$$EX_\infty Y_\infty \le 0. \tag{10}$$

This follows at once from

$$EX_\infty Y_\infty = EN_\infty^2 - EM_\infty^2$$
$$= E\int_{[0,\infty)} (V_t^2 - 1)\,d[M, M]_t.$$

Here the integrand is nonpositive and $[M, M]$ is the nondecreasing quadratic-variation process. From the assumption on $\|M\|_p$ in the statement of the theorem it follows that $\|X_\infty - Y_\infty\|_p > 0$. So $P(X_\infty \ne Y_\infty) > 0$ and, by (10),

$$P(X_\infty Y_\infty \le 0, (X_\infty, Y_\infty) \ne (0,0)) > 0,$$

for otherwise $EX_\infty Y_\infty$ would be strictly positive. Here $p^* > 2$ and it is easy to check that if $xy \le 0$ and $(x, y) \ne (0,0)$, then $v(x, y) < 0$. Therefore,

$$P(v(Z_\infty) < 0) > 0. \tag{11}$$

It is also easy to check (see [2]) that

$$v(x, y) > 0 \Leftrightarrow v(x, y) < u(x, y). \tag{12}$$

We can now prove that $\|N_\infty\|_p^p - (p^* - 1)^p\|M_\infty\|_p^p$ is strictly negative by showing its equivalent:

$$Ev(Z_\infty) < 0. \tag{13}$$

This will give (2) in the case $p > 2$.

By (9) and the fact that u majorizes v, the implication (12) gives (13) if $P(v(Z_\infty) > 0) > 0$. On the other hand, if $P(v(Z_\infty) \le 0) = 1$, then (13) follows from (11). This completes the proof of the theorem in the case $p > 2$.

(ii) Now suppose that $1 < p < 2$ and, with no loss, that $\|N_\infty\|_p > 0$. Let

$$M'_\infty = (\operatorname{sgn} N_\infty)|N_\infty|^{p-1}/\|N_\infty\|_p^{p-1}.$$

Then $\|N_\infty\|_p = EN_\infty M'_\infty$ and $\|M'_\infty\|_q = 1$. Let M' be a right-continuous martingale with left limits satisfying

$$M'_t = E(M'_\infty | \mathscr{F}_t) \quad \text{a.s.}$$

for all $t \geq 0$. Let $N' = V \cdot M'$. Then

$$EN_\infty M'_\infty = EM_\infty N'_\infty,$$

since each side is equal to

$$E\int_{[0,\infty)} V_t d[M, M']_t.$$

Therefore, by (i),

$$\begin{aligned} \|N_\infty\|_p &\leq \|M_\infty\|_p \|N'_\infty\|_q \\ &< (q-1)\|M_\infty\|_p \|M'_\infty\|_q \\ &= (p^* - 1)\|M_\infty\|_p. \end{aligned}$$

This completes the proof of the theorem.

4. A supermartingale. We shall now prove (7) using Itô's formula applied to a smooth approximation of u. For each positive integer n, let g^n be the Gaussian density on $\mathbb{R}^2$ defined by

$$g^n(x, y) = n\exp[-n\pi(x^2 + y^2)].$$

Let u^n denote the convolution of u with g^n. Then u^n is infinitely differentiable and $u^n \to u$ pointwise as $n \to \infty$. Denote its derivatives by $u^n_x, u^n_y, \ldots$. Then

$$|u^n(x, y)| \leq c_p(|x|^p + |y|^p) + c_p, \tag{14}$$

$$|u^n_x(x, y)| \leq c_p(|x|^{p-1} + |y|^{p-1}) + c_p, \tag{15}$$

with a similar bound on u^n_y, where the symbol c_p denotes a positive real number but not necessarily the same number from one use to the next. It is important to note, however, that c_p can be chosen to be independent of n. Furthermore, if $x, y, h, k \in \mathbb{R}$ and $hk \leq 0$, then the mapping

$$s \to u^n(x + hs, y + ks)$$

is concave on $\mathbb{R}$, implying that

$$u^n(x + h, y + k) \leq u^n(x, y) + u^n_x(x, y)h + u^n_y(x, y)k \tag{16}$$

and

$$u^n_{xx}(x, y)h^2 + 2u^n_{xy}(x, y)hk + u^n_{yy}(x, y)k^2 \leq 0. \tag{17}$$

These properties of u^n follow easily from the properties of u that are proved in [2].

By Itô's formula as extended by Kunita and Watanabe [5] and Meyer [6] (see, in particular, the recent treatment in [3]),

$$u^n(Z_t) = u^n(Z_0) + I_t + J_t + \tfrac{1}{2}Q_t + S_t, \tag{18}$$

where

$$I_t = \int_{(0,t]} u_x^n(Z_{s-})(V_s + 1)\,dM_s,$$

$$J_t = \int_{(0,t]} u_y^n(Z_{s-})(V_s - 1)\,dM_s,$$

$$Q_t = \int_{(0,t]} \Big[u_{xx}^n(Z_{s-})(V_s + 1)^2 + 2u_{xy}^n(Z_{s-})(V_s^2 - 1) + u_{yy}^n(Z_{s-})(V_s - 1)^2\Big]\,d[M^c, M^c]_s,$$

and

$$S_t = \sum_{0<s\le t} \Big[u^n(Z_s) - u^n(Z_{s-}) - u_x^n(Z_{s-})(V_s + 1)\,\Delta M_s - u_y^n(Z_{s-})(V_s - 1)\,\Delta M_s\Big].$$

In this formula, M^c denotes the continuous part of the martingale M, and $\Delta M_s = M_s - M_{s-}$.

The product of $(V_s + 1)\,\Delta M_s$ and $(V_s - 1)\,\Delta M_s$ is nonpositive so, by (16), we have that $S_t \le 0$. By (17), the integrand of Q_t is nonpositive. Thus, Q_t is also nonpositive. Now consider I_t. By (15),

$$|u_x^n(Z_{s-})| \le c_p\big[(X^*)^{p-1} + (Y^*)^{p-1}\big] + c_p.$$

Let U^* denote the right-hand side. Then

$$E|I_t| \le cE\left[\int_{(0,t]} [u_x^n(Z_{s-})(V_s + 1)]^2 d[M, M]_s\right]^{1/2} \le 2cEU^*[M, M]_\infty^{1/2}.$$

The square-function inequality for L^p-bounded martingales implies that $[M, M]_\infty^{1/2} \in L^p$. Since

$$[X, X]_\infty = \int_{[0,\infty)} (V_s + 1)^2 d[M, M]_s \le 4[M, M]_\infty,$$

the square-function inequality implies also that $X^* \in L^p$. Similarly, $Y^* \in L^p$ so $U^* \in L^q$. Therefore, by Hölder's inequality, $E|I_t|$ is finite and we have that $(I_t)_{t\ge 0}$ is a martingale starting at 0. Accordingly, $EI_t = 0$ with a similar result for J_t and $Eu^n(Z_t) \le Eu^n(Z_0)$. In view of (14) and the analog of (8), we obtain

$$Eu(Z_t) \le Eu(Z_0).$$

Since $u(x, y) \le 0$ if $xy \le 0$ and $X_0 Y_0 = (V_0^2 - 1)M_0^2 \le 0$, we have that $u(Z_0)$ is nonpositive, so (7) holds.

REMARKS. It is clear from (18) and the fact that both Q_t and S_t are nonincreasing in t that $(u^n(Z_t))_{t \geq 0}$ is a supermartingale. This implies that $(u(Z_t))_{t \geq 0}$ is a supermartingale.

If u: $\mathbb{R}^2 \to \mathbb{R}$ is any function such that the mapping

$$s \to u(x + hs, y + ks) \tag{19}$$

is concave on $\mathbb{R}$ for all $x, y, h, k \in \mathbb{R}$ with $hk \leq 0$, then $(u(Z_t))_{t \geq 0}$ is either a supermartingale or a local supermartingale under a variety of conditions on M, with Z being defined by (4) and (5) as above. Such is the case, for example, if M is bounded or has continuous paths. If V has its values in $\{-1, 1\}$, then it is enough to assume that, for $hk = 0$, the mapping (19) is concave. Thus, for this special class of predictable processes V, it suffices to have u biconcave.

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