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THE CENTRAL LIMIT THEOREM FOR STOCHASTIC PROCESSES

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If $f = \{f_t | t \in T\}$ is a centered, second-order stochastic process with bounded sample paths, it is then known that f satisfies the central limit theorem in the topology of uniform convergence if and only if the intrinsic metric ρ_f^2 (on T) induced by f is totally bounded and the normalized sums are eventually uniformly ρ_f^2-equicontinuous. We show that a centered, second-order stochastic process satisfies the central limit theorem in the topology of uniform convergence if and only if it has bounded sample paths and there exists totally bounded pseudometric ρ on T so that the normalized sums are eventually uniformly ρ-equicontinuous.

1. Introduction. It is well known that if we consider a stochastic process as a function from a probability space into a space of functions L, $L \subseteq \mathbb{R}^T$, then this function, in general, will be nonmeasurable and nonseparably valued in the topology of uniform convergence. One case where this happens is the empirical distribution function, which can be considered as a sum of "independent and identically distributed" stochastic processes. It is well known that the normalized empirical distribution function converges in law to a Brownian Bridge and this is an example of a central limit theorem in which we do not assume either measurability or separability.

Recently Dudley and Philipp [3] have generalized this idea for nonmeasurable and nonseparably valued random elements and proved the central limit theorem. In their paper they have shown that total boundedness of the intrinsic metric, induced by the empirical process, and equicontinuity in the limit of the normalized sums with respect to the intrinsic metric is a sufficient and necessary condition for the central limit theorem. This result has been used by Giné and Zinn [6] to obtain other sufficient conditions for the central limit theorem under certain measurability conditions.

In the present paper we shall use the setting of stochastic processes and we shall use some results from Hoffmann-Jørgensen [8]. In Chapter 7 of [8] the author has developed a theory of convergence for nonmeasurable and nonseparably valued functions, which extends the usual notion of convergence. Since the book is not published yet, we shall prove all the results we need from this book.

The results in [8] have been used by Andersen in [1] where the central limit theorem for Banach-space valued functions is investigated. We shall use some results from this paper.

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Our main section is Section 5. In this section we introduce the set CLT of all functions satisfying the central limit theorem in the topology of uniform convergence. It turns out that a function belonging to CLT must be a centered, second-order stochastic process with a.s. bounded sample paths and that the intrinsic metric induced by the process must be totally bounded. Furthermore, it turns out that the limit measure is concentrated on the bounded functions and therefore can be considered as a Gaussian measure on the space of bounded functions. Our final result gives necessary and sufficient conditions for the central limit theorem. One of the conditions is the condition in [3].

In Sections 2–4 we introduce basic notation and definitions, some results about stochastic processes, and some results about Radon measures.

Let us finally point out that the three settings: empirical processes, stochastic processes, and Banach-space valued functions are, more or less, equivalent and most of the results can be translated from one setting to another, so it is a matter of taste which to prefer. For us the setting of stochastic processes seems the most natural.

2. Notation and definitions. Let $(S, \mathscr{S}, \mu)$ be a probability space and $(M, \mathscr{M})$ a measurable space. We say that a function g from S into M is *μ-measurable* or a *random variable* on $(S, \mathscr{S}, \mu)$ if g is $(\mathscr{S}(\mu), \mathscr{M})$-measurable, where $\mathscr{S}(\mu)$ is the set of all μ-measurable sets.

Since we are going to work with nonmeasurable functions we need the following concepts: We let μ^* and μ_* denote the *outer* and *inner μ-measure* and if h is a function from S into $\overline{\mathbb{R}} = [-\infty, \infty]$ then we denote the *upper* and *lower μ-integral* of h by $\int^* h\,d\mu$ and $\int_* h\,d\mu$ and we denote the *upper* and *lower μ-envelope* of h by h^* and h_*, i.e.,

(2.1) h^* and h_* are μ-measurable functions from S into $\overline{\mathbb{R}}$,

$$h_*(s) \le h(s) \le h^*(s), \qquad \forall\, s \in S, \tag{2.2}$$

(2.3) $\mu_*(h_* < g \le h) = \mu_*(h \le g < h^*) = 0$, for all μ-measurable functions g from S into $\overline{\mathbb{R}}$.

If ξ is an S-valued random variable on a probability space $(\Omega, \mathscr{F}, P)$ with distribution $P\xi = \mu$, then we say that ξ is *P-perfect* if

$$P^*(\xi \in A) = \mu^*(A), \qquad \forall\, A \subseteq S. \tag{2.4}$$

Envelopes and perfect random variables are investigated in [2] and we shall use some results from this paper.

We denote *the set of all μ-measurable functions* from S into $\overline{\mathbb{R}}$ by $L^0(\mu)$. On $L^0(\mu)$ we define the family of functions $\{\|\cdot\|_p \mid 0 \le p \le \infty\}$ by

$$\|g\|_p = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_S \operatorname{arctg}|g|\,d\mu, & \text{if } p = 0, \\ \left\{\displaystyle\int_S |g|^p\,d\mu\right\}^{1 \wedge 1/p}, & \text{if } 0 < p < \infty, \\ \mu - \operatorname*{ess\,sup}_{s \in S} |g(s)|, & \text{if } p = \infty, \end{cases}$$

for $g \in L^0(\mu)$. For all $g \in L^0(\mu)$ we have that

$$\|g\|_p \le \begin{cases} \left(\|g\|_q\right)^{1\wedge(p\vee(p/q))}, & 0 < p \le q \le \infty, \\ \frac{2}{\pi}\|g\|_q, & 0 = p < q \le 1. \end{cases} \tag{2.5}$$

Let T be a set, then we denote *the set of all functions from* T *into* $\mathbb{R}$ by $\mathbb{R}^T$, *the set of all bounded functions from* T *into* $\mathbb{R}$ by $B(T)$ and we let

$$\|\varphi\|_T = \sup_{t\in T} |\varphi(t)|, \qquad \forall\, \varphi \in \mathbb{R}^T.$$

We let $\Gamma(T)$ be *the set of all finite partitions of* T, i.e., if $\alpha \in \Gamma(T)$ then $\alpha = \{A_i\}_{i=1}^n$ where $n \in \mathbb{N}$ and

$$A_i \subseteq T, \qquad A_i \cap A_j = \varnothing, \qquad \forall\, i \neq j, \qquad \bigcup_{i=1}^n A_i = T.$$

If $\varphi \in \mathbb{R}^T$ then *the difference function* $\Delta\varphi$ of φ is defined by

$$\Delta\varphi(u, v) = \varphi(u) - \varphi(v), \qquad \forall\, u, v \in T,$$

and if ρ is a pseudometric on T then we define the following *oscillation functions*

$$w_\rho(\varphi, a) = \sup\left\{|\Delta\varphi(u, v)| \,\middle|\, u, v \in T\colon \rho(u, v) < a\right\}, \qquad a > 0,$$

$$w(\varphi, A) = \sup\left\{|\Delta\varphi(u, v)| \,\middle|\, u, v \in A\right\}, \qquad A \subseteq T.$$

We denote *the set of all bounded and uniformly* ρ*-continuous real-valued functions* on (T, ρ) by $C_u(T, \rho)$. It can easily be checked that

$$C_u(T, \rho) = \left\{\varphi \in B(T) \,\middle|\, \lim_{a\to 0} w_\rho(\varphi, a) = 0\right\}.$$

Let $\{Y_n\}$ be a sequence of $\mathbb{R}^T$-valued functions on a probability space (Ω, F, P). We say that

(2.6) $\{Y_n\}$ is *eventually bounded* if

$$\lim_{a\to\infty} \limsup_{n\to\infty} P^*(\|Y_n\|_T > a) = 0,$$

(2.7) $\{Y_n\}$ is *eventually totally bounded* if $\{Y_n\}$ is eventually bounded and

$$\forall\, \varepsilon > 0, \quad \exists\, \alpha \in \Gamma(T)\colon \limsup_{n\to\infty} P^*\left(\max_{A\in\alpha} w(Y_n, A) > \varepsilon\right) < \varepsilon,$$

(2.8) $\{Y_n\}$ is *eventually uniformly* ρ*-equicontinuous*, where ρ is a pseudometric on T, if

$$\lim_{a\to 0} \limsup_{n\to\infty} P^*\left(w_\rho(Y_n, a) > \varepsilon\right) = 0, \qquad \forall\, \varepsilon > 0.$$

If L is a subset of $\mathbb{R}^T$, then $(L, \|\cdot\|_T)$ denotes L equipped with the topology induced by $\|\cdot\|_T$, and $\mathscr{K}(L)$, $\mathscr{G}(L)$, $\mathscr{F}(L)$, and $\mathscr{B}(L)$ denote the sets of all *compact*, *open*, *closed*, and *Borel* subsets of $(L, \|\cdot\|_T)$. A Borel measure on $(L, \|\cdot\|_T)$ is a measure on $(L, \mathscr{B}(L))$ and a *Radon measure* μ on $(L, \|\cdot\|_T)$ is a finite Borel measure on $(L, \|\cdot\|_T)$ so that

$$\mu(B) = \sup\{\mu(K) | K \subseteq B,\ K \in \mathscr{K}(L)\}, \tag{2.9}$$

for all $B \in \mathscr{B}(L)$.

Let $\{Y_n\}$ be a sequence of L-valued functions on a probability space $(\Omega, \mathscr{F}, P)$. We shall use the following definitions of convergence in law and eventual tightness due to Hoffmann-Jørgensen [8, Chapter 7]:

(2.10) $\{Y_n\}$ *converges in law* to a Borel probability measure γ on $(L, \|\cdot\|_T)$ if

$$\int_L g\,d\gamma = \lim_{n\to\infty} \int_\Omega^* g(Y_n)\,dP, \qquad \forall\, g \in C(L, \|\cdot\|_T),$$

where $C(L, \|\cdot\|_T)$ is the set of all bounded, continuous functions from $(L, \|\cdot\|_T)$ into $\mathbb{R}$,

(2.11) $\{Y_n\}$ is *eventually tight* on $(L, \|\cdot\|_T)$ if, $\forall\ \varepsilon > 0$, $\exists\ K \in \mathscr{K}(L)$: $\limsup_{n\to\infty} P^*(Y_n \notin G) < \varepsilon$, for all $G \in \mathscr{G}(L)$ so that $G \supseteq K$.

If $\{Y_n\}$ is a sequence of P-measurable functions then (2.10) is consistent with the definition of convergence in law for random variables.

The following result can be found in [7] (see Example 7.28) but for completeness of the paper we will give a proof below.

THEOREM 2.12. *Let $\{Y_n\}$ be a sequence of $B(T)$-valued functions on $(\Omega, \mathscr{F}, P)$. Then the following three statements are equivalent.*

(2.12.1) *$\{Y_n\}$ is eventually tight on $(B(T), \|\cdot\|_T)$,*

(2.12.2) *$\{Y_n\}$ is eventually totally bounded,*

(2.12.3) *$\{Y_n\}$ is eventually uniformly ρ-equicontinuous for some totally bounded pseudometric ρ on T, and $\{Y_n\}$ is eventually bounded.*

PROOF. We have by Theorem IV.5.6 in [4] that

$$\forall\, \varepsilon > 0,\quad \forall\, K \in \mathscr{K}(B(T)) \exists\, \alpha \in \Gamma(T):\quad \sup_{A\in\alpha,\ f\in K} w(f, A) < \varepsilon,$$

and from this it follows easily that (2.12.1) implies (2.12.2).

Assume that (2.12.2) holds and choose $\{\alpha_j\} \subseteq \Gamma(T)$ so that

$$\limsup_n P^*\Big(\sup_{A\in\alpha_j} w(Y_n, A) > 2^{-j-1}\Big) < 2^{-j-1}, \qquad \forall\, j \in \mathbb{N}. \tag{2.12.4}$$

Put $\mathscr{F}_j = \sigma(\alpha_1, \dots, \alpha_j)$ and define ρ by

$$\rho(u, v) = \sup\{2^{-n} | u \not\equiv v\ (\mathrm{mod}\ \mathscr{F}_n)\}, \qquad \forall\, u, v \in T,$$

with the convention $\sup \varnothing = 0$. Then one can easily check that ρ is a totally bounded pseudometric on T satisfying

$$\{u \in T | \rho(v, u) < 2^{-j}\} = \mathscr{F}_j(v), \qquad \forall\, v \in T,$$

where $\mathscr{F}_j(v)$ is the $\mathscr{F}_j$-atom containing v. Hence

$$w_\rho(f, 2^{-j}) \le \sup_{A \in \alpha_j} w(f, A), \qquad \forall f \in B(T), \quad \forall j \in \mathbb{N}.$$

Now using (2.12.4) one can show that $\{Y_n\}$ is eventually uniformly ρ-equicontinuous, i.e., (2.12.3) holds.

Assume finally that (2.12.3) holds and let $\varepsilon > 0$. Choose $\{a_j\} \subseteq \mathbb{R}_+$, $a_j \downarrow 0$, $a \in \mathbb{R}$, and $n_0 \in \mathbb{N}$ so that

$$\limsup_n P^*\big(w_\rho(Y_n, a_j) > 2^{-j}\big) < \varepsilon \cdot 2^{-j-2}, \qquad \forall j \ge 1, \tag{2.12.5}$$

$$P^*(\|Y_n\|_T > a) < \varepsilon/2, \qquad \forall n \ge n_0. \tag{2.12.6}$$

Let $K_m = \bigcap_{j=1}^m \{f \in B(T) | w_\rho(f, a_j) \le 2^{-j}, \|f\|_T \le a\}$ and let $K = \bigcap_{m=1}^\infty K_m \subseteq C_u(T, \rho)$. Then by Corollary IV.6.8 in [4] $K \in \mathscr{K}(B(T))$. If $G \supseteq K$ and G is open then by compactness of K there exists $m \in \mathbb{N}$ so that $G \supseteq K_m$. Hence by (2.12.5) and (2.12.6)

$$\limsup_n P(Y_n \notin G) \le \varepsilon.$$

This shows that $\{Y_n\}$ is eventually tight. □

REMARK 2.13. The Portmanteau Theorem (see, e.g., Theorem 8.1 in [9]) can be extended to the nonmeasurable case. The proof of this extension is the same as the ordinary proof.

3. Stochastic processes. For a real valued function f on $S \times T$ we will use the following notation:

$$f_t = f(\cdot, t) \in \mathbb{R}^S, \qquad f(s) = f(s, \cdot) \in \mathbb{R}^T.$$

A *stochastic process f on $(S, \mathscr{S}, \mu)$ with timeset T* is a real valued function on $S \times T$, where $(S, \mathscr{S}, \mu)$ is a probability space and T is an arbitrary set, so that $f_t \in L^0(\mu)$ for all $t \in T$, and it is said to be *centered* if

$$\int_S f_t \, d\mu = 0, \qquad \forall t \in T,$$

and a *second-order process* if

$$\int_S f_t^2 \, du < \infty, \qquad \forall t \in T.$$

A stochastic process f on $(S, \mathscr{S}, \mu)$ with timeset T induces a family of pseudometrics on T, $\{\rho_f^p | 0 \le p \le \infty\}$, by

$$\rho_f^p(u, v) = \|f_u - f_v\|_p, \qquad \forall u, v \in T.$$

Let us point out that ρ_f^2, called the intrinsic metric induced by f, has been considered quite often (see, e.g., [3] and [6]).

THEOREM 3.1. *Let f be a stochastic process on $(S, \mathscr{S}\, \mu)$ with timeset T, ρ a pseudometric on T, and $p \in [0, \infty]$. Then*

(3.1.1) $$\rho_f^p(u, v) \le \| w_\rho(f, a)_* \|_p, \qquad \forall\, u, v \in T\colon \rho(u, v) < a,$$

and, furthermore, if

(3.1.2) $$\exists\, a > 0\colon \| w_\rho(f, a)_* \|_p < \infty$$

and

(3.1.3) $$\mu_*(f \in C_u(T, \rho)) = 1,$$

then

(3.1.4) $$\forall\, \varepsilon > 0, \quad \exists\, \delta > 0\colon \rho(u, v) \le \delta \Rightarrow \rho_f^p(u, v) \le \varepsilon.$$

REMARKS. (1) (3.1.3) says that $f(s)$ is bounded and uniformly ρ-continuous for μ-almost all $s \in S$.

(2) If (3.1.4) holds then total boundedness of ρ implies total boundedness of ρ_f^p.

PROOF. (3.1.1.) follows from Theorem II.1.2 and Example II.1.5 in [2] and (3.1.4) from (3.1.1)–(3.1.3) using the Lebesgue dominated convergence theorem. □

The following theorem is due to Hoffmann-Jørgensen:

THEOREM 3.2. *Let f be a stochastic process on $(S, \mathscr{S}, \mu)$ with timeset T and ρ a totally bounded pseudometric on T so that*

(3.2.1) $$\mu_*(f \in C_u(T, \rho)) = 1.$$

Then

(3.2.2) $$\mu_*\big(f \in C_u(T, \rho_f^p)\big) = 1, \qquad \forall\, p \in [0, \infty].$$

PROOF. By (2.5) we need to show (3.2.2) only for $p = 0$ and since (3.2.1) holds it is no restriction to assume that $f(s) \in C_u(T, \rho)$ for all $s \in S$. Notice that (3.1.2) is satisfied for $p = 0$.

Let $(\hat{T}, \hat{\rho})$ be the completion of (T, ρ). Then $(\hat{T}, \hat{\rho})$ is a compact pseudometric space, T is a dense subset of $\hat{T}$ and $\rho = \hat{\rho}|_{T\times T}$, where $\hat{\rho}|_{T\times T}$ is the restriction of $\hat{\rho}$ to $T \times T$. By Theorem 1.6.17 in [4] $f(s)$ admits a unique, uniformly $\hat{\rho}$-continuous extension $h(s)$ to $\hat{T}$. Since for every $t \in \hat{T}$ there exists a sequence $\{t_n\} \subseteq T$ so that $\hat{\rho}(t_n, t) \to_{n\to\infty} 0$, we have, by continuity of h,

$$f_{t_n}(s) \to h_t(s), \qquad \forall\, s \in S.$$

Therefore h is a stochastic process on $(S, \mathscr{S}, \mu)$ with timeset $\hat{T}$ and $\rho_f^0 = \rho_h^0|_{T\times T}$.

Let $A = \{(u, v) \in \hat{T} \times \hat{T} \,|\, \rho_h^0(u, v) = 0\}$. Then

$$(u, v) \in A \Leftrightarrow h_u = h_v, \quad \mu\text{-a.s.}$$

The product space $(\hat{T} \times \hat{T}, \hat{\rho} \otimes \hat{\rho})$ is a compact pseudometric space and therefore there exists a countable dense subset A_0 of A. Since A_0 is countable we can choose a nullset $N \in \mathscr{S}(\mu)$ so that

$$h(s,u) = h(s,v), \qquad \forall\, s \in S \setminus N, \ \ \forall\, (u,v) \in A_0,$$

and by $\hat{\rho}$-continuity of $h(s)$ for all $s \in S$ we get

$$h(s,u) = h(s,v), \qquad \forall\, s \in S \setminus N, \ \ \forall\, (u,v) \in A. \tag{3.2.3}$$

Now let $s_0 \in S \setminus N$ and assume that $f(s_0)$ is not uniformly ρ_f^0-continuous. Then there exist $\varepsilon > 0$ and a sequence $\{(u_n, v_n)\} \subseteq T \times T$ so that

$$|f(s_0, u_n) - f(s_0, v_n)| \geq \varepsilon, \qquad \forall\, n \geq 1, \tag{3.2.4}$$

$$\rho_f^0(u_n, v_n) \leq 2^{-n}, \qquad \forall\, n \geq 1. \tag{3.2.5}$$

By compactness of $(\hat{T} \times \hat{T}, \hat{\rho} \times \hat{\rho})$ there exists a convergent subsequence $\{(u_{n(k)}, v_{n(k)})\}$ of $\{(u_n, v_n)\}$ and let $(u, v) \in \hat{T} \times \hat{T}$ be its limit point. Theorem 3.1 ensures that ρ_h^0 is $\hat{\rho}$-continuous; thus by (3.2.5) we have that $\rho_h^0(u, v) = 0$, i.e., $(u, v) \in A$ and by (3.2.3) that

$$\lim_{k \to \infty} |f(s_0, u_{n(k)}) - f(s_0, v_{n(k)})| = |h(s_0, u) - h(s_0, v)| = 0;$$

but this contradicts (3.2.4). □

EXAMPLE 3.3. Let γ be a Gaussian Radon measure on $(B(T), \|\cdot\|_T)$. Then there exists a probability space $(\Omega, \mathscr{F}, P)$ and a stochastic process f on $(\Omega, \mathscr{F}, P)$ with timeset T so that f is P-measurable and has the distribution γ. If there exists a totally bounded pseudometric ρ on T so that $\gamma(C_u(T, \rho)) = 1$, then, since all the pseudometrics ρ_f^p, $p \in [0, \infty)$, are equivalent, it follows from Theorems 3.1 and 3.2 that

$$\rho_f^p \text{ is totally bounded, for all } p \in [0, \infty), \tag{3.3.1}$$

$$\gamma\big(C_u(T, \rho_f^p)\big) = 1, \quad \text{for all } p \in [0, \infty]. \tag{3.3.2}$$

4. Results about Radon measures. Since the following two results are stated in a little bit more general form than we need, they form a separate section.

THEOREM 4.1. *Let T be a set, ρ a pseudometric on T, and $\{Y_n\}$ a sequence of $B(T)$-valued functions defined on some probability space $(\Omega, \mathscr{F}, P)$. If Y_n converges in law to a Radon probability measure γ then the following two statements are equivalent.*

$$\gamma(C_u(T, \rho)) = 1, \tag{4.1.1}$$

(4.1.2) $\{Y_n\}$ *is eventually uniformly ρ-equicontinuous.*

PROOF. Let for all $\delta > 0$ and $\varepsilon > 0$

$$F_{\delta\varepsilon} = \{f \in B(T) | w_\rho(f, \delta) \geq \varepsilon\},$$
$$F_\varepsilon = \{f \in B(T) | w_\rho(f, \delta) \geq \varepsilon, \forall\, \delta > 0\},$$
$$G_{\delta\varepsilon} = \{f \in B(T) | w_\rho(f, \delta) > \varepsilon\},$$
$$G_\varepsilon = \{f \in B(T) | w_\rho(f, \delta) > \varepsilon, \forall\, \delta > 0\}.$$

Then $F_{\delta\varepsilon}$ is a closed set and $G_{\delta\varepsilon}$ an open set. By the Portmanteau Theorem (see Remark 2.13) we get that

$$\gamma(G_{\delta\varepsilon}) \leq \liminf_{n\to\infty} P_*(Y_n \in G_{\delta\varepsilon}) \leq \limsup_{n\to\infty} P^*(Y_n \in F_{\delta\varepsilon}) \leq \gamma(F_{\delta\varepsilon}),$$

so since $F_{\delta\varepsilon} \downarrow_{\delta\to 0} F_\varepsilon$ and $G_{\delta\varepsilon} \downarrow_{\delta\to 0} G_\varepsilon$ for all $\varepsilon > 0$ we have

$$\gamma(G_\varepsilon) \leq \lim_{\delta\to 0} \limsup_{n\to\infty} P^*\big(w_\rho(Y_n, \delta) > \varepsilon\big) \leq \gamma(F_\varepsilon), \qquad \forall\, \varepsilon > 0,$$

and then the equivalence follows from the fact that

$$G_\varepsilon \uparrow_{\varepsilon\to 0} \mathbb{R}^T \setminus C_u(T, \rho) \quad \text{and} \quad F_\varepsilon \uparrow_{\varepsilon\to 0} \mathbb{R}^T \setminus C_u(T, \rho). \qquad \square$$

THEOREM 4.2. *Let T be a set and γ a Radon measure on $(\mathbb{R}^T, \|\cdot\|_T)$ so that for all finite subsets T_0 of T the marginals of γ on $\mathbb{R}^{T_0}$ are centered Gaussian measures. Then the following statements hold*:

(4.2.1) $$\gamma(B(T)) = 1,$$

(4.2.2) *if f is a stochastic process induced by γ then ρ_f^p is totally bounded for all $p \in [0, \infty)$ and $\gamma(C_u(T, \rho_f^p)) = 1$ for all $p \in [0, \infty]$.*

REMARK. If (4.1.1) holds γ can be considered as a Gaussian measure on $(B(T), \|\cdot\|_T)$.

PROOF. If $T_0 \subseteq T$ and $\varphi \in \mathbb{R}^T$ then we let φ_{T_0} be the restriction of φ to T_0, i.e., $\varphi_{T_0} \in \mathbb{R}^{T_0}$, and let γ_{T_0} be the marginal of γ on $\mathbb{R}^{T_0}$. We define for all $a > 0$

$$B(\varphi, a, T_0) = \{\psi \in \mathbb{R}^{T_0} | \|\varphi_{T_0} - \psi\|_{T_0} \leq a\} \subseteq \mathbb{R}^{T_0}.$$

By the definition of a Radon measure on $(\mathbb{R}^T, \|\cdot\|_T)$ [see (2.9)] it follows that

(4.2.3) $$\forall\, \varepsilon > 0, \quad \forall\, a > 0, \quad \exists\, A \subseteq \mathbb{R}^T, \quad A \text{ finite: } \gamma\Big(\bigcup_{\varphi\in A} B(\varphi, a, T)\Big) > 1 - \varepsilon$$

and since $(\mathbb{R}^T, \|\cdot\|_T)$ is pseudometrisable and γ is τ-smooth

(4.2.4) $$\exists\, M \in \mathscr{F}(\mathbb{R}^T), \quad M \text{ separable: } \gamma(M) = 1,$$

(4.2.5) $$\forall\, \varphi \in \mathbb{R}^T \text{ we have that}$$
$$\gamma(B(\varphi, a, T)) = \inf\{\gamma_{T_0}(B(\varphi, a, T_0)) | T_0 \subseteq T, T_0 \text{ finite}\}, \qquad \forall\, a > 0.$$

We have, by assumption, that for all finite subsets T_0 of T, γ_{T_0} are centered Gaussian measures and it is well known that

$$\gamma_{T_0}(B(\varphi, a, T_0)) \leq \gamma_{T_0}(B(0, a, T_0)), \qquad \forall\, a \geq 0, \quad \forall\, \varphi \in \mathbb{R}^T. \tag{4.2.6}$$

Using (4.2.3), (4.2.5), and (4.2.6) we have that for all $a > 0$, $\gamma(B(0, a, T)) > 0$ and, consequently,

$$\gamma(B(T)) > 0. \tag{4.2.7}$$

Choose M in (4.2.4) and let $\{\varphi_n\}$ be dense in M. We define $\Phi\colon T \to \mathbb{R}^{\mathbb{N}}$ by

$$\Phi(t) = \{\varphi_n(t)\}.$$

If we put the product topology on $\mathbb{R}^{\mathbb{N}}$, then there exist $\{t_i\} \subseteq T$ so that $\{\Phi(t_i)\}$ is dense in $\Phi(T) \subseteq \mathbb{R}^{\mathbb{N}}$, i.e., for every $\varphi \in M$ and all $\varepsilon > 0$ there exist $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $t \in T$ so that

$$\begin{gathered} \|\varphi - \varphi_m\|_T < \varepsilon, \qquad \|\varphi_m\|_T < \varepsilon + |\varphi_m(t)|, \\ |\varphi_m(t) - \varphi_m(t_n)| < \varepsilon. \end{gathered} \tag{4.2.8}$$

We define the function $q\colon \mathbb{R}^T \to \overline{\mathbb{R}}$ by

$$q(\varphi) = \sup_{n \in \mathbb{N}} |\varphi(t_n)|, \qquad \forall \varphi \in \mathbb{R}^T.$$

Then q is a seminorm on $\mathbb{R}^T$. Using (4.2.8) we have for all $\varphi \in M$ and all $\varepsilon > 0$

$$\begin{aligned} \|\varphi\|_T \leq \varepsilon + \|\varphi_m\| \leq 2\varepsilon + |\varphi_m(t)| &\leq 3\varepsilon + |\varphi_m(t_n)| \\ &\leq 4\varepsilon + |\varphi(t_n)| \leq 4\varepsilon + q(\varphi), \end{aligned}$$

so

$$\|\varphi\|_T \leq q(\varphi), \qquad \forall \varphi \in M.$$

The converse inequality holds for all $\varphi \in \mathbb{R}^T$ so

$$\|\varphi\|_T = q(\varphi), \qquad \forall \varphi \in M. \tag{4.2.9}$$

Now q is measurable with respect to the product σ-algebra on $\mathbb{R}^T$. By Corollary 2.2 in [5] this σ-algebra equals $\mathscr{B}a(\mathbb{R}^T)$. Let $A = \{\varphi \in \mathbb{R}^T | q(\varphi) < \infty\}$. Then, since $A \supseteq B(T)$, we have by Theorem II.3.4 in [6] and (4.2.7) that $\gamma(A) = 1$, so if we let $M_0 = M \cap A$ then $\gamma(M_0) = 1$ and by (4.2.9) $M_0 \subseteq B(T)$ thus (4.2.1) is proved.

If there exists a totally bounded pseudometric ρ on T so that $\gamma(C_u(T, \rho)) = 1$ statement (4.2.2) follows from Example 3.3. Let us find such a ρ. By (2.9) and (4.2.1) there exists a sequence $\{K_n\}$ of closed, compact subsets of $(B(T), \|\cdot\|_T)$ so that $K_n \subseteq K_{n+1}$ and $\gamma(K) = 1$, where $K = \bigcup_{i=1}^{\infty} K_n$. If we define a sequence of functions $\{\rho_n\}$ from $T \times T$ into $\mathbb{R}$ by

$$\rho_n(u, v) = \sup\{|\varphi(u) - \varphi(v)| | \varphi \in K_n\},$$

then Theorem IV.5.6 in [4] states that $\{\rho_n\}$ is a sequence of totally bounded pseudometrics on T. The pseudometric we are looking for is now defined by

$$\rho(u,v) = \sum_{i=1}^{\infty} 2^{-n}(\rho_n(u,v) \wedge 1),$$

since ρ is totally bounded and all $\varphi \in K$ are uniformly ρ-continuous. □

5. The central limit theorem. This is the main section of the present paper.

DEFINITION 5.1. Let $(S, \mathscr{S}, \mu)$ be a probability space, $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ the countable product of $(S, \mathscr{S}, \mu)$ and $\{\pi_n\}$ the sequence of natural projections from $S^{\mathbb{N}}$ into S, i.e., $\pi_n(s) = s_n$ for all $s = \{s_n\} \in S^{\mathbb{N}}$.

Furthermore, let T be a set, L a subset of $\mathbb{R}^T$, and f a real valued function on $S \times T$ such that $\{f(s)|s \in S\} \subseteq L$.

We say that f satisfies the central limit theorem in $(L, \|\cdot\|_T)$ or $f \in \mathrm{CLT}(L, \|\cdot\|_T)$ if there exists a Radon probability measure γ_f on $(L, \|\cdot\|_T)$ so that $1/\sqrt{n}\sum_{i=1}^{n} f(\pi_i)$ converges in law to γ_f.

REMARK. The assumption that γ is a Radon measure is justified by the fact that it is consistent with the usual axioms of set theory to assume that all finite Borel measures on $(\mathbb{R}^T, \|\cdot\|_T)$ are Radon measures.

In the definition of the CLT we use the product space $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ and the sequence of natural projections $\{\pi_n\}$, but one could ask what happens if we replace $\{\pi_n\}$ with a sequence of independent identically distributed S-valued random variables defined on some probability space. The next proposition gives the answer:

PROPOSITION 5.2. *Let $(S, \mathscr{S}, \mu)$ be a probability space, T a set and f a real valued function on $S \times T$ so that $f(s) \in L \subseteq R^T$ for all $s \in S$. Furthermore, let $X = \{X_n\}$ be a sequence of independent, identically distributed S-valued random variables, with common distribution μ, defined on some probability space $(\Omega, \mathscr{F}, P)$. Then*

(5.2.1) *if $f \in CLT(L, \|\cdot\|_T)$, $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} f(X_i)$ converges in law to γ_f,*

(5.2.2) *if X is P-perfect and $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} f(X_i)$ converges in law to some Radon probability measure on $(L, \|\cdot\|_T)$, $f \in CLT(L, \|\cdot\|_T)$.*

PROOF. Let h be any bounded real valued function on $S^{\mathbb{N}}$. Then by II.2.1.1 and I.2.4.4, both in [2], we have

$$-\int^*(-h)\,d\mu^{\mathbb{N}} \le -\int^*(-h(X))dP \le \int^* h(X)\,dP \le -\int^* h\,d\mu^{\mathbb{N}}$$

and if X is P-perfect, by Theorem II.2.2 in [2]

$$\int^* h(X)\, dP = \int^* h\, d\mu^{\mathbb{N}}.$$

The results follow now from the statements above and the definition of convergence in law. □

The next theorem gives some consequences for f and γ_f if $f \in \mathrm{CLT}(\mathbb{R}^T, \|\cdot\|_T)$:

THEOREM 5.3. *Let $(S, \mathscr{S}, \mu)$ be a probability space, T a set and f a real valued function on $S \times T$. If $f \in CLT(\mathbb{R}^T, \|\cdot\|_T)$ the following six statements hold*:

(5.3.1) *f is a centered, second-order stochastic process on $(S, \mathscr{S}, \mu)$ with timeset T.*

(5.3.2) *if $T_0 \subseteq T$ and T_0 is finite, then the marginal of γ_f on $\mathbb{R}^{T_0}$ is a centered Gaussian measure*,

(5.3.3) $$\gamma_f(B(T)) = 1,$$

(5.3.4) $$\rho_f^p \text{ is totally bounded}, \quad \forall\, p \in [0,2],$$

(5.3.5) $$\gamma_f\big(C_u(T, \rho_f^p)\big) = 1, \quad \forall\, p \in [0,\infty],$$

(5.3.6) $$\|f\|_T < \infty, \quad \mu\text{-a.s.}$$

PROOF. Let $T_0 \subseteq T$ be finite, f_{T_0} the restriction of f to T_0, and γ_{T_0} the marginal of γ_f on T_0. From Definition 5.1 it follows that if $f \in \mathrm{CLT}(\mathbb{R}^T, \|\cdot\|_T)$ then $f_{T_0} \in \mathrm{CLT}(\mathbb{R}^{T_0}, \|\cdot\|_T)$ with the limit measure γ_{T_0}. By Proposition 3.7 in [1] we have that (5.3.1) and (5.3.2) hold and that ρ_f^2 equals ρ_h^2, where h is the stochastic process induced by γ. Now using Theorem 4.2, Theorem 3.2, and (2.5) we get (5.3.3)–(5.3.5). Let $\{\pi_n\}$ and $(S^{\mathbb{N}}, \mathscr{S}^{\mathbb{N}}, \mu^{\mathbb{N}})$ be as in Definition 5.1. Then by the Portmanteau Theorem (see Remark 2.13)

(5.3.7) $$\limsup_{n\to\infty} (\mu^{\mathbb{N}})^*\left(\left\|\sum_{i=1}^{n} f(\pi_i)\right\|_T > a\sqrt{n}\right) \le \gamma_f\big(\varphi \in \mathbb{R}^T \,\|\, \|\varphi\|_T > a\big) \to_{a\to\infty} 0,$$

which gives that $\{(1/\sqrt{n})\Sigma_{i=1}^n f(\pi_i)\}$ is eventually bounded.

Perfectness of $\{\pi_n\}$ ensures that

(5.3.8) $$\begin{aligned}\mu^*(\|f\|_T > t) &= (\mu^{\mathbb{N}})^*(\|f(\pi_n)\|_T > t)\\ &\le (\mu^{\mathbb{N}})^*\left(\left\|\sum_{i=1}^{n} f(\pi_i)\right\|_T > \frac{t}{2}\right) + (\mu^{\mathbb{N}})^*\left(\left\|\sum_{i=1}^{n-1} f(\pi_i)\right\|_T > \frac{t}{2}\right),\end{aligned}$$

for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose $n_0 \in \mathbb{N}$ and $a > 0$ in (5.3.7) so that

$$(\mu^{\mathbb{N}})^*\left(\left\|\sum_{i=1}^{n} f(\pi_i)\right\|_T > a\sqrt{n}\right) > \varepsilon/2, \qquad \forall\, n \ge n_0.$$

If we let $t = 2a\sqrt{n_0 + 1}$ then by (5.3.8) we have that

$$\mu^*(\|f\|_T > t) < \varepsilon,$$

which proves (5.3.6). □

In the definition of the CLT on $(\mathbb{R}^T, \|\cdot\|_T)$ we have not required much about f and the limit measure γ_f, but the previous theorem states that f must be a stochastic process with sample paths which are bounded almost sure, and that γ_f is concentrated on the bounded functions. Without loss of generality we shall assume that $f(s) \in B(T)$ for all $s \in S$ and in this case, since $B(T)$ is closed and open in $(\mathbb{R}^T, \|\cdot\|_T)$, we have that $f \in \mathrm{CLT}(\mathbb{R}^T, \|\cdot\|_T)$ if and only if $f \in \mathrm{CLT}(B(T), \|\cdot\|_T)$; consequently we only need to investigate $\mathrm{CLT}(B(T), \|\cdot\|_T)$. Since $(B(T), \|\cdot\|_T)$ is a Banach space we can use all the results from Section 3 in [1], e.g.,

PROPOSITION 5.4. *$CLT(B(T), \|\cdot\|_T)$ is a linear space.*

PROOF. Proposition 3.9 in [1]. □

The next theorem is a main theorem in this paper and it gives necessary and sufficient conditions for $\mathrm{CLT}(B(T), \|\cdot\|_T)$.

THEOREM 5.5. *Let f be a centered, second-order stochastic process on $(S, \mathscr{S}, \mu)$ with timeset T so that $\{f(s) | s \in S\} \subseteq B(T)$. Furthermore, let $X = \{X_n\}$ be a sequence of independent, identically distributed S-valued random variables with common distribution μ and defined on some probability space $(\Omega, \mathscr{F}, P)$, and assume that X is P-perfect. Finally let*

$$S_n = \sum_{i=1}^{n} f(X_i) \colon (\Omega, \mathscr{F}, P) \to \mathbb{R}^T.$$

Then the following four statements are equivalent:

(5.5.1) $$f \in CLT(B(T), \|\cdot\|_T),$$

(5.5.2) $\left\{\frac{1}{\sqrt{n}} S_n\right\}$ *is eventually tight on* $(B(T), \|\cdot\|_T)$,

(5.5.3) $\left\{\frac{1}{\sqrt{n}} S_n\right\}$ *is eventually totally bounded,*

(5.5.4) *there exists a totally bounded pseudometric ρ on T so that* $\left\{\frac{1}{\sqrt{n}} S_n\right\}$ *is eventually uniformly ρ-equicontinuous.*

If one of the statements holds (and hence all) then

(5.5.5) ρ_f^p *is totally bounded,* $\quad \forall\, p \in [0, 2]$,

(5.5.6) $$\gamma_f\big(C_u(T, \rho_f^p)\big) = 1, \quad \forall\, p \in [0, \infty].$$

PROOF. The equivalence between (5.5.1) and (5.5.2) follows from Theorem 3.6 and Proposition 3.2, both in [1], and (2.4). If we can show that (5.5.4) implies that $\{(1/\sqrt{n})S_n\}$ is eventually bounded then the equivalence between (5.5.2), (5.5.3), and (5.5.4) follows from Theorem 2.12.

Now assume that (5.5.4) is fulfilled and let $\varepsilon > 0$. Choose $\delta > 0$ and $n_0 \in \mathbb{N}$ so that

$$P^*\big(w_\rho(S_n, \delta) > \sqrt{n}\big) < \frac{\varepsilon}{2}, \qquad \forall n \geq n_0, \tag{5.5.7}$$

and since ρ is totally bounded there exist $\{t_i\}_{i=1}^k$, where $k \in \mathbb{N}$, so that

$$\forall\, t \in T\colon \inf_{i \leq k} \rho(t, t_i) < \delta. \tag{5.5.8}$$

By the k-dimensional central limit theorem there exists $a > 1$ so that

$$P^*\Big(\max_{i \leq k} |S_n(t_i)| > a\sqrt{n}\Big) < \frac{\varepsilon}{3}, \qquad \forall\, n \geq n_0.$$

Now by (5.5.7) and (5.5.8)

$$\begin{aligned}
P^*\big(\|S_n\|_T > 2a\sqrt{n}\big) &= P^*\Big(\sup_{t \in T} |S_n(t) - S_n(t_j) + S_n(t_j)| > 2a\sqrt{n},\ j = 1 \dots k\Big) \\
&\leq P^*\Big(\sup_{t \in T} |S_n(t) - S_N(t_j)| > a\sqrt{n},\ j = 1 \dots k\Big) \\
&\quad + P\Big(\max_{j \leq k} |S_n(t_j)| > a\sqrt{n}\Big) \\
&\leq P^*\big(w_\rho(S_n, \delta) > \sqrt{n}\big) + P\Big(\max_{i \leq k} |S_n(t_i)| > a\sqrt{n}\Big) \\
&\leq \varepsilon,
\end{aligned}$$

so $\{(1/\sqrt{n})S_n\}$ is eventually bounded.

Finally (5.5.5) and (5.5.6) follow from Theorem 5.3. □

Let us finish this paper by giving some consequences of the previous theorem.

PROPOSITION 5.6. *Let the conditions and setting be as in Theorem 5.5 and let ρ be a totally bounded pseudometric on T so that $\{(1/\sqrt{n})S_n\}$ is eventually uniformly ρ-equicontinuous. Then $f \in CLT(B(T), \|\cdot\|_T)$ and $\gamma_f(C_u(T, \rho)) = 1$. On the other hand, if $f \in CLT(B(T), \|\cdot\|_T)$ and ρ' is pseudometric on T then $\gamma_f(C_u(T, \rho')) = 1$ is equivalent to $\{(1/\sqrt{n})S_n\}$ being eventually uniformly ρ'-equicontinuous.*

PROOF. Theorem 5.5 and Theorem 4.1. □

EXAMPLE 5.7. By Theorem 5.5 and Proposition 5.6 a necessary and sufficient condition for the central limit theorem is that ρ_f^2 is totally bounded and $\{(1/\sqrt{n})S_n\}$ is eventually uniformly ρ_f^2-equicontinuous. As is mentioned in Section 1 this is exactly the necessary and sufficient condition for the central limit theorem found by Dudley and Philipp [3] and used by Giné and Zinn [6]. In some contexts there exist more natural pseudometrics on T than the intrinsic metric ρ_f^2. Theorem 5.5 tells us that if we can show totally boundedness and eventually uniformly equicontinuity of $\{(1/\sqrt{n})S_n\}$ with respect to this more natural pseudometric, then still the central limit theorem for f holds.

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