The Annals of Probability
1987, Vol. 15, No. 1, 115–137

MULTIDIMENSIONAL REFLECTED BROWNIAN MOTIONS HAVING EXPONENTIAL STATIONARY DISTRIBUTIONS[1]

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We are concerned with the stationary distribution of reflected Brownian motion (RBM) in a d-dimensional domain G. Such a process behaves like Brownian motion with a constant drift vector μ in G and is instantaneously reflected at the boundary, with the direction of reflection given by a nontangential vector field v on ∂G. We consider first the case where G is smooth and bounded and v varies smoothly over ∂G. It is shown that the RBM has a stationary density of the exponential form $C(\mu)\exp\{\gamma(\mu)\cdot x\}$ for each $\mu \in R^d$ if and only if v satisfies a certain skew symmetry condition. An explicit formula is given for $\gamma(\mu)$ in terms of v and μ.

Motivated by applications in queueing theory, we next consider the case where G is a convex polyhedral domain and v is constant on each face of the boundary. Postponing for now the treatment of certain foundational questions, we work directly with a basic adjoint relation (BAR) that appears to characterize stationary distributions for a wide class of RBM's in polyhedral domains. This analytic relation is motivated by formal analogy with the smooth case and will be rigorously justified in later work. As in the smooth case, it is found that (BAR) has a solution of exponential form for each $\mu \in R^d$ if and only if v satisfies a certain skew symmetry condition. Moreover, under a mild nondegeneracy condition, it is shown that an exponential solution exists for one $\mu \in R^d$ if and only if such a solution exists for every $\mu \in R^d$.

1. Introduction. This paper is concerned with the stationary distributions of certain d-dimensional diffusion processes. These processes are called *regulated* Brownian motions in [3], but here we revert to the name *reflected* Brownian motion (abbreviated as RBM), which has been used in most previous work [4–6, 8, 10, 11, 16–18, 20–22]. Roughly speaking, an RBM behaves like Brownian motion with constant drift vector μ in a d-dimensional domain and is instantaneously reflected at the boundary of the domain, with the direction of reflection given by a nontangential vector field on the boundary. Two classes of RBM's are considered here. First we consider those in smooth bounded domains with smooth reflection fields, for which there is a well-developed theory of existence and uniqueness. (This will be referred to hereafter as the *smooth case*.) Then, motivated by applications to queueing and storage theory [3, 12, 13, 20], we consider RBM's in convex polyhedral domains with constant direction of reflection on each boundary face. (This will be referred to hereafter as the *polyhedral case*.) Except when (a) $d = 1$, or (b) $d = 2$ and $\mu = 0$ [18], only sufficient conditions for existence of RBM's in polyhedral domains are known [6, 10, 16].

Received March 1985; revised January 1986.
[1]Research supported in part by NSF Grants ECS-8503445 and DMS-8319562.
AMS 1980 *subject classifications*. Primary 60J65, 60J60, 60K25; secondary 35J25.
Key words and phrases. Reflected Brownian motion, drift, polyhedron, stationary distribution, exponential form, skew symmetry, queueing theory.

Here discussion of the polyhedral case is restricted to the study of a certain purely analytic problem, with consideration of probabilistic questions postponed to a subsequent paper [23].

When $d = 1$, the stationary distribution for an RBM on a bounded interval has a density of the exponential form $C\exp\{2\mu x\}$ for some $C > 0$. In general, no explicit formula is known for the stationary distribution of an RBM when $d \geq 2$. However, it is natural to ask under what conditions an analogue of the one-dimensional formula prevails in higher dimensions. It is shown in this paper that, given a smooth bounded domain G for an RBM and a smooth reflection vector field v on ∂G, the following two statements are equivalent:

(i) for each $\mu \in R^d$, the stationary distribution of the RBM with drift μ has a density of the *exponential form*

$$C(\mu)\exp\{\gamma(\mu)\cdot x\}, \tag{1.1}$$

where $\gamma(\mu) \in R^d$ and $C(\mu) > 0$;

(ii) the reflection field v satisfies the *skew symmetry condition*

$$n(\sigma^*)\cdot q(\sigma) + q(\sigma^*)\cdot n(\sigma) = 0, \quad \text{for all } \sigma, \sigma^* \in \partial G, \tag{1.2}$$

where n is the inward unit normal vector field on ∂G, q is a tangential vector field on ∂G, and the reflection vector field is $v = n + q$.

The sharpness of this result is illustrated with examples. First (see Example 5.3), there is a reflection vector field that admits a stationary density of exponential form for one but not all $\mu \in R^d$. Second (see Lemma 3.1 and Examples 5.1 and 5.2), the skew symmetry property implies that the tangential component of the reflection vector field is divergence free, but the converse implication does not hold, unless $d = 1$, or $d = 2$ and ∂G is connected.

Our study of the polyhedral case is motivated by the fact that RBM's in polyhedral domains arise as diffusion approximations for queueing network models [12, 13, 20]. For such approximations to be useful, one must calculate interesting quantities for the approximating diffusion processes, and stationary distributions are the usual focus of attention in queuing theory. From experience to date it is clear that no simple general formula exists for the stationary distribution of an RBM in a polyhedral domain, and so it is natural to look for special cases in which explicit calculations *can* be done. The simplest imaginable type of stationary distribution is one having the exponential form (1.1), and our original goal was to determine necessary and sufficient conditions for the stationary distribution of an RBM to have this form in the polyhedral case.

Note that (1.1) describes a *product form* density function, by which we mean that $p(x)$ is a separable function of $x_1, \dots, x_d$. Virtually all tractable models of queueing networks have product form stationary distributions [9, 14], and preliminary investigation suggests that the RBM's obtained as limits of such product form queueing networks all have stationary distributions of the exponential form (1.1). Much more important, however, is that the RBM approximating a queueing network model may have an exponential stationary distribution when the original model is intractable. Examples of this phenomenon are provided by Peterson [12].

In the smooth case, an RBM can be defined using the machinery of Stroock and Varadhan [15], and a smooth probability density p is a stationary density for the RBM if and only if it satisfies a certain integral relation [see (ii) in Lemma 2.1]. Unfortunately, study of the polyhedral case is complicated by thorny foundational problems. Given a d-dimensional polyhedral domain G, a constant direction of reflection v_i for each face i of the polyhedron, and a drift vector μ, there need not exist a well-defined RBM with this data. Except for the simple cases of normal reflection and $d = 1$ [3, 8, 16], and the two-dimensional case with $\mu = 0$ [4, 18], only sufficient conditions for the existence and uniqueness of RBM's with polyhedral data are known [6, 10]. In this paper, our treatment of the polyhedral case is restricted to the solution of a certain purely analytic problem, whose relationship to the original probabilistic questions of interest will be rigorously established in subsequent work (cf. [23]). Specifically, we seek solutions of exponential form for the following *basic adjoint relation*, which is the polyhedral analogue of the integral relation that characterizes stationary densities in the smooth case:

$$\text{(BAR)} \qquad \int_G p\,Lf\,dx + \tfrac{1}{2}\int_{\partial G} p\,Df\,d\sigma = 0, \quad \text{for all } f \in C_c^2(\overline{G}).$$

Here $C_c^2(\overline{G})$ denotes the set of functions that are twice continuously differentiable and have compact support in some domain containing $\overline{G}$, $L = \Delta/2 + \mu\nabla$, D is a differential operator on ∂G satisfying $D = v_i\nabla$ on the smooth part of boundary face i and $D = 0$ at the intersections of faces, dx denotes integration with respect to Lebesgue measure on R^d, and $d\sigma$ denotes integration with respect to surface measure on ∂G. General conditions under which (BAR) is necessary and sufficient for p to be a stationary density for an RBM in a polyhedral domain are not known. However, this relation is useful as a formal tool for obtaining candidates for stationary densities of RBM's with polyhedral data. Following the formal analysis of (BAR) given here, a discussion of the related probabilistic questions of existence, uniqueness and stationary distribution of an RBM with polyhedral data, will be given in [23]. Results obtained in this paper for the polyhedral case are summarized below.

Given a fixed polyhedral domain G in R^d and a constant direction of reflection on each face, the following two conditions are equivalent:

(i) for each constant drift vector $\mu \in R^d$, there is a vector $\gamma(\mu) \in R^d$ such that $p(x) = \exp\{\gamma(\mu) \cdot x\}$ satisfies (BAR),

(ii) for any two faces of G, numbered i and j,

$$n_i \cdot q_j + q_i \cdot n_j = 0, \tag{1.3}$$

where n_i is the inward unit normal on face i and q_i is the tangential component of the constant reflection vector $v_i = n_i + q_i$ for that face.

If the polyhedron is *simple*, which means that each vertex is contained in precisely d of the faces, then the result can be strengthened as follows. Given a fixed *simple* polyhedron $\overline{G}$ and a constant direction of reflection on each of its faces, the following four conditions are equivalent:

(i) as above,
(ii) as above,
(iii) the constant function $p(x) \equiv 1$ satisfies (BAR) for $\mu = 0$,
(iv) for some $\mu \in R^d$ there exists $p \in C^2(\overline{G})$ such that $p > 0$ on $\overline{G}$ and (BAR) holds.

Here $C^2(\overline{G})$ denotes the set of functions that are twice continuously differentiable in some domain containing $\overline{G}$.

Thus, in the polyhedral case, under the mild nondegeneracy assumption that $\overline{G}$ is simple, we have the following surprising result: if there exists a smooth and strictly positive p satisfying (BAR) for *some* $\mu \in R^d$, then for *each* $\mu \in R^d$ there is a solution of exponential form for (BAR). Example 8.2 shown that if $\overline{G}$ is *not* simple, then (i) need not follow from (iv). In [23], under the assumption that $\overline{G}$ is simple and the skew symmetry condition (1.3) holds, it will be shown that for each $\mu \in R^d$ there is a well-defined RBM associated with $\overline{G}$, $\{v_i\}$, and μ. Moreover, if $\exp\{\gamma(\mu) \cdot x\}$ is integrable over $\overline{G}$, then it will be shown that $p(x) = C(\mu)\exp\{\gamma(\mu) \cdot x\}$ is the unique stationary density for this RBM, where $C(\mu)$ is a suitable normalization constant.

In both the smooth and polyhedral cases, $\gamma(\mu)$ is unique and it is given by the explicit formula (4.9).

The formal statements and proofs of the above results, together with supporting lemmas and examples constitute the remainder of this paper. The smooth case is treated in Sections 2–5, and Sections 6–9 are devoted to the polyhedral case.

2. Smooth case. To facilitate a more precise description of the main result for the smooth case, the following notation is introduced. Further details on terms taken from the theory of partial differential equations can be found in Gilbarg and Trudinger [2]. Let $0 < \varepsilon < 1$ and G be a nonempty bounded domain in R^d of class $C^{2+\varepsilon}$. Let n denote the *inward* unit normal vector field on the boundary ∂G of G. Let v denote a $C^{1+\varepsilon}$ vector field on ∂G such that $v \cdot n = 1$. Then $q \equiv v - n$ is the tangential component of v on ∂G. The Laplacian operator on R^d will be denoted by Δ and the gradient operator will be denoted by ∇. Differentiation in the inward unit normal direction on ∂G will be denoted by $\partial/\partial n \equiv n \cdot \nabla$ and the gradient operator in the tangent space to ∂G will be denoted by $\nabla_T \equiv \nabla - n(n \cdot \nabla)$. For $\mu \in R^d$ fixed, let operators L, L^*, on G and D, D^*, on ∂G be defined as follows:

$$L = \tfrac{1}{2}\Delta + \mu \cdot \nabla,$$

$$L^* = \tfrac{1}{2}\Delta - \mu \cdot \nabla,$$

$$D = v\nabla \equiv \frac{\partial}{\partial n} + q \cdot \nabla_T,$$

$$D^* = \frac{\partial}{\partial n} - q \cdot \nabla_T.$$

Integration with respect to Lebesgue measure on R^d will be denoted by dx and $d\sigma$ will be used to denote integration with respect to surface measure on ∂G. Throughout this paper, vectors will be regarded as column vectors and a prime will be used to denote transpose.

In Sections 2–5, we view G and v as fixed and regard the drift μ as a parameter. Thus, the reflected Brownian motion with domain G, reflection vector field v, and drift μ, will be simply referred to as the RBM with drift μ. This RBM can be characterized (in law) as the solution of a submartingale problem [15]. Indeed, the following is a canonical representation for this process. Let $\Omega = C([0,\infty),\overline{G})$, the space of continuous functions $\omega\colon [0,\infty) \to \overline{G}$. Suppose Ω is endowed with the σ-algebra $\mathscr{F} = \sigma\{\omega(s)\colon 0 \le s < \infty\}$ generated by the coordinate maps, and for each $t \in [0,\infty)$, let $\mathscr{F}_t = \sigma\{\omega(s)\colon 0 \le s \le t\}$. If P is a probability measure on $(\Omega, \mathscr{F})$, a function $M\colon [0,\infty) \times \Omega \to R$ will be called a P-(sub)martingale if and only if $\{M_t \equiv M(t,\cdot), \mathscr{F}_t, t \ge 0\}$ is a (sub)martingale on $(\Omega, \mathscr{F}, P)$. For each $(t,\omega) \in [0,\infty) \times \Omega$, define

$$X(t,\omega) \equiv X_t(\omega) = \omega(t). \tag{2.1}$$

Then for each $x \in \overline{G}$, there is unique probability measure P_x^μ on $(\Omega, \mathscr{F})$, which has the following two properties [15].

(i) $P_x^\mu(X(0) = x) = 1.$
(ii) For each $f \in C^2(\overline{G})$ that satisfies $Df \ge 0$ on ∂G, we have

$$f(X_t) - \int_0^t Lf(X_s)\,ds$$

is a P_x^μ-submartingale.

The uniqueness implies that $\{P_x^\mu, x \in \overline{G}\}$ is Feller continuous and has the strong Markov property [15, page 196]. The RBM with drift μ is then defined to be the strong Markov process associated with (2.1) and the family of probability measures $\{P_x^\mu, x \in \overline{G}\}$.

By the boundedness of $\overline{G}$ and the smoothness assumptions on L, v, and ∂G, the RBM with drift μ is positive recurrent on $\overline{G}$ and has a unique stationary distribution [19, pages 5–6]. With further effort it can be shown that this distribution is absolutely continuous with respect to Lebesgue measure and that its density (the stationary density) is strictly positive on $\overline{G}$ and is the first eigenfunction for the adjoint operator L^* defined on

$$\left\{ f \in C^{2+\varepsilon}(\overline{G})\colon D^*f = (\nabla_T \cdot q + 2\mu \cdot n) f \text{ on } \partial G \right\}.$$

Since we shall not need the later result here, the proof is omitted.

The next theorem is the main result for the smooth case.

THEOREM 2.1. *Given a fixed bounded $C^{2+\varepsilon}$ domain G and $C^{1+\varepsilon}$ reflection vector field v on ∂G, the following two conditions are equivalent.*

(i) *For each $\mu \in R^d$, the stationary distribution of the RBM with drift μ has a density of the exponential form $p(x) = C(\mu)\exp\{\gamma(\mu) \cdot x\}$ where $\gamma(\mu) \in R^d$ and $C(\mu) > 0$.*

(ii) *The reflection vector field satisfies the skew symmetry condition*:

$$n(\sigma^*)\cdot q(\sigma) + q(\sigma^*)\cdot n(\sigma) = 0, \quad \textit{for all } \sigma, \sigma^* \in \partial G. \tag{2.2}$$

When these conditions hold, $\gamma(\mu)$ *is given by formula* (4.9) *and* $C(\mu)$ *is given by*

$$C(\mu) = \left(\int_{\overline{G}} \exp\{\gamma(\mu)\cdot x\}\,dx\right)^{-1}.$$

REMARKS.

1. Note that the condition on q is independent of μ. Indeed, Example 5.3 below shows that the stationary density of the RBM with drift μ can be of exponential form for one μ without being of exponential form for all $\mu \in R^d$, when (2.2) is violated.
2. In (4.9), $\overline{N}$ (resp. $\overline{Q}$) is a $d \times d$ matrix whose ith row in $n(\sigma_i)'$ [resp. $q(\sigma_i)'$] where $\sigma_1, \dots, \sigma_d$ are d points of ∂G such that $n(\sigma_1), \dots, n(\sigma_d)$ are linearly independent. Although it first appears that this formula for $\gamma(\mu)$ depends on the particular choice of $\overline{N}$ and $\overline{Q}$, in fact it does not, because condition (2.2) implies that $\overline{N}^{-1}\overline{Q}$ is the same for all valid choices of $\sigma_1, \dots, \sigma_d$. For further details, see the explanation following (4.9).

Several characterizations of a stationary density for an RBM are given below. The third of these involves partial differential equations and figures prominently in the proof of Theorem 2.1. The other two are in the form of integral relations. The second of these is used for the purposes of analogy in Section 6, where the polyhedral case is discussed.

LEMMA 2.1. *Fix* $\mu \in R^d$. *Suppose* $p \in C^2(\overline{G})$ *is such that* $p > 0$ *on* G *and* $\int_G p\,dx = 1$. *Then* p *is a stationary density for the RBM with drift* μ *if and only if one of the following equivalent conditions holds.*

(i) $\displaystyle\int_G p\,Lf\,dx = 0, \quad \textit{for all } f \in C_c^2(\overline{G}) \textit{ satisfying } Df = 0 \textit{ on } \partial G.$

(ii) $\displaystyle\int_G p\,Lf\,dx + \tfrac{1}{2}\int_{\partial G} p\,Df\,d\sigma = 0, \quad \textit{for all } f \in C_c^2(\overline{G}).$

(iii) $\begin{cases} L^*p = 0 & \textit{in } G,\\ D^*p = (\nabla_T\cdot q + 2\mu\cdot n)p & \textit{on } \partial G.\end{cases}$

Here $\nabla_T \cdot q$ *denotes the divergence of* q *on the surface* ∂G.

REMARK. Note that since $\overline{G}$ is bounded for the smooth case, we could use $C_c^2(\overline{G})$ in place of $C^2(\overline{G})$ in (i) and (ii) above.

PROOF. Because of the smoothness assumptions on ∂G and v, it follows from Gilbarg and Trudinger [2, Theorem 6.31 ff.] that for each $h \in C^{\varepsilon}(\overline{G})$ and

$\lambda > 0$, there is $f \in C^{2+\varepsilon}(\overline{G})$ such that

$$Lf - \lambda f = -h \quad \text{in } G, \tag{2.3}$$

$$Df = 0 \quad \text{on } \partial G. \tag{2.4}$$

Then, by the submartingale characterization for the RBM with drift μ starting from $x \in \overline{G}$,

$$f(X_t) - \int_0^t (\lambda f - h)(X_s)\,ds$$

is a P_x^μ-martingale. By applying the product formula of stochastic calculus to $\exp(-\lambda t) f(X_t)$, it follows that

$$e^{-\lambda t} f(X_t) + \int_0^t e^{-\lambda s} h(X_s)\,ds$$

is a P_x^μ-martingale. Then, taking expectations and letting $t \to \infty$ yields

$$f(x) = E^{P_x^\mu}\left[\int_0^\infty e^{-\lambda s} h(X_s)\,ds\right] \equiv R_\lambda^\mu h(x), \tag{2.5}$$

since h and f are bounded on $\overline{G}$. Here R_λ^μ denotes the resolvent operator associated with λ and the RBM with drift μ.

Now, p is a stationary density for the RBM with drift μ if and only if

$$\int_{\overline{G}} E^{P_x^\mu}[h(X_s)]\, p(x)\,dx = \int_{\overline{G}} hp\,dx, \quad \text{for all } h \in C^\varepsilon(\overline{G}) \text{ and } s > 0. \tag{2.6}$$

Since $s \to E^{P_x^\mu}[h(X_s)]$ is continuous on $[0, \infty)$ whenever $h \in C^\varepsilon(\overline{G})$, it follows by the uniqueness of the Laplace transform that (2.6) is equivalent to

$$\lambda \int_{\overline{G}} R_\lambda^\mu h(x) p(x)\,dx = \int_{\overline{G}} hp\,dx, \quad \text{for all } h \in C^\varepsilon(\overline{G}) \text{ and } \lambda > 0. \tag{2.7}$$

Here the integral over $\overline{G}$ is the same as that over G. Then, by (2.3), (2.5), and the fact that any $f \in C^{2+\varepsilon}(\overline{G})$ is a solution of (2.3) with $h \in C^\varepsilon(\overline{G})$, it follows that (2.7) is equivalent to

$$\text{(i}'\text{)} \quad \int_G p\,Lf\,dx = 0, \quad \text{for all } f \in C^{2+\varepsilon}(\overline{G}) \text{ satisfying } Df = 0 \text{ on } \partial G. \tag{2.8}$$

Thus (i$'$) is necessary and sufficient for p to be a stationary density for the RBM with drift μ.

By Green's identity and the divergence theorem, for $f \in C^2(\overline{G})$,

$$\begin{aligned}
\int_G p\,Lf\,dx &= \int_G f L^* p\,dx + \tfrac{1}{2}\int_{\partial G}\left\{ f\frac{\partial p}{\partial n} - p\frac{\partial f}{\partial n} - 2\mu\cdot n p f\right\} d\sigma \\
&= \int_G f L^* p\,dx + \tfrac{1}{2}\int_{\partial G}\left\{ f\frac{\partial p}{\partial n} + pq\cdot\nabla_T f - 2\mu\cdot npf - pDf\right\} d\sigma \\
&= \int_G f L^* p\,dx + \tfrac{1}{2}\int_{\partial G}\left\{ f\left(\frac{\partial p}{\partial n} - q\cdot\nabla_T p - (\nabla_T\cdot q)p - 2\mu\cdot np\right)\right. \\
&\qquad\qquad \left. + \nabla_T\cdot(qpf) - pDf\right\} d\sigma.
\end{aligned} \tag{2.9}$$

Here n is the inward unit normal vector field on ∂G. By applying the divergence theorem on the compact manifold ∂G, which has no boundary, we see that the last line of (2.9) equals

$$(2.10)\quad \int_G f L^* p\, dx + \tfrac{1}{2}\int_{\partial G}\left\{ f\left(\frac{\partial p}{\partial n} - q\cdot\nabla_T p - (\nabla_T\cdot q + 2\mu\cdot n)p\right) - p\,Df\right\} d\sigma.$$

Thus (i′) holds if and only if the following holds for all $f \in C^{2+\varepsilon}(\overline{G})$ that satisfy (2.4):

$$(2.11)\qquad \int_G f L^* p\, dx + \tfrac{1}{2}\int_{\partial G} f\{D^* p - (\nabla_T\cdot q + 2\mu\cdot n)p\}\, d\sigma = 0.$$

Clearly, this holds if (iii) does. On the other hand, by letting f range over functions in $C^{2+\varepsilon}(\overline{G})$ with compact support in G, we see that the above implies $L^*p = 0$ in G, and then since any $f \in C^{2+\varepsilon}(\partial G)$ can be extended to a function in $C^{2+\varepsilon}(\overline{G})$ satisfying (2.4) [cf. [2, page 130]], it follows that the above also implies

$$D^* p = (\nabla_T\cdot q + 2\mu\cdot n)p \quad \text{on } \partial G.$$

Thus, (i′) is equivalent to (iii). Since the first line of (2.9) equals (2.10) for all $f \in C^2(\overline{G})$, it follows that (iii) implies (ii). The observation that (ii) implies (i), which in turn implies (i′), completes the proof. □

3. Skew symmetric vector fields. In this section, vector fields satisfying the skew symmetry condition (2.2) are characterized. First the following preliminary result is established.

LEMMA 3.1. *Suppose q satisfies* (2.2). *Then,*

$$(3.1)\qquad \nabla_T\cdot q = 0 \quad \textit{on } \partial G.$$

PROOF. Fix $\sigma \in \partial G$ and let $e_1,\dots,e_{d-1}$ be an orthonormal basis for the tangent space to ∂G at σ. For any C^1 vector field w defined on ∂G, let $D_{e_i}w(\sigma)$ denote the covariant derivative of w in the direction e_i at σ. Then for any fixed $\sigma^* \in \partial G$ and $i \in \{1,\dots,d-1\}$, by differentiating (2.2) in the direction e_i at σ we obtain

$$(3.2)\qquad D_{e_i}q(\sigma)\cdot n(\sigma^*) = -q(\sigma^*)\cdot D_{e_i}n(\sigma).$$

For each $i \in \{1,\dots,d-1\}$, let $\sigma_i \in \partial G$ such that $n(\sigma_i) = e_i$. Such points σ_i exist because G is bounded and ∂G is sufficiently smooth. Then, setting $\sigma^* = \sigma_i$ in (3.2) yields

$$(3.3)\qquad D_{e_i}q(\sigma)\cdot e_i = -q(\sigma_i)\cdot D_{e_i}n(\sigma).$$

Now, $D_{e_i}n(\sigma)$ lies in the tangent space to ∂G at σ and so may be written

$$(3.4)\qquad D_{e_i}n(\sigma) = \sum_{j=1}^{d-1} h_{ij}(\sigma)e_j.$$

The $h_{ij}(\sigma)$ define the second fundamental form on ∂G and are symmetric with respect to i and j [7, pages 20–23]. Moreover, by (2.2),

$$q(\sigma_i)\cdot e_j = -q(\sigma_j)\cdot e_i, \quad \text{for } i, j \in \{1,\dots,d-1\}, \tag{3.5}$$

since $n(\sigma_k) = e_k$ for $k = 1,\dots,d-1$. Thus combining the above we have

$$\begin{aligned}(\nabla_T \cdot q)(\sigma) &\equiv \sum_{i=1}^{d-1} D_{e_i} q(\sigma)\cdot e_i \\ &= -\sum_{i=1}^{d-1}\sum_{j=1}^{d-1} q(\sigma_i)\cdot e_j h_{ij}(\sigma),\end{aligned} \tag{3.6}$$

where the ij term in the double sum is skew symmetric with respect to the interchange of i and j. Hence the double sum is zero. Thus (3.1) has been proved. □

If $d = 1$, then $q \equiv 0$, so that (2.2) and (3.1) hold trivially. If $d = 2$ and ∂G is connected, it is shown in Example 5.1 that (2.2) is equivalent to q being a constant multiple of the unit tangent vector field to ∂G, which in turn is equivalent to (3.1). However, if $d = 2$ and ∂G is not connected, or $d \ge 3$ (see Example 5.2), then (3.1) does not imply (2.2) in general. Indeed, for $d \ge 3$, the space of divergence free tangential vector fields on ∂G is infinite dimensional in general, whereas the space of tangential vector fields q satisfying (2.2) has dimension $d(d-1)/2$. A proof of the latter is given below.

Let $\sigma_1,\dots,\sigma_d$ be d points on ∂G such that $n(\sigma_1),\dots,n(\sigma_d)$ are linearly independent (but not necessarily orthogonal). For each $i \in \{1,\dots,d\}$, let $q(\sigma_i)$ be a vector in the tangent space to ∂G at σ_i, i.e.,

$$n(\sigma_i)\cdot q(\sigma_i) = 0.$$

Let $\overline{N}$ (respectively $\overline{Q}$) denote the $d \times d$ matrix whose ith row ($i \in \{1,\dots,d\}$) is the vector $n(\sigma_i)'$ [respectively $q(\sigma_i)'$]. (The reason for the bars in this notation will become apparent later in Sections 6 and 7.)

Lemma 3.2. *Suppose*

$$\overline{N}\overline{Q}' + \overline{Q}\overline{N}' = 0, \tag{3.7}$$

i.e., (2.2) *holds at* $\sigma_1,\dots,\sigma_d$. *Then there is a unique extension of* $q(\sigma_1),\dots,q(\sigma_d)$ *to a* $C^{1+\varepsilon}$ (*tangential*) *vector field* q *on* ∂G *that satisfies* (2.2).

Proof. Uniqueness is established first. Suppose q is an extension of $q(\sigma_1),\dots,q(\sigma_d)$ so that (2.2) holds on all of ∂G. By setting $\sigma^* = \sigma_i$ in (2.2), we see that for each $\sigma \in \partial G \setminus \{\sigma_1,\dots,\sigma_d\}$, $q(\sigma)$ must satisfy

$$n(\sigma_i)\cdot q(\sigma) = -q(\sigma_i)\cdot n(\sigma), \quad \text{for } i = 1,\dots,d. \tag{3.8}$$

This may be rewritten in terms of matrices as

$$\overline{N}q(\sigma) = -\overline{Q}n(\sigma). \tag{3.9}$$

Since $n(\sigma_1), \ldots, n(\sigma_d)$ form a basis for R^d, $\overline{N}$ is invertible, and so (3.9) is equivalent to

$$q(\sigma) = -\overline{N}^{-1}\overline{Q}n(\sigma). \tag{3.10}$$

Thus, q is uniquely determined by $q(\sigma_1), \ldots, q(\sigma_d)$ and the normal vector field n on ∂G.

For the proof of existence, suppose q is extended to $\partial G \setminus \{\sigma_1, \ldots, \sigma_d\}$ by (3.10). Observe that (3.10) also holds for $\sigma = \sigma_1, \ldots, \sigma_d$, by (3.7). Since n is a $C^{1+\varepsilon}$ vector field on ∂G, it follows from (3.10) that the same is true of q. To verify (2.2), let $\sigma, \sigma^* \in \partial G$. Then

$$n(\sigma^*) \cdot q(\sigma) = -n(\sigma^*) \cdot (\overline{N}^{-1}\overline{Q}n(\sigma)). \tag{3.11}$$

By premultiplying (3.7) by $\overline{N}^{-1}$ and postmultiplying by $(\overline{N}')^{-1} = (\overline{N}^{-1})'$, we see that

$$\overline{Q}'(\overline{N}^{-1})' = -\overline{N}^{-1}\overline{Q}. \tag{3.12}$$

Then, substituting $-\overline{N}^{-1}Q$ from this in (3.11) and using (3.10) with σ^* in place of σ, we obtain (2.2). Since any vector field q satisfying (2.2) must be tangential to ∂G, the proof is complete. □

COROLLARY 3.1. *There is a one-to-one correspondence between the set of $d \times d$ matrices $\overline{Q}$ satisfying* (3.7) *and the set of (tangential) vector fields q satisfying* (2.2). *As a vector space, this set has dimension $d(d-1)/2$.*

PROOF. The one-to-one correspondence follows immediately from Lemma 3.2. Now, (3.7) is equivalent to the statement "$\overline{N}\overline{Q}'$ is skew symmetric." Since $\overline{N}$ is invertible, it follows that there is a one-to-one correspondence between the $d \times d$ matrices $\overline{Q}$ satisfying (3.7) and the $d \times d$ skew symmetric matrices. The vector space formed by the latter has dimension $d(d-1)/2$. □

4. Proof of Theorem 2.1.

PROOF. By Lemma 2.1, Theorem 2.1(i) holds if and only if for each $\mu \in R^d$ there is $\gamma = \gamma(\mu) \in R^d$ such that $p = \exp\{\gamma \cdot x\}$ satisfies Lemma 2.1(iii), i.e., such that the following two conditions hold:

$$\begin{aligned} &\text{(a)}\ \tfrac{1}{2}|\gamma|^2 - \mu \cdot \gamma = 0, \\ &\text{(b)}\ (n - q) \cdot \gamma = \nabla_T \cdot q + 2\mu \cdot n \quad \text{on } \partial G. \end{aligned} \tag{4.1}$$

Setting $\mu = 0$ in (4.1)(a) yields $\gamma = 0$. Then by substituting this in (4.1)(b) we see that a necessary condition for Theorem 2.1(i) to hold is

$$\nabla_T \cdot q = 0 \quad \text{on } \partial G. \tag{4.2}$$

As in Section 3, let $\sigma_1, \ldots, \sigma_d$ be d points on ∂G such that $n(\sigma_1), \ldots, n(\sigma_d)$ are linearly independent, and let $\overline{N}$ (respectively $\overline{Q}$) denote the $d \times d$ matrix whose ith row is $n(\sigma_i)'$ [respectively $q(\sigma_i)'$]. Then for $\mu \in R^d$, since $\overline{N}$ is invertible, we have that $\gamma \in R^d$ satisfies (4.1) and (4.2) holds only if

$$(I - \overline{N}^{-1}\overline{Q})\gamma = 2\mu. \tag{4.3}$$

Now, there is a solution $\gamma = \gamma(\mu)$ of (4.3) for each $\mu \in R^d$ if and only if $I - \overline{N}^{-1}\overline{Q}$ is invertible, in which case $\{\gamma(\mu) = 2(I - \overline{N}^{-1}\overline{Q})^{-1}\mu \colon \mu \in R^d\} = R^d$. The result of substituting μ from (4.3) in (4.1)(a) is

$$\gamma'\overline{N}^{-1}\overline{Q}\gamma = 0. \tag{4.4}$$

This holds for all $\gamma \in R^d$ if and only if $\overline{N}^{-1}\overline{Q}$ is skew symmetric. Conversely, if $\overline{N}^{-1}\overline{Q}$ is skew symmetric, then $I - \overline{N}^{-1}\overline{Q}$ is invertible. Thus, (4.2), "$\overline{N}^{-1}\overline{Q}$ is skew symmetric," and $\gamma(\mu)$ is related to μ by (4.3), are all necessary conditions for Theorem 2.1(i) to hold. If these are substituted into (4.1)(b), the remaining condition for (i) to hold becomes

$$\begin{aligned} q \cdot \gamma &= n \cdot \left(\overline{N}^{-1}\overline{Q}\gamma\right) \\ &= -\left(\overline{N}^{-1}\overline{Q}n\right) \cdot \gamma \quad \text{on } \partial G, \text{ for all } \gamma \in R^d, \end{aligned} \tag{4.5}$$

where (4.3) was used to obtain the first equality and skew symmetry was used for the second. Since $\gamma \in R^d$ is arbitrary, this is equivalent to

$$q = -\overline{N}^{-1}\overline{Q}n \quad \text{on } \partial G. \tag{4.6}$$

By combining the above we see that for Theorem 2.1(i) to hold, it is necessary and sufficient that the following three conditions hold:

$$\begin{aligned} &\text{(a)} \quad \nabla_T \cdot q = 0 \text{ on } \partial G; \\ &\text{(b)} \quad \overline{N}^{-1}\overline{Q} \text{ is skew symmetric;} \\ &\text{(c)} \quad q = -\overline{N}^{-1}\overline{Q}n \text{ on } \partial G. \end{aligned} \tag{4.7}$$

By a simple calculation, (4.7)(b) is equivalent to:

$$\text{(b}'\text{)} \quad \overline{N}\,\overline{Q}' \text{ is skew symmetric.} \tag{4.7}$$

By Lemma 3.2 [see especially (3.7) and (3.10)], (4.7)(b′) and (c) together are equivalent to:

$$q \text{ satisfies (2.2) on all of } \partial G. \tag{4.8}$$

Moreover, (4.8) implies (4.7)(a), by Lemma 3.1. Thus, Theorem 2.1(i) holds if and only if (4.8) holds, and in this case, $\gamma(\mu) \in R^d$ is given for each $\mu \in R^d$ by

$$\gamma(\mu) = 2\left(I - \overline{N}^{-1}\overline{Q}\right)^{-1}\mu. \tag{4.9}$$

Under the assumption that (2.2) holds, $\overline{N}^{-1}\overline{Q}$, and hence $\gamma(\mu)$, is independent of the particular choice of $\overline{N}$ and $\overline{Q}$. To see this note that (2.2) implies (3.10) and (3.12) from which it follows that

$$q(\sigma) = \left(\overline{N}^{-1}\overline{Q}\right)'n(\sigma), \text{ for all } \sigma \in \partial G.$$

Applying this at points $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_d$ on ∂G such that $n(\tilde{\sigma}_1), \ldots, n(\tilde{\sigma}_d)$ are linearly independent, one obtains for the corresponding matrices $\tilde{N}$ and $\tilde{Q}$:

$$\tilde{N}^{-1}\tilde{Q} = \overline{N}^{-1}\overline{Q},$$

as desired. □

5. Examples.

EXAMPLE 5.1. Suppose $d = 2$ and ∂G is connected. Let τ denote the unit tangent vector field on ∂G, which points in the positive direction as ∂G is traversed in the counterclockwise sense. Let

$$w = q \cdot \tau. \tag{5.1}$$

Now, if q satisfies (2.2), then by Lemma 3.1 it must satisfy (3.1), or equivalently, if ∂G is parameterized by arc-length σ,

$$\frac{dw}{d\sigma} = 0 \quad \text{on } \partial G, \tag{5.2}$$

i.e., w is constant since ∂G is connected. Thus, the space of vector fields q satisfying (2.2) is contained in the one-dimensional space

$$W = \{c\tau \colon c \in R\}. \tag{5.3}$$

On the other hand, by Corollary 3.1, the space of vector fields satisfying (2.2) is one-dimensional when $d = 2$, and so it is precisely W. Thus, we have

$$W = \{q\colon (2.2) \text{ holds}\} = \{q\colon \nabla_T \cdot q = 0\}. \tag{5.4}$$

If the restriction that ∂G is connected were removed, the first equality in (5.4) would still hold, but the space of divergence free tangential vector fields would be larger.

EXAMPLE 5.2. Suppose G is the open unit ball centered at the origin in R^3. Define a tangential vector field q on the unit sphere ∂G as follows. Let (ϕ, θ) denote spherical polar coordinates on ∂G such that $0 \le \phi \le \pi$ and $0 \le \theta < 2\pi$. Let f be a C^∞ nonnegative function on $[0, \pi]$ with compact support in $(0, \pi)$ such that $f(\pi/2) \neq 0$. Let e_θ denote the unit tangent vector in the θ-direction at each point on ∂G except at the north and south poles (where $\phi = 0, \pi$). So e_θ is tangent to the parallels of latitude. Define

$$q(\phi, \theta) = \begin{cases} f(\phi)e_\theta, & \text{for } \phi \neq 0, \pi, \\ 0, & \text{for } \phi = 0, \pi. \end{cases}$$

Then, q is a divergence free vector field on ∂G, i.e., (3.1) holds. On the other hand,

$$q\left(\frac{\pi}{2}, 0\right) = f\left(\frac{\pi}{2}\right) e_\theta|_{\theta=0} \neq 0,$$

and for $\varepsilon > 0$ sufficiently small, $f(\varepsilon) = 0$, so that

$$q\left(\varepsilon, \frac{\pi}{2}\right) = f(\varepsilon) e_\theta|_{\theta=\pi/2} = 0.$$

Thus, for $\sigma = (\tau/2, 0)$ and $\sigma^* = (\varepsilon, \pi/2)$, we have

$$n(\sigma^*) \cdot q(\sigma) + q(\sigma^*) \cdot n(\sigma) = -f(\pi/2)\sin\varepsilon \neq 0.$$

Hence, q does not satisfy (2.2).

EXAMPLE 5.3. As in Example 5.1, let $d = 2$, ∂G be connected, and τ denote the unit tangent vector field on ∂G. Parametrize ∂G by arc-length $\sigma \in [0, \sigma_0)$. Fix $\mu \in R^2$ and define

$$w(s) = -2\int_0^s \mu \cdot n\, d\sigma, \quad \text{for all } s \in [0, \sigma_0). \tag{5.5}$$

Note that if $\mu \neq 0$, then w is not constant. By the divergence theorem,

$$\int_0^{\sigma_0} \mu \cdot n\, d\sigma \equiv \int_{\partial G} \mu \cdot n\, d\sigma = -\int_G \nabla \cdot \mu\, dx = 0.$$

Thus,

$$q(\sigma) = w(\sigma)\tau(\sigma), \quad \text{for } \sigma \in [0, \sigma_0), \tag{5.6}$$

defines a $C^{1+\varepsilon}$ vector field on ∂G. Moreover,

$$\nabla_T \cdot q = \frac{dw}{d\sigma} = -2\mu \cdot n \quad \text{on } \partial G,$$

so for this μ and q, (4.1) holds with $\gamma = 0$. It follows that the stationary density for the RBM associated with this μ and q is of exponential form (in fact it is constant). However, this does not imply the stationary density is of exponential form for the RBM associated with q given by (5.6) and all other μ's in R^d, since the latter would require w to be constant (cf. Example 5.1).

6. Polyhedral case. The data for a reflected Brownian motion (RBM) in a polyhedral domain are as follows (primes denote transposes, vectors without primes are column vectors, and diag$(\cdot)$ denotes the vector formed by the diagonal elements of a square matrix):

(a) integers $k \geq d \geq 1$,
(b) a $k \times d$ matrix N such that diag$(NN') = 1$ and N contains an invertible $d \times d$ submatrix $\overline{N}$,
(c) a $k \times d$ matrix Q such that diag$(QN') = 0$,
(d) a vector $b = (b_1, \ldots, b_k)' \in R^k$, and
(e) a *drift vector* $\mu \in R^d$.

We denote by n_i' and q_i' the ith rows of the matrices N and Q, respectively ($i = 1, \ldots, k$); thus n_i and q_i are both d-dimensional column vectors. We define the convex polyhedron

$$\overline{G} \equiv \{x \in R^d \colon Nx \geq b\},$$

assuming throughout that the interior G is nonempty. It is also assumed that this representation of $\overline{G}$ is irreducible. That is, for any matrix $\tilde{N}$ and column vector $\tilde{b}$ formed by removing one of the rows of N and the corresponding row element of b, the set $\{x \in R^d\colon \tilde{N}x \geq \tilde{b}\}$ is strictly larger than $\bar{G}$. This is equivalent to the assumption that each of the *faces*

$$F_i \equiv \{x \in \bar{G}\colon x \cdot n_i = b_i\}, \qquad i = 1, \ldots, k,$$

has dimension $d - 1$ (cf. Brondsted [1, Theorem 8.2]). The reader will observe that n_i is a unit vector normal to F_i that points into the interior G, whereas q_i is a vector parallel to F_i. We call $v_i \equiv n_i + q_i$ the *direction of reflection* associated with face F_i; n_i and q_i are called the *normal component* and *tangential component*, respectively, of v_i.

The requirement that N contain an invertible $d \times d$ submatrix means that no line can lie entirely within the polyhedron $\bar{G}$. That is, the boundary of the polyhedron must bound each dimension in at least one direction. This condition is necessary for a function of the exponential form $C\exp\{\gamma \cdot x\}$ to be integrable over $\bar{G}$. It is automatically satisfied if $\bar{G}$ is bounded.

Intuitively speaking, by an RBM associated with these data we mean a strong Markov process with continuous sample paths in $\bar{G}$ that (a) behaves like a d-dimensional Brownian motion with covariance matrix I and drift vector μ in G, (b) is reflected at the boundary of G in the direction v_i on the F_i, and (c) spends zero time (in the sense of Lebesgue measure) on the boundary of G. Without further restrictions on the data, there need not exist a well-defined process satisfying these conditions [18]. Indeed, there is no general theory of existence and uniqueness for such RBM's, although some sufficient conditions are known [6, 10, 16, 18]. In the remaining sections of this paper, we study the purely analytic problem of finding solutions of exponential form to the *basic adjoint relation* (BAR) defined in Section 1. Discussion of related probabilistic questions is postponed to a subsequent paper [23].

Recall that for

$$D_i \equiv v_i \cdot \nabla \quad \text{on } F_i,$$

the differential operator D is defined on ∂G such that (a) $D = D_i$ at all points on face F_i that are not also on some other face, and (b) $D = 0$ at the intersections of faces. The following theorem gives a necessary and sufficient condition for the existence of a solution of exponential form for (BAR) for each $\mu \in R^d$. This is the analogue of the result (Theorem 2.1) obtained for smooth bounded domains.

THEOREM 6.1. *For a fixed polyhedron $\bar{G}$ and reflection vectors $v_i = n_i + q_i$ on F_i, $i = 1, \ldots, k$, the following two conditions are equivalent.*

(i) *For each constant drift vector $\mu \in R^d$, there is a vector $\gamma(\mu) \in R^d$ such that $p(x) = \exp\{\gamma(\mu) \cdot x\}$ satisfies* (BAR).

(6.1) (ii) $n_i \cdot q_j + q_i \cdot n_j = 0$, *for all i and j.*

When these conditions hold $\gamma(\mu)$ is unique and is given by formula (4.9).

REMARKS.

1. In words, condition (ii) says that the matrix NQ' is skew symmetric.
2. As per Remark 2 following Theorem 2.1, the entity $\bar{N}^{-1}\bar{Q}$ appearing in formula (4.9) does not depend on the particular choice of $\bar{N}$ and $\bar{Q}$.

If we further assume that the polyhedron is simple, as defined below, then Theorem 6.1 can be considerably strengthened.

DEFINITION. A *vertex* of $\overline{G}$ is a point $x \in \partial G$ where d or more of the faces F_i intersect. The polyhedron $\overline{G}$ is said to be *simple* if each vertex of $\overline{G}$ is contained in *precisely* d of the faces F_i, $i = 1, \ldots, k$.

THEOREM 6.2. *For a fixed simple polyhedron $\overline{G}$ and reflection vectors $v_i = n_i + q_i$ on F_i, $i = 1, \ldots, k$, the following four conditions are equivalent.*

(i) *of Theorem* 6.1.
(ii) *of Theorem* 6.1.
(iii) *The constant function $p(x) \equiv 1$ satisfies* (BAR) *for $\mu = 0$.*
(iv) *For some $\mu \in R^d$ there exists $p \in C^2(\overline{G})$ such that $p > 0$ on $\overline{G}$ and* (BAR) *holds.*

When these conditions hold $\gamma(\mu)$ is unique and is given by formula (4.9).

It will be shown in [23] that if $\overline{G}$ is simple and (6.1) holds, then there is a well-defined RBM associated with $\overline{G}$, $\{v_i\}$ and μ, and if furthermore,

$$C \equiv \int_G \exp\{\gamma(\mu) \cdot x\}\, dx < \infty, \tag{6.2}$$

then the RBM has a unique stationary distribution with density $p(x) = C^{-1}\exp\{\gamma(\mu) \cdot x\}$. If $\overline{G}$ is bounded, then (6.2) automatically holds.

7. Proofs of Theorems 6.1 and 6.2. The following additional notation is needed in the sequel. For each i, let $v_i^* = n_i - q_i$, the *adjoint* direction of reflection to v_i, and define the adjoint differential operator

$$D_i^* = v_i^* \cdot \nabla \quad \text{on } F_i.$$

Let $L^* = \frac{1}{2}\Delta - \mu \cdot \nabla$, as before. For each i and j, let $F_{ij} \equiv F_i \cap F_j$, the intersection of two (possibly nondistinct) faces, and define

$$F_i^0 = F_i \setminus \bigcup_{j \neq i} F_{ij},$$

so that $\bigcup_i F_i^0$ is the smooth part of the boundary of G. We shall use $\partial/\partial n_i \equiv n_i \cdot \nabla$ to denote differentiation in the inward unit normal direction on F_i^0. For the following important preliminary result, we do not actually need the assumption that N contains an invertible $d \times d$ submatrix.

LEMMA 7.1. *Fix $\mu \in R^d$ and suppose $p \in C^2(\overline{G})$ is such that $p > 0$ on $\overline{G}$. Then p satisfies* (BAR) *if and only if the following hold*:

$$L^* p = 0 \quad \text{in } G, \tag{7.1}$$

$$D_i^* p = 2\mu \cdot n_i p \quad \text{on } F_i, \text{ for } i = 1, \ldots, k, \text{ and} \tag{7.2}$$

$$n_i \cdot q_j + q_i \cdot n_j = 0, \quad \text{whenever } F_{ij} \text{ is } (d-2)\text{-dimensional.} \tag{7.3}$$

PROOF. First observe that since $f \in C_c^2(\overline{G})$, $p \in C^2(\overline{G})$, and the surface measure (with respect to σ) of F_{ij} is zero for $j \neq i$, the integral over ∂G in (BAR) is the same as that over $\cup_i F_i^0$. Then, since the boundary of G is piecewise smooth and $f \in C_c^2(\overline{G})$ has compact support in R^d, we can apply Green's theorem and the divergence theorem to conclude that p satisfies (BAR) if and only if

$$\int_G f L^* p \, dx + \tfrac{1}{2} \sum_{i=1}^{k} \int_{F_i^0} \left\{ \left(f \frac{\partial p}{\partial n_i} - p \frac{\partial f}{\partial n_i} - 2\mu \cdot n_i \, p f \right) + p \left(\frac{\partial f}{\partial n_i} + q_i \cdot \nabla f \right) \right\} d\sigma = 0, \tag{7.4}$$

$$\text{for all } f \in C_c^2(\overline{G}).$$

By letting f range over the functions in $C_c^2(\overline{G})$ having compact support in G, we see that (7.4) implies (7.1). After substituting (7.1) back into (7.4) and using the fact that the divergence $\nabla \cdot q_i$ of the constant vector field q_i is zero for each i, we see that (7.4) implies

$$\sum_{i=1}^{k} \int_{F_i^0} \{ f(D_i^* p - 2\mu \cdot n_i p) + \nabla \cdot (q_i p f) \} \, d\sigma = 0, \quad \text{for all } f \in C_c^2(\overline{G}). \tag{7.5}$$

Since q_i is parallel to F_i^0, the divergence in (7.5) is the same as the divergence taken in the $(d-1)$-dimensional manifold F_i^0. So it follows by applying the divergence theorem on each F_i^0 that (7.5) is equivalent to:

$$\sum_{i=1}^{k} \left\{ \int_{F_i^0} f(D_i^* p - 2\mu \cdot n_i \, p) \, d\sigma - \sum_{j \neq i} \int_{F_{ij}} q_i \cdot n_{ij} \, p f \, d\sigma_{ij} \right\} = 0, \tag{7.6}$$

$$\text{for all } f \in C_c^2(\overline{G}),$$

where for $j \neq i$ such that F_{ij} is $(d-2)$-dimensional, σ_{ij} denotes $(d-2)$-dimensional surface measure on F_{ij}, and n_{ij} denotes the unit vector that is normal to both F_{ij} and n_i, and points into F_i^0 from F_{ij}. For all other $j \neq i$, we define σ_{ij} and n_{ij} to be zero. By letting f range over the functions in $C_c^2(\overline{G})$ such that $f|_{\partial G}$ has compact support in F_i^0, we see that (7.6) implies

$$D_i^* p - 2\mu \cdot n_i \, p = 0 \quad \text{on } F_i^0, \text{ for all } i. \tag{7.7}$$

Since $p \in C^2(\overline{G})$ and n_i, q_i and μ are constant vectors, (7.7) extends by continuity to F_i for all i and so implies (7.2). When (7.7) is substituted back into (7.6), the latter reduces to

$$\sum_{i=1}^{k} \sum_{1 \leq j < i} \int_{F_{ij}} (q_i \cdot n_{ij} + q_j \cdot n_{ji}) p f \, d\sigma_{ij} = 0, \quad \text{for all } f \in C_c^2(\overline{G}). \tag{7.8}$$

By letting f range over functions in $C_c^2(\overline{G})$ such that the support of f intersects at most one of the $(d-2)$-dimensional F_{ij} $(= F_{ji})$, and using the assumption that $p > 0$ on $\overline{G}$, we see that (7.8) implies

$$q_i \cdot n_{ij} - q_j \cdot n_{ji} = 0, \quad \text{whenever } F_{ij} \text{ is } (d-2)\text{-dimensional.} \tag{7.9}$$

Now for $j \neq i$ such that F_{ij} is $(d-2)$-dimensional, since n_{ij} is normal to F_{ij}, it must lie in the two-dimensional space spanned by n_i and n_j. The additional conditions that n_{ij} is a unit vector normal to n_i and that it points into F_i^0 from F_{ij} determine it uniquely:

$$n_{ij} = (n_j - n_i \cdot n_j n_i)\Big/\left(1 - (n_i \cdot n_j)^2\right)^{1/2}. \tag{7.10}$$

Similarly,

$$n_{ji} = (n_i - n_i \cdot n_j n_j)\Big/\left(1 - (n_i \cdot n_j)^2\right)^{1/2}. \tag{7.11}$$

By combining this with the fact that $q_m \cdot n_m = 0$ for $m = i, j$, we see that (7.9) is equivalent to (7.3). Thus we have shown that the conditions (7.1)–(7.3) are necessary for p to satisfy (BAR). Reversing the arguments in the above proof shows that these conditions (together) are also sufficient. □

Note that (7.3) is a local form of the skew symmetry condition (6.1). For the next lemma, the reader should recall that $\overline{N}$ denotes an invertible $d \times d$ submatrix of N; we denote by $\overline{Q}$ the corresponding $d \times d$ submatrix of Q.

LEMMA 7.2. *The skew symmetry condition* (6.1) *holds if and only if* Q *satisfies*

$$Q' = -\overline{N}^{-1}\overline{Q}N'. \tag{7.12}$$

Furthermore, the vector space of Q *matrices satisfying* (7.12) *has dimension* $d(d-1)/2$.

PROOF. This follows by applying the same arguments as in Lemma 3.2 and Corollary 3.1 on the smooth part $\cup_i F_i^0$ of ∂G in place of ∂G. □

PROOF OF THEOREM 6.1. By Lemma 7.1, for fixed $\mu \in R^d$, $\gamma = \gamma(\mu) \in R^d$ is such that $p = \exp(\gamma \cdot x)$ satisfies (BAR) if and only if the following three conditions hold:

$$\tfrac{1}{2}|\gamma|^2 - \mu \cdot \gamma = 0, \tag{7.13}$$

$$(n_i - q_i) \cdot \gamma = 2\mu \cdot n_i, \quad \text{for } i = 1, \dots, k, \quad \text{and} \tag{7.14}$$

$$n_i \cdot q_j + q_i \cdot n_j = 0, \quad \text{whenever } F_{ij} \text{ is } (d-2)\text{-dimensional.} \tag{7.15}$$

By the same reasoning as in the proof of Theorem 2.1, there is a solution $\gamma = \gamma(\mu) \in R^d$ of (7.13)–(7.14) for each $\mu \in R^d$ if and only if the skew symmetry condition (6.1) holds, and in this case $\gamma(\mu)$ is given by formula (4.9). Since (6.1) implies (7.15), Theorem 6.1 follows. □

In preparation for the proof of Theorem 6.2, we now prove that under the nondegeneracy assumption that $\overline{G}$ is simple, the local skew symmetry condition (7.3) implies the skew symmetry condition (6.1). For this we introduce the following terminology.

DEFINITION. A face F_i is said to be *incident* to a vertex x of $\overline{G}$ if $x \in F_i$. Two vertices of $\overline{G}$ are to said to be *adjacent* if they have at least $d - 1$ incident faces in common.

LEMMA 7.3. *For a fixed simple polyhedron $\overline{G}$ and reflection vectors $v_i = n_i + q_i$ on F_i, $i = 1, \ldots, k$, the skew symmetry condition* (6.1) *is equivalent to the local skew symmetry condition* (7.3).

REMARK. When $\overline{G}$ is simple, for $j \neq i$, F_{ij} is either empty or $(d-2)$-dimensional [1, Theorem 12.14] and for $j = i$, $n_i \cdot q_i = 0$, so then (7.3) is equivalent to:

$$n_i \cdot q_j + q_i \cdot n_j = 0, \quad \text{whenever } F_{ij} \neq \varnothing. \tag{7.16}$$

PROOF. Clearly (6.1) implies (7.3). To prove the converse, let $\bar{x}$ be a distinguished vertex of $\overline{G}$. (Note that, by the assumption that N contains an invertible $d \times d$ submatrix, $\overline{G}$ must have at least one vertex). Since $\overline{G}$ is simple, precisely d faces, $F_1, \ldots, F_d$, say, meet at $\bar{x}$. The normals associated with these faces form a linearly independent set and so we may take the matrix $\overline{N}$ to be the $d \times d$ innvertible matrix whose ith row is n_i'. Then $\overline{Q}$ is the $d \times d$ matrix whose ith row is q_i'.

Shortly we shall prove the following proposition.

PROPOSITION 7.1. *Suppose the hypotheses of Lemma* 7.3 *hold. If* (7.3) *holds and $\bar{x} = x_1, \ldots, x_s$ are s distinct vertices of $\overline{G}$ such that x_r and x_{r+1} are adjacent for $r = 1, \ldots, s-1$, then for any i such that F_i is incident to at least one of the vertices $x_1, \ldots, x_s$, we have*

$$q_i = -\overline{N}^{-1}\overline{Q}n_i. \tag{7.17}$$

Assuming this proposition holds, the proof of Lemma 7.3 can be completed as follows. For any vertex x of $\overline{G}$, there is a sequence $\bar{x} = x_1, \ldots, x_s = x$ as described in Proposition 7.1. (When $\overline{G}$ is bounded, this follows directly from Theorem 15.5 of Brondsted [1]. If $\overline{G}$ is unbounded, the assertion can be proved as follows. Introduce a new face $F = \{x \in \overline{G}\colon x \cdot n = c\}$ for some unit vector n and constant c such that $\{x \in \overline{G}\colon x \cdot n > c\}$ is bounded and contains all of the vertices of $\overline{G}$. Then apply Theorem 15.5 of [1] to the bounded polyhedron $\{x \in \overline{G}\colon x \cdot n \geq c\}$, with the distinguished face F, to obtain the existence of a sequence $x_1, \ldots, x_s$ as described above.) The assumption that N contains an invertible $d \times d$ submatrix implies that each face is incident to some vertex. Hence, (7.3) implies (7.17) for all i, which is equivalent to (7.12). But by Lemma 7.2, the latter is equivalent to (6.1). Thus, Lemma 7.3 follows. □

PROOF OF PROPOSITION 7.1. Our proof is by induction. Suppose (7.3) or equivalently (7.16) holds and $\bar{x} = x_1, \dots, x_s$ satisfy the hypotheses of the proposition. First suppose F_i is one of the d faces incident to $\bar{x}$. By applying (7.16) with $j = 1, \dots, d$, we see that the following must hold:

$$\overline{Q}n_i + \overline{N}q_i = 0.$$

But since $\overline{N}$ is invertible, this is equivalent to (7.17). For the induction step, suppose that $1 \le r < s$ and for each i such that F_i is incident to at least one of the vertices $x_1, \dots, x_r$, (7.17) holds. Now suppose F_i is a face that is incident to x_{r+1}, but is not incident to any of the vertices $x_1, \dots, x_r$. Since x_r and x_{r+1} are adjacent, they have $d - 1$ faces in common, $F_{r(1)}, \dots, F_{r(d-1)}$, say. These faces together with $F_{r(d)} \equiv F_i$ are distinct and intersect at x_{r+1}, so the normals $n_{r(1)}, \dots, n_{r(d)}$ are linearly independent. By applying (7.16) with $j = r(1), \dots, r(d)$, we obtain

$$q_i \cdot n_{r(m)} = -n_i \cdot q_{r(m)}, \qquad m = 1, \dots, d. \tag{7.18}$$

The equation for $m = d$ is equivalent to

$$q_i \cdot n_i = 0. \tag{7.19}$$

Since the $n_{r(m)}$, $m = 1, \dots, d$, are linearly independent and the $q_{r(m)}$, $m = 1, \dots, d - 1$, are assumed given by (7.17), it follows that q_i is uniquely determined by (7.18). On the other hand, q_i given by (7.17) is a solution of these equations. To see this, suppose that $q_m = -\overline{N}^{-1}\overline{Q}n_m$ for all m. Then by Lemma 7.2, (6.1) holds, which implies (7.16) and hence (7.18). It follows that q_i given by (7.17) is the unique solution of (7.18). This completes the induction step and the proposition follows. □

PROOF OF THEOREM 6.2. Assume the hypotheses of Theorem 6.2 hold. By Lemma 7.1, (iv) implies the local skew symmetry condition (7.3). By Lemma 7.3, condition (7.3) is equivalent to the skew symmetry condition (6.1). By Theorem 6.1, (6.1) is equivalent to (i) and when either of these conditions holds $\gamma(\mu)$ is given by formula (4.9). In particular, for $\mu = 0$, (4.9) yields $\gamma(0) = 0$. Hence (i) implies (iii). The observation that (iii) clearly implies (iv) completes the proof. □

8. Examples.

EXAMPLE 8.1. If $d = 2$, $\overline{G}$ is a convex polygon and is automatically simple because no more than two sides can meet at a vertex. Our requirement that N have an invertible 2×2 submatrix means that $\overline{G}$ has at least one vertex. (The only convex polygons that do not satisfy this condition are a half-plane and an infinite strip.) It also follows that ∂G is connected and (7.3) is equivalent to (6.1), cf. Lemma 7.3. For a geometric interpretation of skew symmetry, let ∂G be oriented, and let τ_i be the unit tangent vector pointing in the positive direction on side i. (Thus $\tau_i \cdot n_i = 0$.) Now for each side $i = 1, \dots, k$, define an *angle of reflection* $\theta_i \in (-\pi/2, \pi/2)$ by the relationship $q_i = \tau_i \tan\theta_i$. These angles are pictured for a typical case in Figure 1, where the vector pointing into the interior

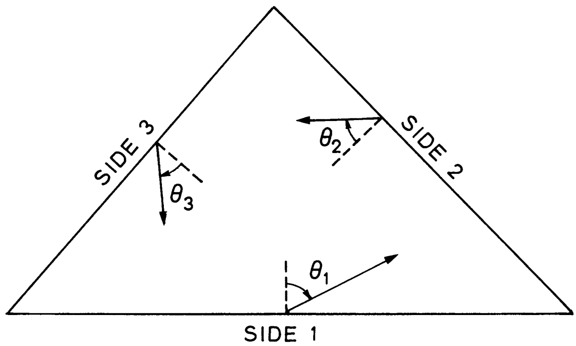

FIG. 1. *Angles of reflection for a typical polygon.*

from side i represents v_i and the positive direction on the boundary is taken to be counterclockwise. By simple geometry, condition (7.3) is equivalent to $\theta_i = \theta_j$ for each pair of sides i and j that intersect. Since the boundary is connected, this is equivalent to a requirement of *constant angle of reflection* over the entire boundary. This is consistent with results obtained in [4] and [22]. In [4] the stationary distribution for driftless RBM in a bounded polygon was obtained. In [22] the density of the invariant measure for null recurrent RBM with zero drift in a two-dimensional wedge was determined. In each of these cases, the density is constant if and only if $\theta_i = \theta_1$ for all i.

EXAMPLE 8.2. The following example shows that when $d \geq 3$ the local skew symmetry condition (7.3) can hold without the skew symmetry condition (6.1) holding, in which case there can be solutions of exponential form for some but not all $\mu \in R^d$. Let $d = 3$ and let N be the 4×3 matrix with rows n_i' defined by

$$\begin{aligned} &n_1' = (1,0,0), \qquad n_2' = (0,1,0), \\ &n_3' = (0,0,1), \qquad n_4' = (1,1,-1)/\sqrt{3}. \end{aligned} \tag{8.1}$$

See Figure 2 for a sketch of the polyhedron $\overline{G} = \{x \in R^3\colon\ Nx \geq 0\}$ in a neighborhood of the origin. There are four faces of $\overline{G}$ and four $(d-2)$-dimensional edges: $F_{13} = F_{31}$, $F_{23} = F_{32}$, $F_{24} = F_{42}$, and $F_{14} = F_{41}$. Consequently, condition (c) of Section 6 and (7.3) impose eight linear conditions on the elements of the matrix Q. These conditions are satisfied if and only if there are are real numbers a, b, c, and d, such that

$$\begin{aligned} &q_1' = (0, a, b), \qquad q_2' = (c, 0, d), \\ &q_3' = (-b, -d, 0), \quad q_4' = (b-a, d-c, b-a+d-c)/\sqrt{3}. \end{aligned} \tag{8.2}$$

Thus, the vector space of matrices Q such that (7.3) holds has dimension 4. On

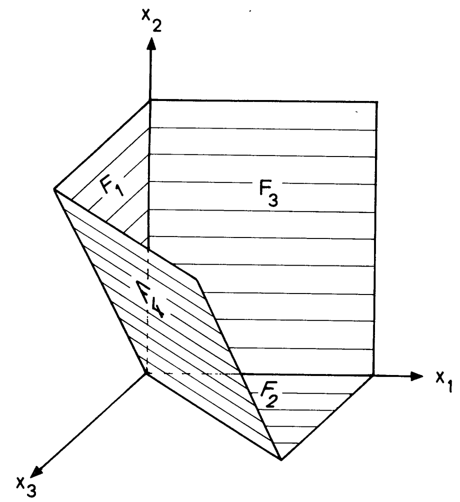

FIG. 2. *Polyhedron $\bar{G}$ of Section 8 in a neighborhood of the origin.*

the other hand, by Lemma 7.2, the vector space of matrices Q satisfying the skew symmetry condition (6.1) has dimension $d(d-1)/2 = 3$ for $d = 3$. Indeed, Q satisfies (6.1) if and only if q_1, q_2, q_3, q_4 are as in (8.2) with $c = -a$.

By Theorem 6.1, the skew symmetry condition (6.1) is a necessary and sufficient condition for (BAR) to have a solution of exponential form for each $\mu \in R^d$. Thus, if the local skew symmetry condition (7.3) holds, but not (6.1), then there is at least one $\mu \in R^d$ such that there is no associated solution of exponential form for (BAR). By Lemma 7.1, for $\mu \in R^d$ fixed, $p = \exp(\gamma \cdot x)$ satisfies (BAR) if and only if (7.3) holds and

$$\tfrac{1}{2}|\gamma|^2 - \mu \cdot \gamma = 0, \tag{8.3}$$

$$(N - Q)\gamma = 2N\mu. \tag{8.4}$$

Since (8.3)–(8.4) clearly hold with $\gamma = 0$ and $\mu = 0$, it follows that $p \equiv 1$ is always a solution of (BAR) when (7.3) holds. The following examples show that when (7.3) holds, but not (6.1), there can be infinitely many γ's (but not all of R^d) such that (8.3)–(8.4) hold for suitable μ's in R^d. Since (8.3) is quadratic, not linear in γ, the collection of all such γ's need not be a vector space.

(a) If the null space of Q is nontrivial, i.e., there is $0 \neq \gamma \in R^d$ such that $Q\gamma = 0$, then (8.3)–(8.4) are satisfied with such a γ and $\mu = \delta/2$. For example, if $b = c = d = 0$ and $a \neq 0$ in (8.2), then the null space of Q is $\{\beta(1,0,-1)'$: $\beta \in R\}$ and so for each $\beta \in R$, $\exp(\beta(x_1 - x_3))$ is a solution of (BAR) with $\mu = (\beta/2)(1,0,-1)'$.

(b) On the other hand, even if the null space of Q is trivial, there can still be nonzero solutions γ of (8.3)–(8.4). For example, if in (8.2), $c = d = 0$ and $a \neq 0$, $b \neq 0$, $b \neq a$, then Q has trivial null space, but (8.3)–(8.4) hold if and only if there is $\beta \in R$ such that either $\gamma = \beta(0,1,-1)'$ and $\mu = (\beta/2)(b-a,1,-1)'$, or $\gamma = \beta(1,0,-1)'$ and $\mu = (\beta/2)(b+1,0,b-1)'$. This can be verified as follows. Observe that if $\overline{N}$ and $\overline{Q}$ denote the 3×3 matrices formed by the first three rows of N and Q, respectively, then (8.3)–(8.4) are equivalent to:

$$\mu = \tfrac{1}{2}(I - \overline{N}^{-1}\overline{Q})\gamma, \tag{8.5}$$

$$\gamma'\overline{N}^{-1}\overline{Q}\gamma = 0, \tag{8.6}$$

$$n_4 \cdot (\overline{N}^{-1}\overline{Q}\gamma) = q_4 \cdot \gamma. \tag{8.7}$$

Here $\overline{N}$ is the 3×3 identity matrix and (8.6) is equivalent to $\gamma_1 = 0$ or $\gamma_2 = 0$, since $a \neq 0$. If $\gamma_1 = 0$, then (8.7) is equivalent to $\gamma_3 = -\gamma_2$, and then (8.5) gives $\mu = (\gamma_2/2)(b-a,1,-1)'$. This corresponds to the first one-parameter family of solutions above. Similarly, by considering $\gamma_2 = 0$, one obtains the other family of solutions. Moreover, for this special case we have obtained all solutions $\gamma \in R^d$ (with associated $\mu \in R^d$) of (8.3)–(8.4). Note that the set of these γ's does *not* form a vector space.

9. Geometric interpretation of local skew symmetry. Since we have already discussed the cases $d = 1, 2$, we suppose $d \geq 3$. Consider $j \neq i$ such that F_{ij} is $(d-2)$-dimensional. Let H be the two-dimensional plane spanned by the vectors n_i and n_j (see Figure 3). This H is orthogonal to F_{ij}, and the vectors n_{ij} and n_{ji} defined in the proof of Lemma 7.1 point away from $H \cap F_{ij}$ along $H_i \equiv H \cap F_i$ and $H_j \equiv H \cap F_j$ respectively. Then, $\tilde{q}_i \equiv n_{ij} \cdot q_i n_{ij}$ denotes the projection of q_i on H and $\tilde{q}_j \equiv n_{ji} \cdot q_j n_{ji}$ denotes the projection of q_j on H. By the proof of Lemma 7.1 [see especially (7.9)], (7.3) holds on F_{ij} if and only if $\tilde{q}_i \cdot n_{ij} = -q_j \cdot n_{ji}$. This is equivalent to saying that the condition (7.3) holds for the RBM in the two-dimensional wedge $\{x \in H\colon\ n_i \cdot x \geq b_i \text{ and } n_j \cdot x \geq b_j\}$ with directions of reflection $n_i + \tilde{q}_i$ and $n_j + \tilde{q}_j$ (cf. the case $d = 2$ in Section 8). Thus, locally, (7.3) reduces to a two-dimensional condition.

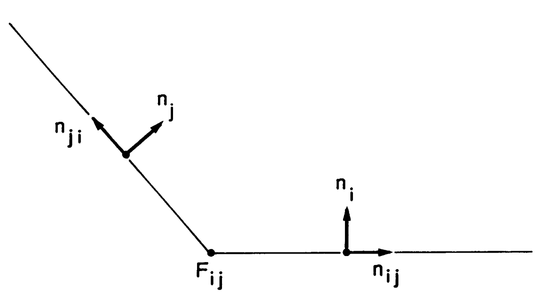

FIG. 3. *Typical cross-section of $\overline{G}$ perpendicular to a $(d-2)$-dimensional edge F_{ij}.*

Acknowledgments. It is a pleasure to thank P. Sarnak and R. Schoen for several helpful conversations. In particular, Example 5.2 is due to P. Sarnak.

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