The Annals of Probability
1987, Vol. 15, No. 1, 102–114

ASYMPTOTIC EXPANSIONS IN BOUNDARY CROSSING PROBLEMS[1]

BY MICHAEL WOODROOFE[2] AND ROBERT KEENER[3]

University of Michigan

Let S_n, $n \geq 1$, be a random walk and $t = t_a = \inf\{n \geq 1: ng(S_n/n) > a\}$. The main results of this paper are two-term asymptotic expansions as $a \to \infty$ for the marginal distributions of t_a and the normalized partial sum $S_t^* = (S_t - t\mu)/\sigma\sqrt{t}$. To leading order, S_t^* has a standard normal distribution. The effect of the randomness in the sample size t on the distribution of S_t^* appears in the correction term of the expansion.

1. Introduction. Let $X_1, X_2, \ldots$ denote i.i.d. random variables with a common distribution function F, a common mean μ and a finite positive variance σ^2; and let

$$S_n = X_1 + \cdots + X_n$$

and

$$S_n^* = (S_n - n\mu)\big/\sigma\sqrt{n}$$

denote the partial sums and standardized partial sums for $n = 1, 2, \ldots$. Next, let $t = t_a$, $a \geq 1$, denote positive integer valued random variables, defined on the same probability space as $X_1, X_2, \ldots$, for which

$$t_a/a \to c, \quad \text{in probability as } a \to \infty, \tag{1}$$

for some finite positive constant c. Then Anscombe's (1952) theorem asserts that the distribution of S_t^* converges to the standard normal distribution Φ as $a \to \infty$,

$$S_t^* \Rightarrow N(0,1).$$

(Here and below t is written for t_a to avoid second order subscripts.) That is, the limiting distribution of the randomly stopped, normalized sums S_t^* as $a \to \infty$ is the same as the limiting distribution of S_n^* as $n \to \infty$. Of course, the randomness in the sample size $t = t_a$ may affect the distribution of S_t^* for each fixed a, as illustrated at the end of this section; but the effect disappears in the limit. To see this effect one must look beyond the limiting distribution.

This paper develops asymptotic expansions for the distributions of S_t^* for a class of stopping times $t = t_a$, $a \geq 1$. Specifically, let g denote a measurable function on $\mathbf{R} = (-\infty, \infty)$ which is twice continuously differentiable near μ and

Received January 1985.
[1]Presented at the Midwest Probability Colloquium.
[2]Research supported by NSF grant DMS-8413452.
[3]Research supported by NSF grant MCS-8102080.
AMS 1980 *subject classifications*. Primary 60F05; secondary 60J15.
Key words and phrases. Nonlinear renewal theory, random walks, excess over the boundary, Edgeworth expansions.

satisfies the conditions $g(\mu) > 0 < g'(\mu)$; let

$$Z_n = ng(S_n/n), \qquad n = 1, 2, \dots, \tag{2}$$

and let

$$t = t_a = \inf\{n \geq 1: Z_n > a\}, \qquad a \geq 1,$$

denote the first passage times for the process Z_n, $n \geq 1$. The main results of this paper determine two-term asymptotic expansions for the distributions of t_a, properly normalized, and S_t^*. The effect of the randomness in the sample size $t = t_a$ on the distribution of S_t^* appears in the correction term of the latter expansion.

These results have potential applications to sequential analysis, where processes of the form Z_n, $n \geq 1$, and stopping times of the form t_a, $a \geq 1$, arise naturally, and t_a represents the sample size of a sequential experiment. See, for example, Woodroofe (1982). The asymptotic expansions of Theorem 1 may provide a better approximation to the distribution of the sample size in such experiments; and, in cases where μ is estimated by S_t/t, the expansions of Theorem 2 may provide more accurate approximations to confidence levels in sequential experiments.

To see the need for improved approximations, consider the simplest case in which X is normally distributed with mean μ and variance 1 and $t = t_a$, $a \geq 1$, are defined by (2) with $g(x) = x$ for all $x \in \mathbf{R}$. Then (1) holds with $c = 1/\mu$ and, therefore, the limiting distribution of S_t^* as $a \to \infty$ is the standard normal distribution, which is also the common distribution of S_n^* for $n = 1, 2, \dots$. Direct use of Anscombe's theorem suggests the approximation $P\{S_t^* \leq 0\} \approx \frac{1}{2}$. On the other hand, Monte Carlo estimates for this probability are 0.408 and 0.424 with a standard error of 0.0025, for $a = 8$ and 12, when $\mu = \frac{1}{2}$. These values of a are modest, but of interest in sequential analysis.

The expansions for the distribution of t and S_t^* are presented in Sections 2 and 3. Some of the terms which appear in the expansions are complicated; these are related to the underlying distribution function F in Section 4. The expansions are compared with simulations and direct normal approximation in Section 5.

The recent paper by Takahashi (1987) is closely related to this one. Using more complicated methods he obtains more detailed expansions in the special case in which F is a normal distribution and $g(x) = x^2/2$ for $-\infty < x < \infty$.

2. First passage times. Some notation and assumptions are required to state the main results. First, it is assumed that F has a finite third central moment ρ and that F is a nonlattice distribution (i.e., not supported by any coset of a proper closed subgroup of $\mathbf{R}$). Let

$$F_n(z) = P\{S_n^* \leq z\} \tag{3}$$

for $z \in \mathbf{R}$ and $n = 1, 2, \dots$; and let Φ and ϕ denote the standard normal distribution function and density function. Then there is a two-term Edgeworth expansion for F_n of the form

$$F_n(z) = \Phi(z) + (1/\sqrt{n})Q_0(z)\phi(z) + o(1/\sqrt{n}), \tag{4}$$

where

$$Q_0(z) = \rho(1 - z^2)/6\sigma^3$$

and $o(1/\sqrt{n})$ is uniform with respect to $z \in \mathbf{R}$ as $n \to \infty$. See, for example, Gnedenko and Kolmogorov (1954, Section 42). This formula is useful below and allows a comparison of the distributions of S_t^* and S_n^*.

Next, it is assumed that there is a closed interval I containing μ in its interior for which g is twice continuously differentiable on I and

$$g(\mu) > 0 < g'(\mu). \tag{5}$$

Let

$$T_n = ng(\mu) + g'(\mu)(S_n - n\mu)$$

for $n = 1, 2, \dots$. Then T_n, $n \geq 1$, is a random walk with i.i.d. increments $Y_k = T_k - T_{k-1}$, $k \geq 1$. The common distribution G of $Y_1, Y_2, \dots$ has mean $\nu = g(\mu) > 0$, standard deviation $\Delta = g'(\mu)\sigma$, and third central moment $g'(\mu)^3\rho$; and G is nonlattice.

The assumptions of the previous two paragraphs are made throughout Sections 2, 3, and 4; they are not repeated in the statements of our lemmas and theorems.

Recall that $Z_n = ng(S_n/n)$ for $n = 1, 2, \dots$ and observe that Z_n may be written in the form $Z_n = T_n + \xi_n$ for $n = 1, 2, \dots$, where ξ_n converges in distribution to $g''(\mu)\sigma^2/2$ times a chi-squared random variable on one degree of freedom. Let

$$Z_n^* = (Z_n - n\nu)/\Delta\sqrt{n}$$

and

$$K_n(z) = P\{Z_n^* \leq z\}$$

for $z \in \mathbf{R}$ and $n = 1, 2, \dots$.

LEMMA 1.

$$K_n(z) = \Phi(z) + Q_1(z)\phi(z)/\sqrt{n} + o(1/\sqrt{n}),$$

where

$$Q_1(z) = Q_0(z) - \tfrac{1}{2}\Delta^{-1}g''(\mu)\sigma^2 z^2, \qquad z \in \mathbf{R}.$$

PROOF. The lemma is a special case of the main result of Bhattacharya and Ghosh (1978); or it may be proved directly from simple Taylor series expansion of the inverse of the restriction of g to a suitable neighborhood of μ. □

In the next lemma, let

$$
(6)\qquad
\begin{aligned}
L_a^-(n, y) &= P\{t_a < n, Z_n - a \le y\},\\
L_a^+(n, y) &= P\{t_a \ge n, Z_n - a > y\},\\
M &= \inf\{T_n\colon n \ge 1\},\\
l_-(y) &= \int_{-\infty}^{y} P\{M < x\}\,dx,\\
l_+(y) &= \int_{y}^{\infty} P\{M \ge x\}\,dx,
\end{aligned}
$$

and

$$a_n = (a - n\nu)/\Delta\sqrt{n}$$

for $y \in \mathbf{R}$ and $n = 1, 2, \dots$.

LEMMA 2. *If $n = n_a \to \infty$ as $a \to \infty$ in such a manner that a_n remains bounded, then*

$$L_a^-(n, y) = \frac{1}{\Delta\sqrt{n}} l_-(y)\phi(a_n) + o\left(\frac{1}{\sqrt{n}}\right)$$

and

$$L_a^+(n, y) = \frac{1}{\Delta\sqrt{n}} l_+(y)\phi(a_n) + o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly on compacts with respect to $y \in \mathbf{R}$ as $a \to \infty$.

Special cases of Lemma 2 appear in Anscombe (1953), without a complete proof, and in Woodroofe (1976), under unnecessary conditions. For $y > 0$, the second assertion of Lemma 2 may be deduced easily from Theorem 7 of Lalley (1984); and Lalley's proof may be extended to all $y \in \mathbf{R}$. The first assertion may be established along similar lines.

It is well known that the normalized first passage times

$$t_a^* = (t_a - a/\nu)/\sqrt{a/\nu}$$

are asymptotically normal with mean zero and standard deviation Δ/ν as $a \to \infty$. The first theorem gives an asymptotic expansion. Its proof follows Anscombe (1953).

THEOREM 1. *If $n = n_a \to \infty$ as $a \to \infty$ in such a manner that a_n remains bounded, then*

$$(7)\qquad P\{t_a \le n\} = (1 - \Phi)(a_n) + (1/\sqrt{n})Q_2(a_n)\phi(a_n) + o(1/\sqrt{n}),$$

where

$$Q_2(z) = \Delta^{-1}l_-(0) - Q_1(z) = \Delta^{-1}l_-(0) + \tfrac{1}{2}\Delta^{-1}g''(\mu)\sigma^2 z^2 - Q_0(z)$$

for $z \in \mathbf{R}$ with Q_1 as in Lemma 1.

PROOF. For all $n \geq 1$ and $a \geq 1$,

$$P\{t \leq n\} = P\{Z_n > a\} + P\{t_a < n, Z_n \leq a\}$$
$$= (1 - K_n)(a_n) + L_a^-(n,0).$$

So, the theorem follows easily from Lemmas 1 and 2 and some simple algebra. □

3. Randomly stopped sums. The following lemma provides a connecting link between the distributions of t_a^* and S_t^*.

LEMMA 3. *For $z \in \mathbf{R}$ and $a \geq 1$, let*

$$\gamma(a, z) = \inf\{x > a/2\nu\colon xg(\mu + \sigma z/\sqrt{x}) > a\}.$$

Then

$$\gamma(a, z) = \frac{a}{\nu} - \frac{z\Delta}{\nu}\sqrt{\frac{a}{\nu}} + \frac{z^2\Delta^2}{2\nu^2} - \frac{1}{2\nu}g''(\mu)\sigma^2 z^2 + o(1)$$

uniformly on compacts with respect to $z \in \mathbf{R}$ as $a \to \infty$.

PROOF. Let I denote a compact subinterval of $\mathbf{R}$. If a is sufficiently large, the $xg(\mu + \sigma z/\sqrt{x})$ is increasing in $x > a/2\nu$ and has a value which is less than a when $x = a/2\nu$, simultaneously for all $z \in I$. So, $\gamma(a, z)$ is the unique solution to the equation $xg(\mu + \sigma z/\sqrt{x}) = a$ for all $z \in I$ when a is sufficiently large. This equation may be written

$$a = x\nu + z\Delta\sqrt{x} + \tfrac{1}{2}g''(\mu)\sigma^2 z^2 + o(1),$$

where $o(1)$ is uniform with respect to $z \in I$ and $x > a/2\nu$ as $a \to \infty$. The equation is quadratic in $\sqrt{x}$. So,

$$\sqrt{\gamma(a, z)} = \frac{1}{2\nu}\left\{\sqrt{4a\nu + \Delta^2 z^2 - 2g''(\mu)\nu\sigma^2 z^2 + o(1)} - \Delta z\right\}.$$

The lemma follows by squaring this relation. □

In the next theorem $N = N(a, z)$ and $\delta = \delta(a, z)$ denote the integral and fractional parts of $\gamma(a, z)$; that is,

$$N = \lfloor \gamma(a, z) \rfloor \quad \text{and} \quad \delta = \gamma(a, z) - N,$$

for $z \in \mathbf{R}$ and $a \geq 1$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Then it is easily verified that

$$a_N = z + \frac{1}{\Delta\sqrt{N}}\left[\frac{1}{2}g''(\mu)\sigma^2 z^2 + \delta\nu\right] + o\left(\frac{1}{\sqrt{N}}\right) \tag{8}$$

uniformly on compacts with respect to $z \in \mathbf{R}$ as $a \to \infty$.

THEOREM 2. *If*

$$P\{t_a \leq a/2\nu\} = o(1/\sqrt{a}), \quad \text{as } a \to \infty, \tag{9}$$

then

$$P\{S_t^* \le z\} = \Phi(z) + \frac{1}{\sqrt{N}} Q(a, z)\phi(z) + o\left(\frac{1}{\sqrt{N}}\right)$$

uniformly on compacts with respect to $z \in \mathbf{R}$ *as* $a \to \infty$, *where*

$$(10)\qquad \begin{aligned} Q(a, z) &= \frac{1}{\Delta}\left[\frac{1}{2}g''(\mu)\sigma^2 z^2 + \delta\nu\right] - Q_2(z) - \frac{1}{\Delta}\sum_{k=1}^{\infty} l_+[(k-\delta)\nu] \\ &= Q_0(z) - \frac{1}{\Delta}[l_-(0) - \delta\nu] - \frac{1}{\Delta}\sum_{k=1}^{\infty} l_+[(k-\delta)\nu] \end{aligned}$$

for $z \in \mathbf{R}$ *and* $a \ge 1$, *with* Q_2 *as in Theorem* 1.

PROOF. Let $\varepsilon > 0$ be so small that g is twice continuously differentiable and $g' > 0$ on $[\mu - \varepsilon, \mu + \varepsilon]$. Then it follows easily from standard maximal inequalities that

$$(11)\qquad P\left\{t_a \le \frac{a}{2\nu} \text{ or } \left|\frac{S_t}{t} - \mu\right| \ge \varepsilon\right\} = o\left(\frac{1}{\sqrt{a}}\right)$$

as $a \to \infty$. Let I denote a compact subinterval of $\mathbf{R}$. If a is sufficiently large and $z \in I$, then the intersection of the events $t_a > a/2\nu$, $|S_t/t - \mu| < \varepsilon$, and $S_t^* \le z$ implies $a < tg(S_t/t) \le tg(\mu + \sigma z/\sqrt{t})$ and, therefore, $t_a > N = N(a, z)$. So,

$$(12)\qquad \begin{aligned} P\{S_t^* \le z\} &= P\left\{t_a > N, \left|\frac{S_t}{t} - \mu\right| < \varepsilon\right\} \\ &\quad - P\left\{t_a > N, S_t^* > z, \left|\frac{S_t}{t} - \mu\right| < \varepsilon\right\} + o\left(\frac{1}{\sqrt{a}}\right), \\ &= I_a - II_a + o\left(\frac{1}{\sqrt{a}}\right), \quad \text{say,} \end{aligned}$$

uniformly with respect to $z \in I$ as $a \to \infty$. These two terms are considered separately.

By (11), $I_a = P\{t_a > N\} + o(1/\sqrt{a})$ uniformly with respect to $z \in I$ as $a \to \infty$. So, I_a may be approximated from Theorem 1. Using (7) and (8) in the computation, one finds that

$$\begin{aligned} I_a &= \Phi(a_N) - \frac{1}{\sqrt{N}} Q_2(z)\phi(z) + o\left(\frac{1}{\sqrt{N}}\right) \\ &= \Phi(z) + \frac{1}{\sqrt{N}}\left\{\frac{1}{\Delta}\left[\frac{1}{2}g''(\mu)\sigma^2 z^2 + \delta\nu\right] - Q_2(z)\right\}\phi(z) + o\left(\frac{1}{\sqrt{N}}\right) \end{aligned}$$

uniformly with respect to $z \in I$ as $a \to \infty$.

To estimate the remaining term, write

$$II_a = \sum_{n > N} l(a, n, z),$$

where

$$l(a, n, z) = P\{t_a = n, S_n > n\mu + z\sigma\sqrt{n}, |S_n/n - \mu| < \varepsilon\}$$

for $n > N$, $z \in I$, and $a \geq 1$. Let

$$G_n = G_n(a, z) = ng(\mu + z\sigma/\sqrt{n})$$

for $n > N$, $z \in I$ and $a \geq 1$. Then $G_n > a$ for all $n > N$ and $z \in I$ for all sufficiently large a, by definition of N; and $P\{|S_n/n - \mu| > \varepsilon\} = o(1/\sqrt{n}^3)$ as $n \to \infty$, since F has a finite third moment. So,

$$l(a, n, z) = L_a^+(n, G_a - a) + o\left(1/\sqrt{n}^3\right)$$

uniformly for $n > N$ and $z \in I$ as $a \to \infty$, where L_a^+ is as in Lemma 2. If $n = N + k$, where k remains fixed as $a \to \infty$, then a simple Taylor series expansion shows that

$$G_n - a = (k - \delta)\nu + o(1), \tag{13}$$

uniformly with respect to $z \in I$ as $a \to \infty$; and it then follows from Lemma 2 that

$$L_a^+(n, G_n - a) = \frac{1}{\sqrt{n}} l_+[(k - \delta)\nu][\phi(a_N) + o(1)]$$

uniformly with respect to $z \in I$ as $a \to \infty$. It is shown in Lemma 4 below that the operations of summation and limit may be interchanged. So,

$$II_a = \frac{1}{\sqrt{N}}\phi(a_N)\sum_{k=1}^{\infty} l_+[(k - \delta)\nu] + o\left(\frac{1}{\sqrt{N}}\right),$$

uniformly with respect to $z \in I$ as $a \to \infty$. The theorem then follows by substituting the asymptotic values of I_a and II_a in (12). This completes the proof, except for Lemma 4.

The main effect of the optional stopping appears as the difference between $Q_0(z)$ and $Q(a, z)$ in (3) and (10). Optional stopping introduces some skewness and the small sawtooth δ. Observe that the curvature $g''(\mu)$ does not appear in (10).

LEMMA 4. *If I is any compact subinterval of R, then there is a summable sequence c_k, $k \geq 1$, for which*

$$L_a^+[n, G_n - a] \leq c_k/\sqrt{n} + o\left(1/\sqrt{n}^3\right)$$

whenever $n = N + k$ with $k \geq 1$, $N = N(a, z)$, $z \in I$, and a is sufficiently large.

PROOF. Let $0 < \varepsilon < 1$ be so small that $g'(\mu)/2 < g'(x) < 2g'(\mu)$ for all $|x - \mu| \leq \varepsilon$, and let B_n be the event

$$B_n = \{|S_{n-1}/(n-1) - \mu| < \varepsilon, |X_n - \mu| < \varepsilon\sqrt{n}\}$$

for $n \geq 2$. Then $P(B_n') = o(1/\sqrt{n}^3)$ as $n \to \infty$, since F has a finite third moment. So,

$$L_a^+(n, y) = P\{t_a \geq n, Z_a > a + y, B_n\} + o\left(1/\sqrt{n}^3\right)$$
$$= l_a(n, y) + o\left(1/\sqrt{n}^3\right), \quad \text{say},$$

uniformly for $y > 0$ and $a \geq 1$ as $n \to \infty$. To estimate l_a, first observe that $l_a(n,y) \leq P\{Z_{n-1} \leq a, Z_n > a + y, B_n\}$ for all $y > 0$, $n \geq 2$, and $a \geq 1$. If B_n occurs then $Z_n \leq Z_{n-1} + c|X_n| + b$ for some positive constants c and b, by the mean value theorem. So, $l_a(n, y) = 0$ when $y > c\varepsilon\sqrt{n} + b$; and, letting H denote the distribution function of $c|X| + b$,

$$\begin{aligned} l_a(n, y) &= P\{Z_{n-1} \leq a, Z_n > a + y, B_n\} \\ &\leq P\{c|X_n| + b > y, a + y - (c|X_n| + b) < Z_{n-1} \leq a, B_n\} \\ &\leq \int_y^\infty P\{a + y - x < Z_{n-1} \leq a\} H(dx) \end{aligned}$$

for $0 < y \leq c\varepsilon\sqrt{n} + b$, $n \geq 2$, and all sufficiently large a.

Here interest centers on the case $n = N + k$ and $y = G_n - a$, where $z \in I$ and a is large. By expanding g in a Taylor series about μ and using the definition of γ, as in (13), it is easily seen that there is an $\eta > 0$ for which $G_n - a > \eta k$ for all $n = N + k$ with $k > 1$ for all $z \in I$ and all sufficiently large a. In particular, $l_a(n, G_n - a) = 0$ when $n = N + k$ with $k > (c\sqrt{n} + b)/\eta$ for $z \in I$ for all sufficiently large a; and if $n = N + k$ with $k \leq (c\sqrt{n} + b)/\eta$, then

$$l_a(n, G_n - a) \leq l_a(n, \eta k)$$

and

$$\begin{aligned} l_a(n, \eta k) &\leq \int_{\eta k}^\infty P\{a - (\eta k - x) < Z_{n-1} \leq a\} H(dx) \\ &\leq \frac{C}{\sqrt{n}} \int_{\eta k}^\infty \left[1 + (x - \eta k)^2\right] H(dx) + o\left(\frac{1}{\sqrt{n}}\right)[1 - H(\eta k)], \end{aligned}$$

by Lemma 1, where $o(1/\sqrt{n})$ is uniform with respect to $z \in I$ and $a \geq 1$ as $n \to \infty$. The lemma now follows easily. □

Condition (9) may be related to moment conditions on X_1 and the sequence $g(S_n/n)$, $n \geq 1$, by writing

$$P\{t \leq a/(2\nu)\} \leq \sum_{k=1}^{M} P\{g(S_k/k) > a/k\} + P\{|S_n/n - \mu| > \eta, \text{ for some } n > M\}$$

for appropriate M and η. For example, if $E|X_1|^p < \infty$ and $E|g(S_n/n)|^q$ remains bounded as $n \to \infty$ for some $p \geq 3$ and $q > (1 + p/2)/(p - 1)$, then letting $M = \lfloor a^{1/p} \rfloor$ shows that (9) is satisfied. If $p = 3$, then the condition requires $q > \frac{5}{4}$. If G is convex then better conditions may be obtained by using the submartingale inequality in place of Boole's inequality in the bound for $P\{t \leq a/(2\nu)\}$, as in Hagwood and Woodroofe (1982). While inelegant, the condition is not very restrictive.

4. Computational formulas. Use of the expansions of Theorems 1 and 2 requires the calculation of $l(0)$ and

$$\mathscr{L}(\delta) = \sum_{k=1}^{\infty} l((k-\delta)\nu)$$

for $0 \le \delta < 1$. These terms are related to the underlying distribution function F of $X_1, X_2, \ldots$, in this section. Closed form expressions are possible only in special cases; but numerical calculations are possible more generally.

Recall that $M = \min\{T_k: k \ge 1\}$ denotes the minimum of the random walk T_k, $k \ge 1$. Let $M^+ = \max\{0, M\}$ and $M^- = \max\{0, -M\}$ denote the positive and negative parts of M; and let ψ, C, C_+, and C_- denote the characteristic functions of $Y_1 = T_1$, M, M^+, and M^-. Then

$$C_-(s) = \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k} \int_{\{T_n \le 0\}} (\exp(isT_k^-) - 1)\, dP \right\} \tag{14}$$

for all $s \in \mathbf{R}$ by Theorem 2 of Feller (1971, page 576). Next, since $M = Y_1 + \min\{T_k - T_1: k \ge 1\}$ and $M = M^+ - M^-$, it is easily seen that

$$C(s) = \psi(s) C_-(-s)$$

and

$$1 - C_+(s) = [1 - \psi(s)] C_-(-s) \tag{15}$$

for all $s \in \mathbf{R}$.

It is easy to compute $l_-(0)$ and $l_+(0)$ from these relations. Indeed, since $l_-(0) = E(M^-)$ and $l_+(0) = E(M^+)$,

$$\begin{aligned} l_-(0) &= -iC_-'(0) = \sum_{k=1}^{\infty} E(T_k^-)/k, \\ l_+(0) &= -iC_+'(0) = \nu, \end{aligned} \tag{16}$$

and

$$E(M^{+2}) = (\Delta^2 + \nu^2) - 2\nu l_-(0).$$

Let H denote the distribution function of M/ν. Then routine calculations show that

$$\begin{aligned} \mathscr{L}(\delta) &= \nu \int_0^{\infty} \lfloor x + \delta \rfloor P\{M/\nu > x\}\, dx \\ &= \nu \int_0^{\infty} \left(\int_0^{y+\delta} \lfloor x \rfloor\, dx \right) H(dy) \\ &= \frac{1}{2\nu} E(M^{+2}) + \left(\delta - \frac{1}{2} \right) E(M^+) - \frac{1}{2} \nu \delta (1 - \delta) \\ &\quad + \frac{\nu}{2} \int_0^{\infty} \langle y + \delta \rangle (1 - \langle y + \delta \rangle) H(dy) \end{aligned} \tag{17}$$

for $0 \le \delta < 1$, where $\lfloor x \rfloor$ and $\langle x \rangle = x - \lfloor x \rfloor$ denote the integer and fractional parts of x for $x > 0$. The penultimate line may be computed from (16). So, it remains to compute the final integral. Let e denote the function defined by $e(x) = x(1 - x)$ for $0 \le x < 1$ and $e(x) = e(1 + x)$ for $x \in \mathbf{R}$. Then e has the Fourier series expansion

$$e(x) = \frac{1}{6} - \sum_{j=1}^{\infty} \cos(2\pi jx)/(\pi j)^2$$

for $x \in \mathbf{R}$. Next, fix $0 \le \delta < 1$ and let $W = \delta + M_+/\nu$. Then the final integral in (17), $I(\delta) = E[e(W)]$, is given by

$$I(\delta) = \frac{1}{6} - \sum_{j=1}^{\infty} \frac{e^{2\pi\delta ji} C_+(2\pi j/\nu) + e^{-2\pi\delta ji} C_+(-2\pi j/\nu)}{2(\pi j)^2} \tag{18}$$

To summarize, $l_-(0)$ is given by (16); and

$$\mathscr{L}(\delta) = \frac{1}{2\nu}(\Delta^2 + \nu^2) - l_-(0) + \frac{\nu}{2}(\delta + \delta^2 - 1) + \nu I(\delta),$$

where $I(\delta)$ is given by (18) for $0 \le \delta < 1$. When these relations are substituted, one finds that

$$Q(a, z) = \frac{\rho(1 - z^2)}{6\sigma^3} - \frac{\Delta^2 + \nu^2}{2\delta\nu} - \frac{\nu\{\delta^2 + 3\delta - 1 + I(\delta)\}}{2\Delta}. \tag{19}$$

To compute $I(\delta)$ using (16), C_+ must be repeatedly evaluated. When $\int|\psi|^p < \infty$ for some p, there is an alternative approach to these calculations that may be more efficient than use of (14) and (15). Let $\tau = \inf\{n \ge 1: T_n > 0\}$ and let χ denote the characteristic function of T_τ. By a duality argument (see Theorem 2.7 of Woodroofe (1982)), $P\{T_\tau > r\} = E(\tau)P\{M^+ > r\}$ for $r > 0$, and hence $C_+ = 1 + (\chi - 1)/E(\tau)$. χ may be computed by quadrature from the formula

$$\chi(s) = 1 - \sqrt{(1 - \psi(s))\left(1 + \frac{i}{\nu s}\right)}$$
$$\times \exp\left\{\frac{i}{2\pi} \lim_{\varepsilon \downarrow 0} \int_{|y-s|>\varepsilon} \log\left((1 - \psi(y))\left(1 + \frac{i}{\nu y}\right)\right) \frac{dy}{y - s}\right\}$$

for $s \in \mathbf{R}$. This identity follows easily from Lemma 2.5 of Keener (1984) (the factor $1 + i/\nu y$ was introduced to make the integrand continuous at $y = 0$).

5. Simulations. In this section the approximations of Theorems 1 and 2 are compared with simulations in the special case that F is a normal distribution. In this case, T_k has a normal distribution for each $k = 1, 2, \ldots$, and the formulas of Section 4 may be easily implemented. Two special cases are studied, the cases in which

$$t_a = \inf\{n \ge 1: S_n > a\}$$

and

$$t_a = \inf\{n \ge 1: |S_n| > \sqrt{2an}\}$$

for $a \geq 1$. Below these are called the cases of straight line and square root boundaries. In both cases, G is taken to be normal with positive mean ν and unit variance, so differences between cases are due to curvature of the boundary. In the curved boundary case, F is normal with mean $\mu = \sqrt{2\nu}$ and variance $\sigma^2 = 1/(2\nu)$, and in the straight line case $F = G$. By focusing on the normal case, the effect of optional stopping on the asymptotic normality of S_t^* may be studied directly, without confounding by such other effects as skewness in the distribution F.

In order to compare the approximations of Theorems 1 and 2 with direct use of normal approximation, a substantial simulation study was conducted. In this study 40,000 independent replications of $t = t_a$ and S_t were generated for selected values of a and μ, and the distribution functions of t_a and S_t^* were estimated from relative frequencies. Thus, the standard deviation of the simulation estimate is at most 0.0025.

Table 1 below compares the approximation to the distribution of t_a given in Theorem 1 with the simulations and direct use of normal approximation. For

TABLE 1

Simulated and approximate values for $P\{t_a \leq n\}$, with $a = 12$. The upper entry is the simulated value; the second entry is the difference between normal approximation and the simulated value; the lower entry is the difference between equation (7) and the simulated value. Each entry is based on 40,000 replications.

	$\nu = 0.50$			$\nu = 1.00$	
n	Straight line	Square root	n	Straight line	Square root
6	0.000	0.008	6	0.008	0.023
	0.037	0.029		0.048	0.033
	0.000	−0.006		0.000	−0.003
12	0.044	0.092	8	0.080	0.106
	0.076	0.028		0.076	0.050
	0.011	0.002		0.005	0.005
18	0.264	0.291	10	0.268	0.285
	0.023	−0.004		0.065	0.048
	0.015	0.006		0.009	0.002
24	0.538	0.543	12	0.511	0.518
	−0.018	−0.023		0.046	0.039
	0.005	0.000		0.004	−0.003
30	0.746	0.753	14	0.715	0.725
	0.000	−0.007		0.050	0.040
	−0.005	−0.002		0.000	−0.003
36	0.870	0.886	16	0.850	0.868
	0.029	0.013		0.053	0.035
	−0.007	−0.003		−0.001	−0.004
42	0.936	0.956	18	0.927	0.945
	0.034	0.014		0.043	0.025
	−0.006	−0.003		−0.001	−0.002

TABLE 2

Simulated and approximate values for $P\{S_t^* \leq z\}$, with $a = 12$. The upper entry is the simulated value, based on 40,000 replications; the second entry is the difference between normal approximation and the simulated value; and the lower entry is the difference between the expansion of Theorem 2 and the simulated value.

	$\nu = 0.50$		$\nu = 1.00$	
z	**Straight line**	**Square root**	**Straight line**	**Square root**
−1.80	0.022	0.023	0.025	0.027
	0.014	0.013	0.011	0.009
	0.003	0.001	0.001	−0.001
−1.20	0.079	0.083	0.089	0.093
	0.036	0.032	0.026	0.022
	0.004	0.000	0.002	0.000
−0.60	0.211	0.215	0.234	0.229
	0.073	0.069	0.050	0.055
	0.004	−0.001	−0.003	0.000
0.00	0.424	0.421	0.440	0.435
	0.076	0.079	0.060	0.065
	−0.007	−0.004	−0.005	0.000
0.60	0.662	0.652	0.683	0.668
	0.064	0.074	0.043	0.058
	−0.013	−0.003	−0.013	−0.004
1.20	0.849	0.836	0.864	0.853
	0.036	0.049	0.021	0.032
	−0.015	−0.003	−0.010	−0.002
1.80	0.951	0.942	0.954	0.952
	0.013	0.022	0.010	0.012
	−0.010	−0.003	−0.005	−0.002

$a = 12$ and selected values of n and μ, the top line lists the relative frequency with which $t_a \leq n$; the second line lists the difference between normal approximation and the simulation estimate; and the third line lists the difference between the approximation of Theorem 1 and the estimate. In the second line, normal approximation means treating t_a as a normally distributed random variable with mean a/ν and variance $a\Delta^2/\nu^3$, after a continuity correction. The results are impressive. The expansions of Theorem 1 are closer to the simulated values than the direct normal approximations in all but two cases, where both are very close.

Table 2 provides similar comparisons for the distribution of S_t^*. The improvements here are also substantial. The expansions of Theorem 2 provide better approximations for negative values of z than for positive ones. This may reflect the relation between small values of t and large values of S_t^*.

Similar results were obtained for $a = 8$, although the agreement was not quite as good. The expansions of Theorems 1 and 2 become less accurate as μ increases for fixed a, since increasing μ decreases t_a. For $a = 8$ and 12, this effect begins to appear at about $\mu = 2$. Similarly, for small values of μ, a diffusion approximation may be adequate and possibly better than the expansions. This effect begins to appear at about $\mu = \frac{1}{4}$, when $a = 8$ for the distribution of t_a.

REFERENCES

ANSCOMBE, F. J. (1952). Large sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

ANSCOMBE, F. J. (1953). Sequential estimation. *J. Roy. Statist. Soc. Ser. B* **15** 1–21.

BHATTACHARYA, R. N. and GHOSH, J. K. (1978). On the validity of the formal Edgeworth expansion. *Ann. Statist.* **6** 434–451.

FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

GNEDENKO, B. and KOLMOGOROV, A. (1954). *Limit Theorems for Sums of Independent Random Variables*. Addison-Wesley, Reading, Mass.

HAGWOOD, C. and WOODROOFE, M. (1982). On the expansion for expected sample size in non-linear renewal theory. *Ann. Probab.* **10** 844–848.

KEENER, R. W. (1984). Asymptotic expansions in non-linear renewal theory. Technical Report 125, Univ. of Michigan.

LALLEY, S. P. (1984). Limit theorems for first passage times in linear and nonlinear renewal theory. *Adv. Appl. Probab.* **16** 766–803.

TAKAHASHI, H. (1987). Asymptotic expansions in Anscombe's theorem for repeated significance tests and estimation after sequential testing. *Ann. Statist.* **15**. To appear.

WOODROOFE, M. (1976). A renewal theorem for curved boundaries and moments of first passage times. *Ann. Probab.* **4** 67–80.

WOODROOFE, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis*. SIAM, Philadelphia.

DEPARTMENT OF STATISTICS
THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109-1027