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NONCENTRAL LIMIT THEOREMS AND APPELL POLYNOMIALS

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Let X_i be a stationary moving average with long-range dependence. Suppose $EX_i = 0$ and $EX_i^{2n} < \infty$ for some $n \geq 2$. When the X_i are Gaussian, then the Hermite polynomials play a fundamental role in the study of noncentral limit theorems for functions of X_i. When the X_i are not Gaussian, the relevant polynomials are Appell polynomials. They satisfy a multinomial-type expansion that can be used to establish noncentral limit theorems.

1. Introduction. Let $G(x)$ be an entire function, X_0 a random variable and let

$$e_k = E\left[\frac{d^k}{dx^k}G(x)\bigg|_{x=X_0}\right]. \tag{1.1}$$

Surgailis (1982) considered a strictly stationary moving average sequence $X_0, X_1, X_2, \dots$ with $EX_i = 0$ and moments of all order that exhibits a long-range dependence. Assuming some additional analytic conditions on G, he proved that if $e_k = 0$ for $k = 0, 1, \dots, n-1$, with $e_n \neq 0$, then $\sum_{i=1}^{[Nt]} G(X_i)$, properly normalized, converges weakly to the Hermite process $Z_n(t)$ (see Section 3 below for details).

We shall focus on the simplest function $P_n(x)$ that is in the domain of attraction of $Z_n(t)$, i.e., a function for which $e_k = 0$ for all $k \neq n$, and which is normalized to satisfy $e_n = n!$. This function is a polynomial of order n, with leading coefficient 1, and it can be defined recursively by

$$P_n'(x) = nP_{n-1}(x), \qquad P_0(x) = 1, \tag{1.2}$$

$$EP_n(X) = \delta_{0,n}, \tag{1.3}$$

where X has the same distribution as X_0. The function is known as the *Appell polynomial* (of order n) associated with the distribution of X. It is also known as a *generalized power*, because it shares many properties with the usual powers. [The Appell polynomials become the powers $P_n(x) = x^n$ when $X = 0$.]

An important particular case, and in fact the first for which limit theorems of the type considered by Surgailis have been established, is $X \sim N(0,1)$ [see Taqqu (1975) and (1979) and Dobrushin and Major (1979)]. When $X \sim N(0,1)$, the Appell polynomials reduce to the Hermite polynomials and in that case

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$EP_n(x)P_m(x) = 0$ for $n \neq m$. The polynomials P_n, however, are in general not orthogonal.

Some of the properties of the Appell polynomials are reviewed in Section 2. We show that they satisfy a multinomial-type expansion and we use that expansion in Section 3 in order to provide a convenient alternative approach to Surgailis' result concerning the convergence to the process $Z_n(t)$. For $P_n(X_i)$ to satisfy such a noncentral limit theorem, it is not necessary to require that all of the moments of X_i be finite.

2. Wick products and Appell polynomials. Let $X_1, X_2, \dots$ be random variables. Following Surgailis (1986), define the *Wick powers* $\langle X_1, X_2, \dots, X_k \rangle$ inductively on k as follows:

Start with $\langle X_1, X_2, \dots, X_k \rangle = 1$ for $k = 0$. Then for any $k > 0$, $\langle X_1, X_2, \dots, X_k \rangle$ is defined recursively for $k = 1, 2, \dots,$ by

$$E\langle X_1, \dots, X_k \rangle = 0 \tag{2.1}$$

and

$$\frac{\partial \langle X_1, \dots, X_k \rangle}{\partial X_i} = \langle X_1, \dots, X_{i-1}, \hat{X}_i, X_{i+1}, \dots, X_k \rangle, \tag{2.2}$$

where $\hat{X}_i$ denotes the absence of the X_i variable. The *Appell polynomials* $P_n(x)$ are then defined by

$$P_{X,n}(X) \equiv P_n(X) = \underbrace{\langle X, \dots, X \rangle}_{n \text{ times}}. \tag{2.3}$$

Relations (1.2) and (1.3) are an immediate consequence of (2.2) and (2.1).

The Wick product $\langle X_1, \dots, X_k \rangle$ is well defined if the joint moments $E\prod_{i \in A} X_i$, exist for all $A \subset \{1, \dots, k\}$. When considering $\langle X_1, \dots, X_k \rangle$, we shall implicitly suppose that the corresponding joint moments exist.

It is easy to see that $\langle X_1, \dots, X_k \rangle$ is a symmetric multilinear form of order k, which involves the random variables $X_1, \dots, X_k$ and their joint moments up to order k. The first two Wick products are

$$\langle X_1 \rangle = X_1 - EX_1,$$
$$\langle X_1, X_2 \rangle = X_1X_2 - X_1EX_2 - X_2EX_1 + 2EX_1EX_2 - EX_1X_2.$$

Denote by X'^n the sequence $(X, X, \dots, X)$ of length n. We start first with a factorization lemma for Wick powers.

FACTORIZATION LEMMA. *If $X_1, X_2, \dots, X_k$ are independent random variables, then*

$$\langle X_1'^{n_1}, X_2'^{n_2}, \dots, X_k'^{n_k} \rangle = \langle X_1'^{n_1} \rangle \langle X_2'^{n_2} \rangle \cdots \langle X_k'^{n_k} \rangle. \tag{2.4}$$

PROOF. For the sake of brevity, we shall prove the lemma in the case where both the generating function (defined below) and the moment generating function $E\exp(\sum_{i=1}^k s_iX_i)$ exist and have a positive radius of convergence. The result holds even if not all moments exist and this can be verified either by induction or by using the algebraic interpretation of generating functions [see Niven (1969)].

Let $A = \{1, \dots, k\}$. Then the generating function of $\langle X_1'^{n_1}, \dots, X_k'^{n_k} \rangle$ is

$$(2.5) \qquad \phi_A(s_i, X_i, i \in A) = \sum_{n_1, \dots, n_k = 0}^{\infty} \frac{s_1^{n_1}}{n_1!} \cdots \frac{s_k^{n_k}}{n_k!} \langle X_1'^{n_1}, \dots, X_k'^{n_k} \rangle.$$

It satisfies

$$(2.6) \qquad \phi_A(s_i, x_i, i \in A) = \frac{\exp\left(\sum_{i=1}^k s_i x_i\right)}{E\left(\exp\left(\sum_{i=1}^k s_i X_i\right)\right)}.$$

To see this, note that (2.1) and (2.2) yield

$$(2.7) \qquad \frac{\partial}{\partial x_1} \phi_A(s_i, x_i, i \in A) = s_1 \phi_A(s_i, x_i, i \in A)$$

and

$$(2.8) \qquad E\phi_A(s_i, x_i, i \in A) = 1.$$

Then integrate (2.7) to get

$$\phi_A(s_i, x_i, i \in A) = \exp\left(\sum_{i=1}^k s_i x_i \right) C(s_1, \dots, s_k)$$

and use (2.8) in that last relation to determine $C(s_1, \dots, s_k)$. This proves (2.6).

Now, if $A = \{1, \dots, k\}$ has a decomposition $A = A_1 \cup A_2$ with $\{X_i, i \in A_1\}$ and $\{X_i, i \in A_2\}$ independent, then it follows from (2.6) that

$$\phi_A(s_i, x_i, i \in A) = \phi_{A_1}(s_i, x_i, i \in A_1) \phi_{A_2}(s_i, x_i, i \in A_2).$$

Identifying the coefficients of $(s_1^{n_1} \cdots s_k^{n_k})/(n_1! \cdots n_k!)$ yields the result. □

Note about generating functions. If the generating function does not have a positive radius of convergence or has undefined coefficients, it cannot be interpreted as an analytic object. However, it can be defined as a formal series, i.e., an algebraic object which allows differentiation, exponentiation, etc. The algebraic structure of formal power series is isomorphic to that of sequences endowed with the convolution product, each series corresponding to the sequence of its coefficients. Thus, equality of two formal power series is interpreted as the equality of their corresponding coefficients. If the coefficients are defined only up to some finite M, then one works with sequences of M elements only.

In the following, we gather some properties of the Appell polynomials $P_n(X) = \langle X, \dots, X \rangle = \langle X'^n \rangle$ which are scattered in the literature [see, for example, Rota (1975), Feinsilver (1978) and Roman (1984)].

Facts about Appell polynomials.

(1) In view of (2.6), the generating function for the Appell polynomials $P_n(X)$,

$$\phi_P(s, x) = \sum_{n=0}^{\infty} \frac{s^n}{n!} P_n(x),$$

satisfies

$$\phi_P(s,x) = \frac{e^{sx}}{Ee^{sX}} = e^{sx-\phi_\kappa(s)},$$

where $\phi_\kappa(s)$ is the generating function of the cumulants κ_n of X, i.e.,

$$\phi_\kappa(s) = \sum_{n=1}^{\infty} \frac{s^n}{n!}\kappa_n = \ln\left(\sum_{n=0}^{\infty} \frac{s^n}{n!} EX^n\right).$$

(2) If $m_n = EX^n$, and $m_1 = EX = 0$, then (1.2) and (1.3) yield

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= x,\\
P_2(x) &= x^2 - m_2,\\
P_3(x) &= x^3 - m_3 - 3m_2 x,\\
P_4(x) &= x^4 - 6m_2x^2 - 4m_3x + 6m_2^2 - m_4,\\
P_5(x) &= x^5 - 10m_2x^3 - 10m_3x^2 + 5x(6m_2^2 - m_4) + 20m_2m_3 - m_5.
\end{aligned}$$

(3) One has

$$P_{n+1}(x) = xP_n(x) - \sum_{k=0}^{n} \binom{n}{k} \kappa_{n-k+1} P_k(x)$$

and

$$x^n = \sum_{k=1}^{n} \binom{n}{k} m_{n-k} P_k(x).$$

(4) If $X \sim N(0,1)$, then all the above relations reduce to the familiar relations characterizing the Hermite polynomials. The generating function becomes $\phi_P(s,x) = e^{sx-s^2/2}$.

We now establish a multinomial-type expansion.

THEOREM 1. *Let $M \geq 1$ be an integer and let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of real numbers satisfying $\sum_{i=1}^{\infty}\lambda_i^2 < \infty$. Let*

$$X = \sum_{i=1}^{\infty} \lambda_i \xi_i,$$

where the ξ_i are i.i.d. with $E|\xi_i|^M < \infty$. [If infinitely many of the λ_i's are not zero, assume also $E\xi_i = 0$ and $E\xi_i^{2M} < \infty$.] Let $P_n(x)$ and $Q_n(x)$, $n \leq M$, denote the Appell polynomials associated with X and ξ, respectively. Then

$$P_n(X) = \sum_{\pi} \binom{n}{p_1, \ldots, p_l} T_\pi = \sum_{\pi} \frac{n!}{p_1! \cdots p_l!} \sum_{(i)_l}{}' \prod_{j=1}^{l} \lambda_{i_j}^{p_j} Q_{p_j}(\xi_{i_j}). \tag{2.9}$$

Here $\sum_\pi$ runs over all $\pi = (p_1, \dots, p_l)$ where $p_1, \dots, p_l$ are integers satisfying $1 \le p_1 < \cdots \le p_l$ and $p_1 + \cdots + p_l = n$. The sum $\sum'_{(i)_l}$ runs over all l-tuples $(i_1, \dots, i_l)$ such that $i_j \ne i_k$ for all $j \ne k$, and, in addition, if $p_j = p_{j+1}$, then $i_j < i_{j+1}$.

EXAMPLE.

$$P_3(\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3) = \sum_{i=1}^{3} \lambda_i^3 Q_3(\xi_i) + 3 \sum_{\substack{i \ne j\\ i,j=1,\dots,3}} \lambda_i \lambda_j^2 Q_1(\xi_i) Q_2(\xi_j)$$
$$+ 6\lambda_1\lambda_2\lambda_3 Q_1(\xi_1)Q_2(\xi_2)Q_3(\xi_3).$$

Here π is in turn (3), (1, 2) and (1, 1, 1).

PROOF. We consider first the finite case

$$X_N = \sum_{i=1}^{N} \lambda_i \xi_i,$$

and we let $P_{X_N, n}$ denote the nth generalized power associated with the distribution of X_N. The multilinearity and symmetry of the Wick product imply

$$P_{X_n, n}(X_N) = \underbrace{\langle X_N, \dots, X_N \rangle}_{n \text{ times}}$$
$$= \sum_\pi \binom{n}{p_1, \dots, p_l} \sum_{(i_l)}' \lambda_{i_1}^{p_1} \cdots \lambda_{i_l}^{p_l} \langle \underbrace{\xi_{i_1}, \dots, \xi_{i_1}}_{p_1 \text{ times}}, \underbrace{\xi_{i_2}, \dots, \xi_{i_2}}_{p_2 \text{ times}}, \dots, \underbrace{\xi_{i_l}, \dots, \xi_{i_l}}_{p_l \text{ times}} \rangle.$$

Since the ξ_i are independent, we have by (2.4)

$$\langle \underbrace{\xi_{i_1}, \dots, \xi_{i_1}}_{p_1 \text{ times}}, \underbrace{\xi_{i_2}, \dots, \xi_{i_2}}_{p_2 \text{ times}}, \dots, \underbrace{\xi_{i_l}, \dots, \xi_{i_l}}_{p_l \text{ times}} \rangle$$
$$= \langle \underbrace{\xi_{i_1}, \dots, \xi_{i_1}}_{p_1 \text{ times}} \rangle \langle \underbrace{\xi_{i_2}, \dots, \xi_{i_2}}_{p_2 \text{ times}} \rangle \cdots \langle \underbrace{\xi_{i_l}, \dots, \xi_{i_l}}_{p_l \text{ times}} \rangle$$
$$= Q_{p_1}(\xi_1) Q_{p_2}(\xi_2) \cdots Q_{p_l}(\xi_l),$$

and thus

$$P_{X_N, n}(X_N) = \sum_\pi \binom{n}{p_1, \dots, p_l} \sum_{\substack{(i)_l\\ i_1, \dots, i_l \le N}}' \prod_{j=1}^{l} \lambda_{i_j}^{p_j} Q_{p_j}(\xi_{i_j}). \tag{2.10}$$

We now show that expression (2.10) continues to hold as $N \to \infty$ if $E\xi_i = 0$ and $E\xi_i^{2M} < \infty$.

Note first that X_N is a martingale in N, and for any even $k \le 2M$,

$$\begin{aligned} EX_N^k &= E\left(\sum_{i=1}^{N} \lambda_i \xi_i\right)^k \\ &= k! \sum_{\substack{p_1 + \cdots + p_N = k \\ p_1, \ldots, p_N \ge 0}} \prod_{j=1}^{N} \frac{\lambda_j^{p_j}}{p_j!} E(\xi_j)^{p_j} \qquad (2.11) \\ &\le k! \alpha^k E|\xi|^k l_k, \end{aligned}$$

where $\alpha = \max_{2 \le p \le k} \sum_{j=1}^{\infty} |\lambda_j|^p$, and l_k is the number of partitions of the integer k. Thus, the martingale X_N converges almost surely and in L^k.

It is sufficient to check that both the left-hand side and right-hand side of (2.10) converge a.s. as $N \to \infty$. Focusing on the left-hand side, we note that since $P_n \equiv P_{X, n}$ is a polynomial in the $n + 1$ variables $EX, EX^2, \ldots, EX^n$ and X, then the random variable $P_{X_N, n}(X_N)$ converges a.s. as $N \to \infty$ to the random variables $P_{X, n}(X)$. As for the right-hand side of (2.10), note that

$$\sum_{\substack{(i)_l \\ i_1, \ldots, i_l \le N}}' \lambda_{i_1}^{p_1} \cdots \lambda_{i_l}^{p_l} Q_{p_1}(\xi_{i_1}) \cdots Q_{p_l}(\xi_{i_l})$$

are martingales in N. A computation similar to (2.11) shows that they are L^2 bounded since $E\xi_i^{2M} < \infty$ and $n \le M$. Therefore the right-hand side of (2.10) also converges a.s., and this concludes the proof. □

3. Noncentral limit theorem. Let $n \ge 1$ be an integer and let $\{\xi_i\}$ be an i.i.d. sequence of random variables with $E\xi_i = 0$ and $E\xi_i^{2n} < \infty$. Let $\frac{1}{2} < \beta < \frac{1}{2}(1 + 1/n)$, $L(x)$ be a slowly varying function as $x \to \infty$, and consider the moving average

$$X_k = \sum_{i=-\infty}^{k} c_{k-i} \xi_i,$$

with

$$c_k \sim k^{-\beta} L(k)$$

as $k \to \infty$. Such a moving average is said to exhibit a long-range dependence. Let $\to_{w^*}$ denote convergence in the CADLAG space $D[0,1]$ endowed with the topology induced by sup-norm convergence.

THEOREM 2. *Let P_n be the nth Appell polynomial associated with the distribution of X_0 and let*

$$H = 1 - n\left(\beta - \tfrac{1}{2}\right).$$

Then as $N \to \infty$,

$$\frac{1}{N^H L^n(N)} \sum_{k=1}^{[Nt]} P_n(X_k) \to_{w^*} Z_n(t),$$

where $Z_n(t)$ is the Hermite process defined by

$$Z_n(t) = n! \int\int_{\{-\infty < u_1 < u_2 < \cdots < u_n < t\}} \cdots$$

$$\times \int \left\{ \int_0^t \prod_{j=1}^{n} \{(v - u_j)_+\}^{-\beta} \, dv \right\} dB(u_1)\, dB(u_2) \cdots dB(u_n)$$

and $B(\cdot)$ is standard Brownian motion.

Theorem 2 follows from Surgailis (1982), but it can also be understood in terms of the multinomial formula as follows:

STEP 1 (multinomial formula). Let

$$T_{k,\pi} = \sum_{(i)_l}{}' c_{k-i_1}^{p_1} \cdots c_{k-i_l}^{p_l} Q_{p_1}(\xi_{i_1}) \cdots Q_{p_l}(\xi_{i_l}),$$

and let

$$U_k^{(n)} = n! \sum_{i_1 < \cdots < i_n} c_{k-i_1} \cdots c_{k-i_l} \xi_{i_1} \cdots \xi_{i_l}$$

be the $T_{k,\pi}$ that corresponds to the π given by $p_1 = \cdots = p_n = 1$. Denote this π as $\pi = (1, \dots, 1)$. The multinomial expansion formula (2.9) becomes

$$P_n(X_k) = U_k^{(n)} + \sum_{\pi \neq (1,\dots,1)} \binom{n}{p_1, \dots, p_l} T_{k,\pi}.$$

STEP 2 [Surgailis (1982), Lemma 5].

$$Z_{n,N}(t) \equiv \frac{1}{N^H L^n(N)} \sum_{k=1}^{[Nt]} U_k^{(n)} \to_{w^*} Z_n(t), \quad \text{as } N \to \infty. \tag{3.1}$$

Surgailis establishes only convergence of the finite-dimensional distributions. However, weak convergence in the w^* sense can be easily established by proving that

$$E|Z_{n,N}(t_2) - Z_{n,N}(t_1)|^{\alpha-\eta} \le M|t_2 - t_1|^{H(\alpha-\eta)},$$

where η is small enough so that $H(\alpha - \eta) > 1$ and M is a constant. For more details, see Fox and Taqqu (1985), Section 5.2.

STEP 3. Using computations similar to those in Lemma 2 of Surgailis (1982), we get

$$E\left[\sum_{k=1}^{N} T_{k,\pi}\right]^2 = O(N), \tag{3.2}$$

for all $\pi = (p_1, \dots, p_l)$ with $p_1 \le p_2 \le \cdots \le p_l$ and $p_l \ge 2$.

Indeed, since $\Sigma_i c_i^2 < \infty$ and $E\xi_1^{2n} < \infty$, it is enough to prove that as $N \to \infty$,

$$(3.3)\quad F \equiv \sum_{(i)_l}{}' \left(\sum_{k=1}^{N} |c_{k-i_1}|^{p_1} \cdots |c_{k-i_l}|^{p_l} \right) \left(\sum_{k=1}^{N} |c_{k-i_1}|^{p_{\sigma_1}} \cdots |c_{k-i_l}|^{p_{\sigma_l}} \right) = O(N),$$

for every $(\sigma_1, \dots, \sigma_l)$ which is a permutation of $(1, \dots, l)$. But

$$F \le \sum_{(i)_l}{}' \left[\max_{\sigma} \left(\sum_{k=1}^{N} |c_{k-i_1}|^{p_{\sigma_1}} \cdots |c_{k-i_l}|^{p_{\sigma_l}} \right) \right]^2$$

$$\le \sum_{(i)_l}{}' \max_{\sigma} \left(\sum_{k=1}^{N} c_{k-i_{j_0}}^2 \right) \left(\sum_{k=1}^{N} \prod_{j=1}^{l} c_{k-i_j}^{2p'_{\sigma_j}} \right),$$

where j_0 is an index such that $p_{\sigma_{j_0}} \ge 2$, and

$$p'_{\sigma_j} = \begin{cases} p_{\sigma_j}, & \text{for } j \ne j_0, \\ p_{\sigma_{j_0}} - 1, & \text{for } j = j_0. \end{cases}$$

Assume now w.l.o.g. that $\Sigma_i c_i^2 < 1$ (and hence $|c_i| < 1$). Then, we also have

$$F \le \sum_{(i)_l}{}' \sum_{k=1}^{N} \prod_{j=1}^{l} c_{k-i_j}^2 \le \sum_{k=1}^{N} \left(\sum_i c_i^2 \right)^l \le N,$$

and thus (3.3) holds.

Since $H > \frac{1}{2}$, relation (3.2) implies

$$\frac{1}{N^H L^n(N)} \sum_{k=1}^{N} T_{k,\pi} \to 0,$$

in probability as $N \to \infty$, and therefore $P_n(X_k)$ satisfies the same noncentral theorem (3.1) as $U_k^{(n)}$. This establishes Theorem 2.

COROLLARY. *If $a_n, a_{n+1}, \dots, a_{n'}$ are real numbers with $n \ne 0$, then $\Sigma_{m=n}^{n'} a_m P_m(X_k)$, $a_n P_n(X_k)$ and $a_n U_k^{(n)}$ all satisfy the same noncentral limit theorem.*

REMARKS. (1) For more information about the limiting process $Z_n(t)$, see Taqqu (1981). (2) Recently Giraitis (1985) and Giraitis and Surgailis (1986) have used Appell polynomials to establish convergence to the finite-dimensional distributions of Brownian motion, in the case where the random variables $X_0, X_1, \dots$ are weakly dependent.

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