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A RATIO LIMIT THEOREM FOR THE TAILS OF WEIGHTED SUMS[1]

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Let $\{Z_\lambda;\ \lambda = 0, \pm 1, \dots\}$ be i.i.d. random variables which have a density f which satisfies $f(z) \sim Kz^\alpha \exp\{-z^p\}$ as $z \to \infty$ for some constants $p > 1$, $K > 0$, and α. Further let q be defined by $p^{-1} + q^{-1} = 1$, and let $\{c_\lambda\}$ be constants with $c_\lambda = O(|\lambda|^{-\theta})$ for some $\theta > \max\{1, 2/q\}$. Then, e.g., if f is symmetric

$$\frac{P\left(\Sigma c_\lambda Z_\lambda > z + x/z^{p/q}\right)}{P(\Sigma c_\lambda Z_\lambda > z)} \to \exp\left\{-p\|c\|_q^{-p} x\right\}, \quad \text{as } z \to \infty,$$

for $\|c\|_q = (\Sigma |c_\lambda|^q)^{1/q}$, and similar results are obtained also for nonsymmetric cases, under some mild further smoothness restrictions. In addition, an order bound for $P(\Sigma c_\lambda Z_\lambda > z)$ itself is obtained, and precise estimates of this quantity are found for the special case of finite sums. In the companion paper [7], the results are crucially used to study extreme values of moving average processes.

1. Introduction. In this paper we study the tail of the distribution of a weighted sum $\Sigma c_\lambda Z_\lambda$ of independent identically distributed (i.i.d.) random variables $\{Z_\lambda;\ \lambda = 0, \pm 1, \dots\}$ for the case when the tail of the marginal distribution of the Z_λ's decreases smoothly, and approximately as $\exp\{-z^p\}$, as $z \to \infty$, for some $p > 1$. In the companion paper [7], the results obtained here are crucially used in studying the intricate and interesting behavior of extremes of a moving average process $\{X_t = \Sigma c_\lambda Z_{t-\lambda}\}$ (continuing the investigation started in [6]). This was the motivation for the present work, but it might be useful also in other contexts—to find the distribution of convolutions is one of the basic problems of probability theory. Also in "geometrical" terms it seems a natural problem to try to find the tail of the distribution of $\Sigma c_\lambda Z_\lambda$; e.g., it includes finding asymptotic estimates for the integral of $\exp\{-\|z\|_p^p\}$ outside an infinite-dimensional hyperplane.

More specifically, we will throughout assume that the distribution of the Z_λ's has a density f which satisfies

$$f(z) \sim Kz^\alpha e^{-z^p}, \quad \text{as } z \to \infty, \text{ with } p > 1, \tag{1.1}$$

for some constants $K > 0$, α and p. Further, we let q be the conjugate exponent of p, defined by $1/p + 1/q = 1$, and let $\{c_\lambda;\ \lambda = 0, \pm 1, \dots\}$ be real constants

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with at least one of the c_λ's strictly positive, which satisfy

$$(1.2)\qquad |c_\lambda| = O(|\lambda|^{-\theta}),\quad \text{as } \lambda \to \pm\infty, \text{for some } \theta > \max(1, 2/q).$$

Let $c_\lambda^+ = \max(0, c_\lambda)$, $c_\lambda^- = \max(0, -c_\lambda)$, and write $\|c\|_q = (\Sigma|c_\lambda|^q)^{1/q}$, $\|c^+\|_q = (\Sigma|c_\lambda^+|^q)^{1/q}$. If some of the c_λ's are negative then the (right) tail of the distribution of $\Sigma c_\lambda Z_\lambda$ is influenced also by the left tail of the marginal distribution of the Z_λ's, and this influence is determined by how a combination of $\{c_\lambda^-\}$ and the left tail of the Z_λ's compares with the corresponding combination of $\{c_\lambda^+\}$ and the right tail of the Z_λ's. For this reason we have to separate three different cases, specified by conditions B.1–B.3 in Section 2 below. We will refer to these as the case of *positive* c_λ's, the case of a *dominating right tail*, and the case of *balanced tails*. (Of course corresponding results for the potential remaining cases of *negative* c_λ's and of a *dominating left tail* are immediate consequences of the results for positive c_λ's and for a dominating right tail.) The main result of this paper is the following description of the local behavior of the tail of the distribution of $\Sigma c_\lambda Z_\lambda$.

THEOREM 1.1. *If assumption* B.1 *or* B.3 *from Section* 2 *is satisfied, then*

$$(1.3)\qquad \frac{P(\Sigma c_\lambda Z_\lambda > z + x/z^{p/q})}{P(\Sigma c_\lambda Z_\lambda > z)} \to \exp\{-p\|c\|_q^{-p}x\},\quad \text{as } z \to \infty,$$

for fixed x, *and if instead* B.2 *holds, then*

$$(1.4)\qquad \frac{P(\Sigma c_\lambda Z_\lambda > z + x/z^{p/q})}{P(\Sigma c_\lambda Z_\lambda > z)} \to \exp\{-p\|c^+\|_q^{-p}x\},\quad \text{as } z \to \infty.$$

Conditions on the tails of the distribution of the Z_λ's are not sufficient to determine the precise global behavior of the tail of $\Sigma c_\lambda Z_\lambda$, but in Section 6 Theorem 1.1 is complemented by order bounds for $P(\Sigma c_\lambda Z_\lambda > z)$, and precise estimates are obtained for the special case of finite sums.

The present problem seems hard, and our proofs are correspondingly long. Some heuristics which originally suggested the results are given in the introduction to [7]. The actual proof uses the "conjugate distributions" introduced by Esscher (1932) and further developed by Cramér (1938), Feller (1969) and many other authors in the context of large deviations in the central limit theorem. The present situation is, however, qualitatively different since it involves infinite sums of nonidentically distributed random variables, rather than finite sums of (more or less) identically distributed variables. Accordingly, it requires a somewhat different use of conjugate distributions, involving sharp estimates of a "local limit" type.

For convenience of notation, let Z be a further random variable which has the same distribution as the Z_λ's. It follows from (1.1), e.g., by partial integration that

$$(1.5)\qquad P(Z > z) \sim \frac{K}{p} z^{\alpha - p + 1} e^{-z^p},\quad \text{as } z \to \infty.$$

To find the tail of the distribution of $\Sigma c_\lambda Z_\lambda$ when (1.5) is satisfied for some $p \in [0,1]$ turns out to be substantially simpler than when $p > 1$. For $0 < p \le 1$ this is done in Sections 7 and 8 of [7], while the case $p = 0$, where necessarily $\alpha < 0$, can be obtained from [3].

An overview of the proof of Theorem 1.1, including a brief discussion of some basic properties of conjugate distributions, is given in Section 3. The conditions and notation needed are collected in Section 2, and the theorem is proved in Sections 4 and 5, with some technicalities postponed to the Appendix. Finally, as discussed above, Section 6 contains some complements, and a brief discussion of the conditions.

2. Conditions. If Theorem 1.1 is known to hold for $\|c\|_q = 1$ [or in case B.2, for $\|c^+\|_q = 1$], then the general result follows at once by dividing by $\|c\|_q$ [or by $\|c^+\|_q$]. Accordingly, *throughout the proofs below we assume that* $\|c\|_q = 1$, for notational convenience.

In addition to (1.1) and (1.2) we always assume that second moments exist, $EZ^2 < \infty$. In particular, then $E|Z_\lambda| = E|Z| < \infty$ for any λ, and since (1.2) implies that $\Sigma|c_\lambda| < \infty$, this ensures that $\Sigma c_\lambda Z_\lambda$ converges a.s. We also need the further assumption that

$$e^{cz} f'(z) \text{ is bounded for } z \in (-\infty, 0], \tag{2.1}$$

for some constant $c \ge 0$. Moreover, defining $D(z) = f(z)e^{z^p}$ for $z \ge 0$, and $D(z) = f(z)$ otherwise so that

$$f(z) = \begin{cases} D(z)e^{-z^p}, & \text{for } z \ge 0, \\ D(z), & \text{for } z < 0, \end{cases} \tag{2.2}$$

with

$$D(z) \sim Kz^\alpha, \quad \text{as } z \to \infty, \tag{2.3}$$

we assume that

$$\limsup_{z\to\infty} \left| \frac{zD'(z)}{D(z)} \right| < \infty. \tag{2.4}$$

Here, of course, f' and D' are the derivatives of f and D. It may be noted that (2.4), e.g., is satisfied if $D(z)$ for large z is a rational function.

The three cases, of positive c_λ's, of a dominating right tail, and of balanced tails, which were discussed in the Introduction, are delineated in the following three conditions,

B.1 (1.1), (1.2), (2.1) and (2.4) hold, and all c_λ's are nonnegative;

B.2 (1.1), (1.2) and (2.4) hold, and in addition $f(-z)$ satisfies (1.1), (2.4), with p in (1.1) replaced by some $p' > p$, and possibly with different D, α, K; and

B.3 (1.1), (1.2) and (2.4) hold, and in addition $f(-z)$ satisfies (1.1), (2.4) with the same p as in (1.1), but possibly with different D, α, K.

In the sequel, C and γ often will be generic constants, whose values may change from one appearance to the next. When limits of summation or integration are deleted then the summation or integration is always from $-\infty$ to $+\infty$. Further $N(0, \sigma^2)$ denotes the normal distribution with mean zero and variance σ^2, the indicator function is denoted I, i.e., $I\{\cdot\}$ is one if the event in curly brackets occurs, and zero otherwise, and finally, $\to_d$ stands for convergence in distribution.

3. Outline of the proof of Theorem 1.1. The distribution $\bar{F}_h$ conjugate to a distribution F is defined by

$$\bar{F}_h(dz) = e^{hz}F(dz)\Big/\int e^{hy}F(dy), \tag{3.1}$$

for $h > 0$ such that $\int e^{hy}F(dy)$ is finite. If Z and $\bar{Z}_h$ are random variables with the distribution of $\bar{Z}_h$ conjugate to the distribution of Z, we write $Z \Leftrightarrow_h \bar{Z}_h$. In particular, if $\bar{F}_h$ is the distribution of $\bar{Z}_h$ we have with this notation that

$$Eg(\bar{Z}_h) = \int g(z)\bar{F}_h(dz) = Eg(Z)e^{hZ}/Ee^{hZ}, \tag{3.2}$$

for any measurable function g. The basic facts we will use about conjugate distributions are that the relation (3.1) of course can be inverted, to yield $F(dz) = e^{-hz}\bar{F}_h(dz)\int e^{hy}F(dy)$, or equivalently that

$$P(Z \in A) = E\left(e^{-h\bar{Z}_h}I\{\bar{Z}_h \in A\}\right)Ee^{hZ}, \tag{3.3}$$

for Borel sets A, if $Z \Leftrightarrow_h \bar{Z}_h$, and that, as can be seen, e.g., from (3.2), the correspondence $\Leftrightarrow_h$ commutes with convolutions, i.e., if Z_λ and $\bar{Z}_{h,\lambda}$, $\lambda = 0, \pm 1, \dots$, are sequences of independent variables and $Z_\lambda \Leftrightarrow_h \bar{Z}_{h,\lambda}$ for each λ, then

$$\sum Z_\lambda \Leftrightarrow_h \sum \bar{Z}_{h,\lambda}, \tag{3.4}$$

provided both sides are well defined. Further, we will make use of the fact that if $c > 0$ is a constant with $Ee^{chZ} < \infty$, and $Z \Leftrightarrow_s \bar{Z}_s$, then

$$cZ \Leftrightarrow_h c\bar{Z}_s, \quad \text{for } s = ch. \tag{3.5}$$

[This follows from (3.2) and the trivial identity

$$Eg((cZ))e^{h(cZ)}/Ee^{h(cZ)} = Eg(cZ)e^{sZ}/Ee^{sZ},$$

which is valid for any measurable g.]

Throughout the rest of the paper we will use the following definitions. The notation above is specialized to assume that $\bar{Z}_h$ and $\bar{Z}_{h,\lambda}$ are defined by requiring that the $\bar{Z}_{h,\lambda}$, $\lambda = 0, \pm 1, \dots$ are mutually independent and that

$$Z \Leftrightarrow_h \bar{Z}_h, \qquad c_\lambda Z_\lambda \Leftrightarrow_h \bar{Z}_{h,\lambda}, \tag{3.6}$$

so that in particular $\Sigma c_\lambda Z_\lambda \Leftrightarrow_h \Sigma \bar{Z}_{h,\lambda}$, for Z, $\{Z_\lambda\}$ and $\{c_\lambda\}$ as defined in Sections 1 and 2. Further, let Z have the moment generating function $\psi(s) = Ee^{sZ}$, which by (1.1) exists for $s \geq 0$, and define

$$\begin{aligned} \Phi_\lambda(h) &= Ee^{hc_\lambda Z_\lambda} = \psi(c_\lambda h), \\ \Phi(h) &= \prod_\lambda \Phi_\lambda(h) = Ee^{h\Sigma c_\lambda Z_\lambda}. \end{aligned} \tag{3.7}$$

The following constants will appear repeatedly in the derivations,

$$z_h = (h/p)^{q/p}, \qquad z_{h,\lambda} = |c_\lambda|^q z_h,$$
$$g_0 = q^{-1}p^{-q/p}, \qquad g_2 = p(p-1)p^{-q}, \tag{3.8}$$
$$\sigma_h = h^{-1/2+q/(2p)}p^{-q/p}g_2^{-1/2}.$$

The major part of the proof is to establish the result for positive c_λ's, and the general case then follows from this by a simple argument. Thus, until further notice is given, we assume that $c_\lambda \geq 0$ for all λ. Now, if Z is replaced by $\Sigma c_\lambda Z_\lambda$ and $\bar{Z}_h$ by $\Sigma \bar{Z}_{h,\lambda}$, for $A = (z_h, \infty)$ the relation (3.3) becomes

$$\begin{aligned} P(\Sigma c_\lambda Z_\lambda > z_h) &= E\Big(e^{-h\Sigma \bar{Z}_{h,\lambda}} I\Big\{\sum \bar{Z}_{h,\lambda} > z_h\Big\}\Big)\Phi(h) \\ &= E\Big(e^{-h\sigma_h \Sigma(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h} \\ &\qquad \times I\Big\{\sum\big(\bar{Z}_{h,\lambda} - z_{h,\lambda}\big)/\sigma_h > 0\Big\}\Big)e^{-hz_h}\Phi(h), \end{aligned} \tag{3.9}$$

since $\Sigma z_{h,\lambda} = z_h \Sigma c_\lambda^q = z_h$, by the assumption that $\|c\|_q = 1$. We will prove (1.3) by estimating the factors in (3.9) separately. The first main step is to show that

$$\sum\big(\bar{Z}_{h,\lambda} - z_{h,\lambda}\big)/\sigma_h \to_d N(0,1), \quad \text{as } h \to \infty. \tag{3.10}$$

Then, if one could replace the distribution of $\Sigma(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h$ by its limiting distribution, this would give that

$$\begin{aligned} &E\Big(e^{-h\sigma_h\Sigma(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h} I\Big\{\sum\big(\bar{Z}_{h,\lambda} - z_{h,\lambda}\big)/\sigma_h > 0\Big\}\Big) \\ &\quad \sim \int_0^\infty e^{-h\sigma_h x}e^{-x^2/2}\,dx/\sqrt{2\pi} \\ &\quad = \big(h\sigma_h\sqrt{2\pi}\big)^{-1}\int_0^\infty e^{-y}e^{-(h\sigma_h)^{-2}y^2/2}\,dy \\ &\quad \sim \big(h\sigma_h\sqrt{2\pi}\big)^{-1}, \quad \text{as } h \to \infty, \end{aligned} \tag{3.11}$$

since $h\sigma_h \to \infty$, as $h \to \infty$. The next step is to show that (3.11) indeed is valid, by proving the "local" limit result that the density of $\Sigma(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h$ converges uniformly to the standard normal density.

Now, let h_* be defined through

$$z_{h_*} = z_h + x/z_h^{p/q}, \tag{3.12}$$

for x fixed. By (3.8), $hz_h = pz_h^p$ and similarly for h_*, and thus

$$\begin{aligned} h_* z_{h_*} - hz_h &= p\big(z_{h_*}^p - z_h^p\big) \\ &= pz_h^p\big((1 + x/z_h^p)^p - 1\big) \\ &= p^2x + o(1), \quad \text{as } h \to \infty. \end{aligned} \tag{3.13}$$

The last main step is to show that

$$\Phi(h_*)/\Phi(h) \to \exp\{p(p-1)x\}, \quad \text{as } h \to \infty. \tag{3.14}$$

It then follows by inserting (3.11), (3.13) and (3.14) into (3.9), that

$$\frac{P\left(\Sigma c_\lambda Z_\lambda > z_h + x/z_h^{p/q}\right)}{P(\Sigma c_\lambda Z_\lambda > z_h)}$$

$$(3.15)\qquad = \frac{P(\Sigma c_\lambda Z_\lambda > z_{h_*})}{P(\Sigma c_\lambda Z_\lambda > z_h)}$$

$$\sim \left(h_*\sigma_{h_*}\sqrt{2\pi}\right)^{-1}\left(h\sigma_h\sqrt{2\pi}\right)\exp\{-p^2x\}\exp\{p(p-1)x\}$$

$$\sim \exp\{-px\}, \quad \text{as } h \to \infty,$$

since $h\sigma_h/(h_*\sigma_{h_*}) \to 1$, as is easily seen. Clearly $z_h \to \infty$ continuously as $h \to \infty$, and thus (3.15) is equivalent to (1.3) (with $\|c\|_q = 1$). Hence, we will prove (1.3) by verifying (3.11) and (3.14).

4. Proof of (3.11). Since Z has the density function f, the conjugate variable $\bar{Z}_h$ has a density $e^{hz}f(z)/\int e^{hy}f(y)\,dy = e^{hz}f(z)/\psi(h)$. Thus $(\bar{Z}_h - z_h)/\sigma_h$ also has a density, say f_h, which is given by

$$(4.1)\qquad f_h(z) = \sigma_h e^{h(z\sigma_h + z_h)}f(z\sigma_h + z_h)/\psi(h).$$

Defining g by

$$(4.2)\qquad g(x) = -p^{-q}\{(1+x)^p - 1 - px\},$$

it follows from (3.8) that

$$h(\sigma_h z + z_h) - (\sigma_h z + z_h)^p = h^q\left(p^{-q/p}(z\sigma_h/z_h + 1) - p^{-q}(z\sigma_h/z_h + 1)^p\right)$$

$$= h^q(g_0 + g(z\sigma_h/z_h)),$$

with $g_0 = q^{-1}p^{-q/p}$, and we can then by (2.2) write f_h, for $\sigma_h z + z_h \ge 0$, as

$$(4.3)\qquad f_h(z) = \sigma_h D(\sigma_h z + z_h)\exp\{h^q(g_0 + g(z\sigma_h/z_h))\}/\psi(h).$$

LEMMA 4.1. *Suppose the density f of the Z_λ's satisfies* (1.1), (2.1) *and* (2.4). *Then*

$$(4.4)\qquad f_h(z) \to e^{-z^2/2}/\sqrt{2\pi}, \quad \text{as } h \to \infty,$$

for fixed z, and hence, by Scheffé's theorem,

$$(4.5)\qquad (\bar{Z}_h - z_h)/\sigma_h \to_d N(0,1), \quad \text{as } h \to \infty.$$

PROOF. According to (3.8),

$$(4.6)\qquad \sigma_h/z_h = h^{-q/2}g_2^{-1/2}.$$

Further, (1.1) implies that (2.3) holds, and thus, for z fixed, $D(\sigma_h z + z_h)/D(z_h) \to 1$ as $h \to \infty$ (since then also $z_h \to \infty$). By Taylor's formula, the function g from (4.2) satisfies

$$g(x) \sim -g_2x^2/2, \quad \text{as } x \to 0,$$

and hence it follows from (4.3) and (4.6) that

$$\frac{f_h(z)}{f_h(0)} = \frac{D(\sigma_h z + z_h)}{D(z_h)} e^{h^q g(z\sigma_h/z_h)} \tag{4.7}$$

$$\to e^{-z^2/2}, \quad \text{as } h \to \infty.$$

By (A.6) of the Appendix, $f_h(z)/f_h(0)$ is bounded by a fixed integrable function for large h, and hence

$$1 = \int f_h(z)\,dz = f_h(0)\int (f_h(z)/f_h(0))\,dz$$

$$\sim f_h(0)\int e^{-z^2/2}\,dz$$

$$= f_h(0)\sqrt{2\pi},$$

as $h \to \infty$, which proves (4.4) for $z = 0$. The rest of (4.4) then follows at once from (4.7). □

We next estimate the first two moments of $(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h$. The estimate for the first moment will be used also in the following section.

Lemma 4.2. *Suppose f satisfies* (1.1), (2.1) *and* (2.4), *and assume that* $c_\lambda \geq 0$. *Then, for some constant $C > 0$ which does not depend on h or λ,*

$$|E(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h| \leq \begin{cases} Cc_\lambda/\sigma_h, & \text{for } c_\lambda < 1/h, \\ C/(h\sigma_h), & \text{for } c_\lambda \geq 1/h, \end{cases} \tag{4.8}$$

and

$$E\left\{\left((\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h\right)^2\right\} \leq \begin{cases} Cc_\lambda^2/\sigma_h^2, & \text{for } c_\lambda < 1/h, \\ Cc_\lambda^q, & \text{for } c_\lambda \geq 1/h. \end{cases} \tag{4.9}$$

Proof. The result trivially holds if $c_\lambda = 0$, so we may assume $c_\lambda > 0$. By definition, $c_\lambda Z \Leftrightarrow_h \bar{Z}_{h,\lambda}$, and hence, according to (3.5), $\bar{Z}_{h,\lambda}$ has the same distribution as $c_\lambda \bar{Z}_s$ for $s = c_\lambda h$ (notation: $\bar{Z}_{h,\lambda} =_d c_\lambda \bar{Z}_s$). Further, by (3.8), $z_{h,\lambda} = c_\lambda^q z_h = c_\lambda z_s$ and hence

$$E(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h = c_\lambda E(\bar{Z}_s - z_s)/\sigma_h \tag{4.10}$$

for $s = c_\lambda h$. Here,

$$E\bar{Z}_s = \int z e^{sz} f(z)\,dz \Big/ \int e^{sz} f(z)\,dz,$$

which by standard properties of moment generating functions is bounded in the bounded interval $0 \leq s = c_\lambda h \leq 1$. Since also z_s is bounded in this interval, this proves the first part of (4.8). The proof of the first part of (4.9) is entirely similar.

It also follows at once that the second part of (4.8), with a suitable choice of C, holds for s in any bounded interval, and similarly for the second part of (4.9), since $c_\lambda^q \geq \text{constant} \times c_\lambda^2/\sigma_h^2$, for $c_\lambda h \geq 1$. Since $s = c_\lambda h$ by definition $c_\lambda \sigma_s \sigma_h^{-1} O(s^{-1}\sigma_s^{-1}) = O(h^{-1}\sigma_h^{-1})$, and thus the second part of (4.8) follows if we prove that

$$\left|\frac{E\bar{Z}_s - z_s}{\sigma_s}\right| = O\big(s^{-1}\sigma_s^{-1}\big) = O\big(s^{-q/2}\big), \quad \text{as } s \to \infty. \tag{4.11}$$

In the same way the second part of (4.9) will be established if we show that

$$E\left(\frac{\bar{Z}_s - z_s}{\sigma_s}\right)^2 = O(1), \quad \text{as } s \to \infty, \tag{4.12}$$

since $c_\lambda^2 \sigma_s^2 \sigma_h^{-2} = c_\lambda^q$. However, it follows from Lemma 4.1 and (A.6), by dominated convergence, that

$$E\left(\frac{\bar{Z}_s - z_s}{\sigma_s}\right)^2 \to 1, \quad \text{as } s \to \infty,$$

which proves (4.12). The proof of (4.11) is similar, but more intricate, and is relegated to the Appendix. □

This result gives estimates for the mean and mean square of the sum of the $\bar{Z}_{h,\lambda}$'s, as follows.

LEMMA 4.3. *Suppose the assumptions of Lemma 4.2 are satisfied and that* $\{c_\lambda\}_{\lambda=-\infty}^{\infty}$ *are nonnegative constants which satisfy* (1.2). *Then*

(i) $$\sum \left| E\big(\bar{Z}_{h,\lambda} - z_{h,\lambda}\big)/\sigma_h \right| \leq C' h^{-q/2+1/\theta},$$

with θ *defined by* (1.2), *for some constant* C', *and*

(ii) $$\limsup_{h\to\infty} E\left\{ \sum_{|\lambda|>\lambda_0} \frac{\bar{Z}_{h,\lambda} - z_{h,\lambda}}{\sigma_h} \right\}^2 \to 0, \quad \text{as } \lambda_0 \to \infty.$$

PROOF. (i) Choose D such that $c_\lambda \leq D|\lambda|^{-\theta}$, for $\lambda \neq 0$, and define $\bar{\lambda} = [D^{1/\theta} h^{1/\theta}]$ so that $c_\lambda < D(D^{1/\theta}h^{1/\theta})^{-\theta} = h^{-1}$ for $|\lambda| > \bar{\lambda}$. Then by (4.8)

$$\begin{aligned}
\sum |E\bar{Z}_{h,\lambda} - z_{h,\lambda}|/\sigma_h &= \left\{ \sum_{|\lambda|\leq\bar{\lambda}} + \sum_{|\lambda|>\bar{\lambda}} \right\} |E\bar{Z}_{h,\lambda} - z_{h,\lambda}|/\sigma_h \\
&\leq C\left\{ (2\bar{\lambda}+1)/h + D \sum_{|\lambda|>\bar{\lambda}} |\lambda|^{-\theta} \right\} \Big/ \sigma_h \\
&= O\big(\bar{\lambda}/h + \bar{\lambda}^{1-\theta}\big)/\sigma_h \\
&= O\big(h^{-1+1/\theta}\sigma_h^{-1}\big),
\end{aligned}$$

which proves part (i) since $h^{-1+1/\theta}\sigma_h^{-1} = \text{constant} \times h^{-q/2+1/\theta}$, by the definition, (3.8).

(ii) Since the $\bar{Z}_{h,\lambda}$'s are independent,

$$E\left\{\sum_{|\lambda|>\lambda_0}\frac{\bar{Z}_{h,\lambda}-z_{h,\lambda}}{\sigma_h}\right\}^2 \le \sum_{|\lambda|>\lambda_0} V\left(\frac{\bar{Z}_{h,\lambda}-z_{h,\lambda}}{\sigma_h}\right) + \left\{\sum_{\lambda}\left|\frac{E\bar{Z}_{h,\lambda}-z_{h,\lambda}}{\sigma_h}\right|\right\}^2.$$

Here the last term tends to zero by (i) and the assumption (1.2) that $\theta > 2/q$. Further, with $\bar{\lambda}$ as above, it follows from (4.9) that

$$\begin{aligned}\sum_{|\lambda|>\lambda_0} V\left(\frac{\bar{Z}_{h,\lambda}-z_{h,\lambda}}{\sigma_h}\right) &\le \sum_{|\lambda|>\lambda_0} E\left(\frac{\bar{Z}_{h,\lambda}-z_{h,\lambda}}{\sigma_h}\right)^2\\ &\le C\left\{\sum_{|\lambda|>\lambda_0} c_\lambda^q + D^2\sum_{|\lambda|>\bar{\lambda}}|\lambda|^{-2\theta}/\sigma_h^2\right\}\\ &= C\sum_{|\lambda|>\lambda_0} c_\lambda^q + O\left(\bar{\lambda}^{1-2\theta}\sigma_h^{-2}\right)\\ &= C\sum_{|\lambda|>\lambda_0} c_\lambda^q + O(h^{1/\theta-q}).\end{aligned}$$

Again by assumption, $h^{1/\theta-q}\to 0$, as $h\to\infty$, and hence it follows that

$$\limsup_{h\to\infty} E\left\{\sum_{|\lambda|>\lambda_0}\frac{\bar{Z}_{h,\lambda}-z_{h,\lambda}}{\sigma_h}\right\}^2 \le C\sum_{|\lambda|>\lambda_0} c_\lambda^q \to 0, \quad \text{as } \lambda_0\to\infty. \qquad \square$$

We are now in a position to prove (3.10) and (3.11).

LEMMA 4.4. *Suppose the assumptions of Lemma* 4.3 *are satisfied. Then*

(4.13) $$\sum\left(\bar{Z}_{h,\lambda}-z_{h,\lambda}\right)/\sigma_h \to_d N(0,1), \quad \text{as } h\to\infty,$$

and (3.11) *holds, i.e.,*

(4.14) $$E\left(e^{-h\sigma_h\Sigma(\bar{Z}_{h,\lambda}-z_{h,\lambda})/\sigma_h} I\left\{\sum\left(\bar{Z}_{h,\lambda}-z_{h,\lambda}\right)/\sigma_h > 0\right\}\right) \sim \left(h\sigma_h\sqrt{2\pi}\right)^{-1},$$

as $h\to\infty$.

PROOF. As in the proof of Lemma 4.2

(4.15) $$\left(\bar{Z}_{h,\lambda}-z_{h,\lambda}\right)/\sigma_h =_d c_\lambda\sigma_s\sigma_h^{-1}\left(\bar{Z}_s-z_s\right)/\sigma_s,$$

for $s = c_\lambda h$, and since $c_\lambda\sigma_s\sigma_h^{-1} = c_\lambda^{q/2}$, it follows from (4.5) that

$$\left(\bar{Z}_{h,\lambda}-z_{h,\lambda}\right)/\sigma_h \to_d N(0, c_\lambda^q), \quad \text{as } h\to\infty.$$

Hence, since the $\bar{Z}_{h,\lambda}$'s are independent,

$$\sum_{|\lambda|\le\lambda_0}\left(\bar{Z}_{h,\lambda}-z_{h,\lambda}\right)/\sigma_h \to N\left(0, \sum_{|\lambda|\le\lambda_0} c_\lambda^q\right), \quad \text{as } h\to\infty,$$

for any λ_0. Combining this with Lemma 4.3(ii) gives [see, e.g., Billingsley (1968), Theorem 4.2] that

$$\sum\left(\bar{Z}_{h,\lambda} - z_{h,\lambda}\right)/\sigma_h \to_d N\left(0, \|c\|_q^q\right), \quad \text{as } h \to \infty.$$

Since $\|c\|_q = 1$, this is the same as (4.13).

We shall next use the following standard facts: If G_1 and G_2 are d.f.'s where G_1 has a bounded continuous density G_1' and G_2 is arbitrary, then the convolution of G_1 and G_2 also has a continuous density which is bounded by the same constant. Similarly, if G_1' has a bounded continuous derivative, then the density of the convolution has a derivative which is bounded by the same constant. [The first result may be obtained by applying the mean value and dominated convergence theorems to show that $\int h^{-1}(G_1(z + h - y) - G_1(z - y))\, dG_2(y) \to \int G_1'(z - y)\, dG_2(y) \le \sup_x |G_1'(x)|$. The second result follows by the same argument].

By (A.5) of the Appendix and Lemma 4.1, $(\bar{Z}_s - z_s)/\sigma_s$ has a uniformly bounded continuously differentiable density, which has a uniformly bounded derivative, and it then follows from (4.15) that $(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h$ has the same property for any λ with $c_\lambda > 0$. Let $\bar{\lambda}$ be such a value. Then, since $\Sigma(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h = (\bar{Z}_{h,\bar{\lambda}} - z_{h,\bar{\lambda}})/\sigma_h + \Sigma_{\lambda \ne \bar{\lambda}}(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h$ it follows that $\Sigma(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h$ has a continuously differentiable density, r_h, say, with both $r_h(z)$ and $|r_h'(z)|$ bounded uniformly in z, $h > h_0$, with h_0 as in Lemma A.1. This together with (4.13) can be seen to imply that $r_h(z)$ converges to $e^{-z^2/2}/\sqrt{2\pi}$, uniformly for z in bounded intervals [this follows, e.g., from the Arzelà-Ascoli theorem, see [1], page 221: we leave the details of the argument to the reader]. Thus, since $h\sigma_h \to \infty$, in particular $r_h(z/(h\sigma_h)) \to 1/\sqrt{2\pi}$, as $h \to \infty$, for fixed z, and in addition r_h is for large h bounded by some constant. Now, the expression on the left in (4.14) equals $\int_0^\infty \exp\{-h\sigma_h z\} r_h(z)\, dz$ and, by dominated convergence,

$$h\sigma_h \int_0^\infty e^{-h\sigma_h z} r_h(z)\, dz = \int_0^\infty e^{-x} r_h(x/(h\sigma_h))\, dx \to \int_0^\infty e^{-x}\, dx/\sqrt{2\pi} = 1/\sqrt{2\pi},$$

which proves (4.14). □

5. Proof of (3.14). We now turn to the result (3.14) on the asymptotic behavior of the moment generating function Φ of $\Sigma c_\lambda Z_\lambda$, recalling that h_* is defined by (3.12), i.e., by

$$z_{h_*} = z_h + x/z_h^{p/q}, \tag{5.1}$$

for x fixed. In the result we include two additional pieces of information, for use in Section 6 and in the companion paper [7], respectively.

LEMMA 5.1. *Suppose the assumptions of Lemma* 4.3 *are satisfied, and let* $\Phi(h) = \Pi_\lambda \Phi_\lambda(h)$ *be as defined by* (3.7). *Then,*

(i) $$\Phi(h_*) \sim \Phi(h)\exp\{p(p-1)x\}, \quad \textit{as } h \to \infty,$$

for fixed x, and the same result holds for general $\|c\|_q$ *provided x is replaced by* $\|c\|_q^q x$.

(ii)
$$\Phi(h) = \exp\{p^{-q/p}q^{-1}h^q + \eta h^{1/\theta}\},$$

where $|\eta| = |\eta(h)|$ *is bounded by some constant which does not depend on* $h > 0$, *and the result holds for general* $\|c\|_q$ *if h is replaced by* $\|c\|_q h$.

(iii) *Let* $\overline{\Phi}_n(h) = \Pi_{n<\lambda}\Phi_\lambda(h)$. *Then, for* $0 < h < n^\theta$,

$$\overline{\Phi}_n(h) \le \exp\Big\{C \sum_{n<\lambda} c_\lambda h\Big\},$$

for some constant C which does not depend on n or h, for h in the specified range, and the same result holds for general $\|c\|_q$.

PROOF. (i) It is straightforward to see that $|\psi(s) - 1| \le \text{constant} \times s$, for $s \ge 0$ in bounded intervals, and since $\Phi_\lambda(h) = \psi(c_\lambda h)$ convergence of the infinite product which defines Φ is assured by $\sum|c_\lambda| < \infty$, which in turn is a consequence of (1.2).

By standard arguments [cf. Feller (1969)] it follows from (3.2) that

$$E\overline{Z}_{h,\lambda} = \Phi_\lambda^{-1}(h)\frac{d}{dh}\Phi_\lambda(h) = \frac{d}{dh}\log\Phi_\lambda(h)$$

and hence

$$\frac{d}{dh}\log\Phi(h) = \sum\frac{d}{dh}\log\Phi_\lambda(h) = \sum E\overline{Z}_{h,\lambda}, \tag{5.2}$$

the interchange of the order of summation and differentiation being permissible since the $E\overline{Z}_{h,\lambda}$'s can be majorized uniformly in bounded h-intervals along the lines of (5.3) below. From Lemma 4.3(i) and (3.8)

$$\begin{aligned} \sum E\overline{Z}_{h,\lambda} &= \sum z_{h,\lambda} + O\Big(\sum|E\overline{Z}_{h,\lambda} - z_{h,\lambda}|\Big) \\ &= z_h + O\big(h^{-q/2+1/\theta}\sigma_h\big) \\ &= z_h + O(h^{-1+1/\theta}), \quad \text{as } h \to \infty. \end{aligned} \tag{5.3}$$

Now, $z_{h_*} = z_h(1 + x/z_h^p)$ by (5.1), and since $z_h = (h/p)^{q/p}$, we have that

$$h_*^{q/p} = h^{q/p}(1 + xp^q/h^q)$$

and thus

$$\begin{aligned} h_* - h &= \{(1 + xp^q/h^q)^{p/q} - 1\}h \\ &\sim p^{q+1}q^{-1}xh^{1-q} \\ &= p^{q/p}p(p-1)xh^{-q/p} \\ &= p(p-1)x/z_h, \quad \text{as } h \to \infty. \end{aligned}$$

Inserting (5.3) into (5.2) and integrating then yields that

$$\begin{aligned}\log\Phi(h_*) - \log\Phi(h) &= \int_h^{h_*} z_s\,ds + O(|h_* - h|h^{-1+1/\theta})\\ &= (h_* - h)z_h + O(|h_* - h|\,|z_{h_*} - z_h|) + o(1)\\ &\to p(p-1)x, \quad \text{as } h \to \infty,\end{aligned}$$

which proves part (i).

(ii) According to (5.2) and (5.3) there is a bounded $\gamma = \gamma(h)$ such that

$$\begin{aligned}\frac{d}{dh}\log\Phi(h) &= z_h + \gamma h^{-1+1/\theta}\\ &= p^{-q/p}h^{q/p} + \gamma h^{-1+1/\theta}.\end{aligned}$$

Since $\Phi(0) = 1$, (ii) follows at once after integration.

(iii) It follows from (4.8) (as was explicitly used in the proof of that inequality), that

$$|E\overline{Z}_{h,\lambda}| \le Cc_\lambda,$$

for $0 < h < n^\theta$. The result then follows from integrating

$$\frac{d}{dh}\log\Phi(h) \le C\sum_{n<\lambda} c_\lambda,$$

in the same way as for part (ii). □

We now have all the ingredients needed to prove the main result.

PROOF OF THEOREM 1.1. Suppose first assumption B.1 from Section 2 is satisfied. Then the hypotheses of Lemma 4.4 and Lemma 5.1 are satisfied, so that (3.11) and (3.14) hold. However, in Section 3 it was shown that this is sufficient to establish (1.3).

Next, assume that B.3 is satisfied. Let

$$\Phi^+(h) = \prod{}^+ \Phi_\lambda(h), \qquad \Phi^-(h) = \prod{}^- \Phi_\lambda(h),$$

where Π^+ and Π^- signify products over λ for which $c_\lambda \ge 0$ and $c_\lambda < 0$, respectively. By Lemma 5.1(i),

$$\Phi^+(h_*) \sim \Phi^+(h)\exp\{p(p-1)\|c^+\|_q^q x\}, \quad \text{as } h \to \infty, \tag{5.4}$$

and since for $c_\lambda < 0$ we may write $c_\lambda Z_\lambda = (-c_\lambda)(-Z_\lambda) = c_\lambda^-(-Z_\lambda)$, and since the density $f(-z)$ of $-Z_\lambda$ is assumed to satisfy the hypothesis of Lemma 5.1(i), it also holds that

$$\Phi^-(h_*) \sim \Phi^-(h)\exp\{p(p-1)\|c^-\|_q^q x\}, \quad \text{as } h \to \infty. \tag{5.5}$$

Since $\Phi(h) = \Phi^+(h)\Phi^-(h)$ and $\|c^+\|_q^q + \|c^-\|_q^q = \|c\|_q^q = 1$, it follows that

$$\Phi(h_*) \sim \Phi(h)\exp\{p(p-1)x\}, \quad \text{as } h \to \infty. \tag{5.6}$$

Similarly, with Σ^+ and Σ^- denoting summation over λ with $c_\lambda \geq 0$ and $c_\lambda < 0$, respectively, we have that, as in the proof of Lemma 4.4,

$$\sum{}^{+}(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h \to_d N(0, \|c^+\|_q^q),$$

$$\sum{}^{-}(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h \to_d N(0, \|c^-\|_q^q), \quad \text{as } h \to \infty.$$

Thus, by independence

$$\sum(\bar{Z}_{h,\lambda} - z_{h,\lambda})/\sigma_h \to_d N(0,1), \quad \text{as } h \to \infty.$$

The remainder of the argument of Lemma 4.4 can now be repeated to show that the same conclusion holds also in the present situation. Thus, since this and (5.6) were the only results needed in the proof of the first part of the present theorem, it follows that the result also holds under assumption B.3.

Finally, if B.2 is satisfied, then again (5.4) holds and, writing $q' = (1 - 1/p')$, with $p' > p$ as specified in B.2, we have that $z_h' = (h/p')^{q'/p'} = (p/p')^{q'/p'} z_h^{pq'/(p'q)}$ and thus, by (5.1),

$$\begin{aligned} z_{h_*}' &= z_h'(1 + x/z_h^p)^{pq'/(p'q)} \\ &= z_h'(1 + O(1/z_h^p)) \\ &= z_h'\left(1 + O\left(1\Big/\left(z_h^{pq'/(p'q)}\right)^{p'q/q'}\right)\right) \\ &= z_h'(1 + O(1)/(z_h')^{p'}), \end{aligned}$$

since $q > q'$ and $z_h \to \infty$.

It then follows from Lemma 5.1(i), with Φ replaced by Φ^- and p by p' that

$$\Phi^-(h_*) \sim \Phi^-(h), \quad \text{as } h \to \infty,$$

since $\Phi^-(h)$ is monotone for large h-values. Similar reasoning shows that the conclusion of Lemma 4.4 holds, with $\|c\|_q = 1$ replaced by $\|c^+\|_q = 1$, and the validity of the result under assumption B.2 now follows in the same way as above. □

6. Complements. Of course the final goal is to find not only the local behavior of the tail of the distribution of $\Sigma c_\lambda Z_\lambda$ as in Theorem 1.1, but to describe the global behavior as well. However, it can be seen that restrictions on the tails of Z alone, as in B.1–B.3, are not enough to determine the tail behavior of $\Sigma c_\lambda Z_\lambda$, in general. Here we will first give a rough estimate for the global behavior which is generally valid, and then discuss a special case, viz. finite sums, where precise estimates can be found.

THEOREM 6.1. *Suppose* B.1 *or* B.3 *holds. Then*

$$P(\Sigma c_\lambda Z_\lambda > z) = \exp\left\{-\left(z/\|c\|_q\right)^p + O(z^\gamma)\right\},$$

for $\gamma = p/(\theta q)$, *and for any constant* $D > 0$ *this is uniform in all* $\{c_\lambda\}$ *satisfying*

$|c_\lambda| \leq D|\lambda|^{-\theta}$, $\lambda \neq 0$. *If instead* B.2 *is satisfied, then*

$$P\left(\sum c_\lambda Z_\lambda > z\right) = \exp\left\{-\left(z/\|c^+\|_q\right)^p + O(z^\gamma)\right\},$$

with $\gamma = p\max(1/(\theta q), q'/q)$, *for* $q' = (1 - 1/p')^{-1}$, *with* p' *given by* B.2, *and the O-term is uniform in the same way as above.*

PROOF. Assume B.1 holds and as usual that $\|c\|_q = 1$. By (3.9), the definition of σ_h, and Lemmas 4.4 and 5.1(ii) we have that

$$P\left(\sum c_\lambda Z_\lambda > z_h\right) \sim \Phi(h)e^{-hz_h}/\left(\sqrt{2\pi}\,h\sigma_h\right)$$
$$= \exp\left\{-hz_h + p^{-q/p}q^{-1}h^q + O(h^{1/\theta})\right\}, \quad \text{as } h \to \infty.$$

As before it follows from (3.8) that $hz_h = pz_h^p$, that $p^{-q/p}q^{-1}h^q = pq^{-1}z_h^p$, and that $h^{1/\theta} = O(z_h^{p/(\theta q)})$, and thus, replacing z_h by z,

$$P\left(\sum c_\lambda Z_\lambda > z\right) = \exp\left\{-\left(p - pq^{-1}\right)z^p + O(z^\gamma)\right\}$$
$$= \exp\left\{-z^p + O(z^\gamma)\right\}, \quad \text{as } z \to \infty.$$

The claimed uniformity can be verified by inspection of the proof. The proof under B.3 is similar. To establish the result under B.2, the proof has to be modified in the same way as in the last part of Theorem 1.1. Since this is straightforward we omit the details. □

Next, we show that if only finitely many, say $k > 0$, of the c_λ's are nonzero and B.1 holds, then

$$P\left(\sum c_\lambda Z_\lambda > z\right) \sim \hat{K}\left(z/\|c\|_q\right)^{\hat{\alpha}}\exp\left\{-\left(z/\|c\|_q\right)^p\right\}, \quad \text{as } z \to \infty, \tag{6.1}$$

with

$$\begin{aligned}\hat{\alpha} &= k\left\{\alpha + \tfrac{1}{2} - p/(2q)\right\} - p/2,\\ \hat{K} &= K^k(2\pi/g_2)^{(k-1)/2}p^{(q/p-1)/2-kq/2}\prod_\lambda\left(c_\lambda/\|c\|_q\right)^{(\alpha+1/2)q/p-1/2}.\end{aligned} \tag{6.2}$$

If instead B.2 or B.3 are satisfied, then (6.1), (6.2) are replaced by slightly more complicated expressions, which we leave to the reader to derive. However, in the special case of B.2 when $f(z)$ is symmetric, (6.1), (6.2) remain unchanged.

These relations can be proved directly, e.g., by partial integration in convolution formulas, but are also readily deduced from the methods used to prove Theorem 1.1. As before, assume for simplicity that $\|c\|_q = 1$ and suppose B.1 holds with only $k < \infty$ nonzero c_λ's. It follows from (4.3) and (4.4), with $z = 0$, that

$$\psi(h) \sim (2\pi)^{1/2}\sigma_h D(z_h)e^{h^q g_0},$$

and hence, writing $s_\lambda = c_\lambda h$ and using that $\|c\|_q = 1$, we have that

$$\Phi(h) = \prod\psi(s_\lambda) \sim (2\pi)^{k/2}\left(\prod\sigma_{s_\lambda}D(z_{s_\lambda})\right)e^{h^q g_0}.$$

Hence, by (3.9) and (4.14),

$$P\Big(\sum c_\lambda Z_\lambda > z_h\Big) \sim \big(h\sigma_h\sqrt{2\pi}\big)^{-1} e^{-hz_h}(2\pi)^{k/2}\big(\Pi \sigma_{s_\lambda} D(z_{s_\lambda})\big)e^{h^q g_0} \tag{6.3}$$

$$\sim K^k (2\pi)^{(k-1)/2}(h\sigma_h)^{-1}\Big(\prod \sigma_{s_\lambda} z_{s_\lambda}^{\alpha}\Big)e^{-(hz_h - h^q g_0)}.$$

Here, by (3.8), $hz_h - h^q g_0 = z_h^p$, and for $\gamma = -\frac{1}{2} + q/(2p)$ also $(h\sigma_h)^{-1} = (h/p)^{-1-\gamma}p^{-1-\gamma+q/p}g_2^{1/2} = z_h^{-(1+\gamma)p/q}p^{(q/p-1)/2}g_2^{1/2}$ and

$$\sigma_{s_\lambda} z_{s_\lambda} = (h/p)^{\gamma} p^{\gamma - q/p} g_2^{-1/2} c_\lambda^{\gamma + \alpha q/p} z_h^{\alpha}$$

$$= p^{\gamma - q/p} g_2^{-1/2} z_h^{\alpha + \gamma p/q} c_\lambda^{\gamma + \alpha q/p}.$$

Thus, writing z for z_h, (6.3) becomes, with $\hat{\alpha}$ as in (6.2),

$$P\Big(\sum c_\lambda Z_\lambda > z\Big) \sim K^k (2\pi/g_2)^{(k-1)/2} p^{(q/p-1)/2 + k(\gamma - q/p)}\Big(\prod c_\lambda^{\gamma+\alpha q/p}\Big) z^{\hat{\alpha}} e^{-z^p},$$

which is the same as (6.1), for $\|c\|_q = 1$. The general result then follows after division by $\|c\|_q$. □

There is of course one more case when $P(\Sigma c_\lambda Z_\lambda > z)$ can be computed explicitly, viz. when Z is normal with mean zero and variance $\frac{1}{2}$, and thus has a density $f(z) = \pi^{-1/2}\exp\{-z^2\}$ which satisfies (1.1) with $K = \pi^{-1/2}$, $\alpha = 0$ and $p = q = 2$. Then $\sqrt{2}\,\Sigma c_\lambda Z_\lambda / \|c\|_q$ is standard normal, and it follows from the estimate $1 - \Phi(z) \sim (2\pi)^{-1}z^{-1}\exp\{-z^2/2\}$ for the standard normal distribution function Φ that

$$P\Big(\sum c_\lambda Z_\lambda > z\Big) \sim 2^{-1}\pi^{-1/2}\big(z/\|c\|_2\big)^{-1} e^{-(z/\|c\|_2)^2}.$$

For the case of a finite sum this agrees with (6.1) as it should since the normal distribution is symmetric.

Finally, the conditions used in Theorem 1.1 and in this section are to some extent used for technical reasons, in order to make proofs work. In particular this is so for the smoothness restriction (2.4), and it seems likely it can be relaxed, and even the existence of a density might not be crucial, but nevertheless it is probably not possible to dispense completely with all smoothness restrictions. The reader is referred to [7], Section 9 for a more detailed discussion of this and of the conditions in general.

APPENDIX

Here we need more precise information on the exponent function (4.2), i.e., on

$$g(x) = -p^{-q}\{(1+x)^p - 1 - px\} \tag{A.1}$$

than the estimate

$$g(x) \sim -g_2 x^2/2, \quad \text{as } x \to 0,$$

for $g_2 = p(p-1)p^{-q}$, which was used in Lemma 4.1. The term in curly brackets in (A.1) is just the remainder term in a first-order Taylor expansion of $(1+x)^p$, i.e.,

$$g(x) = g_2 \int_0^x (y-x)(1+y)^{p-2}\,dy.$$

Hence for any x_0, x_1 with $-1 < x_0 < 0 < x_1$ and for $x \in [x_0, x_1]$ there is a $y^* \in [x_0, x_1]$ with

$$\begin{aligned} g(x) &= g_2(1+y^*)^{p-2} \int_0^x (y-x)\,dx \\ &= g_2(1+y^*)^{p-2}(-x^2/2). \end{aligned}$$

Hence, for $A = 2^{-1} g_2 \min\{(1+y)^{p-2};\ x_0 \le y \le x_1\}$, we have that $A > 0$ and that

$$g(x) < -Ax^2, \quad \text{for } x_0 \le x \le x_1. \tag{A.2}$$

Further, it is seen at once from (A.1) that if x_1 is chosen sufficiently large, then

$$g(x) < -Ax, \quad \text{for } x_1 < x. \tag{A.3}$$

Finally it follows readily from a second-order Taylor expansion that there is a $B > 0$ with

$$|g(x) + g_2 x^2/2| \le B|x|^3, \quad \text{for } x_0 \le x \le x_1. \tag{A.4}$$

LEMMA A.1. *Suppose f satisfies* (1.1), (2.1) *and* (2.4) *and let f_h be as in* (4.1). *Let $-1 < x_0 < -1/p$ and $1 < x_1$ be such that* (A.2)–(A.4) *hold, and put $l_h = x_0 z_h/\sigma_h$, $u_h = x_1 z_h/\sigma_h$, with z_h, σ_h given by* (3.8). *Then there are constants $h_0, C, D > 0$ such that*

$$\left|\frac{f_h'(z)}{f_h(0)}\right| \le \begin{cases} Ce^{-|z|}, & z \notin [l_h, u_h], \\ C(|z|+1)e^{-Dz^2}, & z \in [l_h, u_h], \end{cases} \tag{A.5}$$

and

$$\frac{f_h(z)}{f_h(0)} \le \begin{cases} Ce^{-|z|}, & z \notin [l_h, u_h], \\ C(e^{-Dz^2} + e^{-|z|}), & z \in [l_h, u_h], \end{cases} \tag{A.6}$$

for all $h \ge h_0$.

PROOF. It is seen from (2.1) and (4.1) that $f_h(\pm\infty) = 0$ for $h > c$, and hence (A.6) follows readily from (A.5) by integration, and we only have to show that (A.5) is valid. Let γ and $C > 0$ be generic constants, whose values may change from one appearance to another. It follows by differentiating (4.1) that

$$\begin{aligned} f_h'(z)/f_h(0) &= \left(e^{h(z\sigma_h + z_h)}/\left(f(z_h)e^{hz_h}\right)\right) \\ &\quad \times \{h\sigma_h f(z\sigma_h + z_h) + \sigma_h f'(z\sigma_h + z_h)\}, \end{aligned} \tag{A.7}$$

and for $z > 0$, using (2.3) and (4.3), this can be written as

$$f_h'(z)/f_h(0) = \big(D(z\sigma_h + z_h)/(D(z_h))\big)e^{h^q g(z\sigma_h/z_h)} \tag{A.8}$$
$$\times\sigma_h\big\{h - p(z\sigma_h + z_h)^{p-1} + D'(z\sigma_h + z_h)/D(z\sigma_h + z_h)\big\}.$$

We first consider the interval $z \le l_h$. Since $pz_h^{p-1}/h = 1$, by (3.8), we have that $x \le -|x_0|pz_h^{p-1}/h$, for $x \le x_0$ (< 0), and thus also

$$x + z_h^{p-1}/h \le \big(1 - 1/(p|x_0|)\big)x$$
$$\le Dx,$$

with $D > 0$. Inserting $x = z\sigma_h/z_h \le x_0$, this gives that

$$zh\sigma_h + z_h^p \le Dzh\sigma_h, \quad \text{for } z < l_h.$$

It then follows from (1.1) and (2.1) that

$$\left|\frac{e^{h(z\sigma_h + z_h)}\sigma_h f'(z\sigma_h + z_h)}{f(z_h)e^{hz_h}}\right| \le Ch^\gamma e^{zh\sigma_h + z_h^p}|f'(z\sigma_h + z_h)|$$
$$\le Ch^\gamma e^{Dzh\sigma_h}|f'(z\sigma_h + z_h)|$$
$$\le Ch^\gamma e^{Dzh\sigma_h}e^{-c(z\sigma_h + z_h)}$$
$$= Ch^\gamma e^{-cz_h}e^{z(Dh\sigma_h - c\sigma_h - 1)}e^{-|z|}$$
$$= O(e^{-|z|}),$$

since $h^\gamma \exp\{-cz_h\} \to 0$, and $z(Dh\sigma_h - c\sigma_h - 1) \le 0$ for large h and $z \le l_h \le 0$. Since (2.1) implies that also $f(z) \le ce^{-cz}$, for $z \le 0$, similar considerations for the first part of (A.7) lead to the same conclusion, and hence (A.5) holds for $z \le l_h$.

Next, for the central interval $l_h \le z \le u_h$ we have to be more careful. To estimate the first part of (A.8), we use (2.3), (A.2), (3.8) and that $\sigma_h/z_h = h^{-q/2}g_2^{-1/2}$ by (3.8), in the first step, then Taylor's formula for the second, and again (3.8) for the last step,

$$\left|\frac{D(z\sigma_h + z_h)}{D(z_h)}e^{h^q g(z\sigma_h/z_h)}\sigma_h\big\{h - p(z\sigma_h + z_h)^{p-1}\big\}\right|$$
$$\le Ce^{-Az^2/g_2}\sigma_h h\left|1 - (z\sigma_h/z_h + 1)^{p-1}\right| \tag{A.9}$$
$$\le Ce^{-Az^2/g_2}\sigma_h h|z\sigma_h/z_h|$$
$$\le Ce^{-Az^2/g_2}|z|.$$

Similarly, for the second part of (A.8), using the same arguments as in the first step above, together with (2.4) we obtain that

$$\text{(A.10)}\quad \begin{aligned}&\frac{D(z\sigma_h + z_h)}{D(z_h)} e^{h^q g(z\sigma_h/z_h)}\sigma_h \left|\frac{D'(z\sigma_h + z_h)}{D(z\sigma_h + z_h)}\right| \\ &\leq Ce^{-Az^2/g_2}\sigma_h/(z\sigma_h + z_h) \\ &\leq C\sigma_h z_h^{-1}(x_0 + 1)^{-1} e^{-Az^2/g_2}.\end{aligned}$$

Since $\sigma_h z_h^{-1} \to 0$, it follows from (A.9) and (A.10) that (A.5) holds for $l_h \leq z \leq u_h$.

Finally, by the same arguments, but now using (A.3) instead of (A.2), i.e., the estimate $g(z\sigma_h/z_h) \leq -Az\sigma_h/z_h$, which is valid for $z > u_h$, we obtain that

$$\begin{aligned}\left|\frac{f_h'(z)}{f_h(0)}\right| &\leq Ch^\gamma e^{-Ah^q\sigma_h z_h^{-1} z} \\ &= Ch^\gamma e^{-z(Ah^q\sigma_h z_h^{-1} - 1)} e^{-|z|} \\ &= O(e^{-|z|}),\end{aligned}$$

for $z > u_h$, since then

$$z\left(Ah^q\sigma_h z_h^{-1} - 1\right) \geq x_1\left(Ah^q - \sigma_h^{-1} z_h\right) = x_1\left(Ah^q - h^{q/2} g_2^{1/2}\right).$$

This concludes the proof. □

PROOF OF (4.11). Let x_0, x_1 be as in Lemma A.1 and again write $l_s = x_0 z_s/\sigma_s$, $u_s = x_1 z_s/\sigma_s$. First, using the upper bound in (A.6) and that $f_s(0) \to 1/\sqrt{2\pi}$ by (4.4), we have that

$$\text{(A.11)}\quad \begin{aligned}|E(\bar{Z}_s - z_s)/\sigma_s| &= \left|\int z f_s(z)\,dz\right| \\ &= \left|\int_{l_s}^{u_s} z f_s(z)\,dz\right| + O\left(|l_s| e^{-|l_s|} + |u_s| e^{-|u_s|}\right) \\ &= \left|\int_{l_s}^{u_s} z f_s(z)\,dz\right| + O(s^{-q/2}),\end{aligned}$$

by the definition of l_s and u_s, since

$$\text{(A.12)}\qquad z_s/\sigma_s = s^{q/2} g_2^{1/2}.$$

Next, it follows from (A.12) and (A.4) that for $l_s \leq z \leq u_s$,

$$\begin{aligned}|s^q g(z\sigma_s/z_s) + z^2/2| &= s^q \left|g(z\sigma_s/z_s) + g_2(z\sigma_s/z_s)^2/2\right| \\ &\leq Bg_2^{-3/2} s^{-q/2}|z|^3.\end{aligned}$$

Hence, by (4.3), for $s^{-q/2}|z|^3 \leq 1$,

$$\begin{aligned}&\left|f_s(z) - f_s(0)e^{-z^2/2} D(\sigma_s z + z_s)/D(z_s)\right| \\ &= f_s(z)\left|1 - \exp\left\{-\left(s^q g(z\sigma_s/z_s) + z^2/2\right)\right\}\right| \\ &= f_s(z)|z|^3 O(s^{-q/2}),\end{aligned}$$

and, using (A.6) and (4.4) to obtain dominated convergence of the last integral,

$$\left|\int_{-s^{q/6}}^{s^{q/6}} zf_s(z)\,dz - \int_{-s^{q/6}}^{s^{q/6}} zf_s(0)e^{-z^2/2}D(\sigma_s z + z_s)D(z_s)^{-1}\,dz\right|$$

$$\text{(A.13)}\qquad = O(s^{-q/2})\int_{-s^{q/6}}^{s^{q/6}} z^4 f_s(z)\,dz$$

$$= O(s^{-q/2}).$$

Since $f_h(0)$ is bounded, it follows readily from (A.6) that

$$\int_{s^{q/6}}^{u_s} zf_s(z)\,dz = O(s^{-q/2}),$$

and since $D(\sigma_s z + z_s)/D(z_s)$ is bounded for $l_s \le z \le u_s$, also

$$\int_{s^{q/6}}^{u_s} zf_s(0)e^{-z^2/2}D(\sigma_s z + z_s)D(z_s)^{-1}\,dz = O(s^{-q/2}).$$

Of course, the same estimates hold for the integrals over $(l_s, -s^{q/6}]$, and hence, by (A.13)

$$\text{(A.14)}\qquad \left|\int_{l_s}^{u_s} zf_s(z)\,dz - \int_{l_s}^{u_s} zf_s(0)e^{-z^2/2}D(\sigma_s z + z_s)D(z_s)^{-1}\,dz\right| = O(s^{-q/2}).$$

Further, using first the mean value theorem and then (2.3), (2.4) and (A.12) we have for $z \in (u_s, l_s]$ and for some $z^* \in (u_s, l_s]$ that

$$|D(z\sigma_s + z_s) - D(z_s)| = |z\sigma_s D'(z^*\sigma_s + z_s)|$$

$$= |z|\sigma_s\left|\frac{D'(z^*\sigma_s + z_s)}{D(z^*\sigma_s + z_s)}\right|\frac{D(z^*\sigma_s + z_s)}{D(z_s)}D(z_s)$$

$$= |z|O(\sigma_s z_s^{-1}D(z_s))$$

$$= |z|O(s^{-q/2}D(z_s)),$$

uniformly for z in the prescribed range. Hence, since $f_s(0) \to 1/\sqrt{2\pi}$,

$$\left|\int_{l_s}^{u_s} zf_s(0)e^{-z^2/2}D(\sigma_s z + z_s)D(z_s)^{-1}\,dz - \int_{l_s}^{u_s} zf_s(0)e^{-z^2/2}\,dz\right|$$

$$\text{(A.15)}\qquad = O(s^{-q/2})f_s(0)\int z^2e^{-z^2/2}\,dz$$

$$= O(s^{-q/2}).$$

Since $\int ze^{-z^2/2}\,dz = 0$, it is trivial to see that

$$\left|\int_{l_s}^{u_s} zf_s(0)e^{-z^2/2}\,dz\right| = O(s^{-q/2})$$

and it hence follows from (A.11), (A.14) and (A.15) that

$$|E(\bar{Z}_s - z_s)/\sigma_s| = O(s^{-q/2}),$$

i.e., (4.11) holds. □

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