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AN IDEAL METRIC AND THE RATE OF CONVERGENCE TO A SELF-SIMILAR PROCESS

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A new metric is introduced which is suitable for estimating the rate of convergence of processes related to stable random variables. It is shown that it has an upper bound depending on the difference pseudomoments, but not on the absolute moments. This new metric is then applied to get some rates of convergence to a self-similar process constructed from a stable process.

1. Introduction. Denote by $\mathscr{X} = \mathscr{X}(\Omega, \mathscr{A}, P)$ the vector space of all random variables defined on a probability space $(\Omega, \mathscr{A}, P)$ and by $\mathscr{L}(\mathscr{X})$ the space of laws P_X, $X \in \mathscr{X}$. The mapping μ: $\mathscr{L}(\mathscr{X}) \times \mathscr{L}(\mathscr{X}) \to [0, \infty]$ is called a simple probability metric in $\mathscr{X}$ if it has the following pseudodistance properties: For any $P_X, P_Y, P_Z \in \mathscr{L}(\mathscr{X})$,

(1) if $P_X = P_Y$, then $\mu(P_X, P_Y) = 0$,
(2) $\mu(P_X, P_Y) = \mu(P_Y, P_X)$,
(3) $\mu(P_X, P_Y) \le \mu(P_X, P_Z) + \mu(P_Z, P_Y)$.

By abuse of notation we occasionally write $\mu(X, Y)$ for $\mu(P_X, P_Y)$. For convenience, we shall call it only "metric" in this paper. We refer to the survey paper by Zolotarev (1983) for the general idea of the metric.

Most metrics in this paper take the form

$$\mu(P_X, P_Y) = \nu(P_X - P_Y), \quad \text{for } P_X, P_Y \in \mathscr{L}(\mathscr{X}),$$

where ν is a subadditive $[0, \infty]$-valued functional on the space $\mathscr{M}$ of bounded signed measures on $\mathbb{R}$. Throughout the first three sections of this paper, we write

$$\rho = P_X - P_Y,$$

$$F_\rho(x) = F_X(x) - F_Y(x), \quad \text{where } F_X(x) = P(X \le x),$$

$$F_{r,\rho}(x) = \int_{-\infty}^{x} \frac{(x-t)^{r-1}}{(r-1)!}\, dF_\rho(t), \qquad r = 1, 2, \dots, \tag{1.1}$$

$$\rho(f) = \int f\, d\rho = Ef(X) - Ef(Y).$$

A subadditive $[0, \infty]$-valued functional ν on $\mathscr{M}$ is called ideal of order s if the following two conditions are satisfied:

(1) Regularity: $\nu(\rho * \sigma) \le \nu(\rho)|\sigma|(\mathbb{R})$ for $\rho, \sigma \in \mathscr{M}$, where $|\sigma|$ is the total variation measure of σ.

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(2) Homogeneity of order $s \geq 0$: $\nu(\rho_c) = |c|^s \nu(\rho)$ for $c \neq 0$, $\rho \in \mathscr{M}$, where $\rho_c(\cdot) = \rho(\cdot / c)$.

If $\mu(X, Y) = \nu(P_X - P_Y)$ and ν is ideal of order s, then the metric μ is called so. Regularity now amounts to $\mu(X + Z, Y + Z) \leq \mu(X, Y)$ in case Z is independent of X and Y, and homogeneity of order s to $\mu(cX, cY) = |c|^s \mu(X, Y)$.

Let

$$\nu(\rho) = \sup\{|\rho(f)|: f \in \mathscr{F}\}, \tag{1.2}$$

with $\mathscr{F}$ some class of measurable functions on $\mathbb{R}$. If $\mathscr{F}$ is closed for $f \mapsto f(y + \cdot)$, $y \in \mathbb{R}$, then ν is regular, because

$$\begin{aligned}|(\rho * \sigma)(f)| &= \left|\int\int f(x+y)\rho(dx)\sigma(dy)\right| \leq \int |\rho(f(y + \cdot))| \|\sigma\|(dy) \\ &\leq \sup_{y \in \mathbb{R}} |\rho(f(y + \cdot))| \|\sigma\|(\mathbb{R}) \\ &\leq \nu(\rho) \|\sigma\|(\mathbb{R}).\end{aligned}$$

If $\mathscr{F}$ is closed for $f \mapsto |c|^{-s} f(c \cdot)$ for $c \neq 0$, then ν is homogeneous of order s [see also Zolotarev (1976a), Section 1.5].

The existence of an ideal metric of a given order $s \geq 0$ was shown by Zolotarev (1976a, 1976b). He defined a metric, which will be called the Zolotarev metric and denoted by $\zeta_s(\rho) = \zeta_s(P_X, P_Y)$, by choosing $\mathscr{F}$ in (1.2) as

$$\mathscr{F}_s = \left\{f: |f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^{1/p}, x, y \in \mathbb{R}\right\},$$

where m (≥ -1) is an integer and $p \in [1, \infty)$ is such that

$$s = m + \frac{1}{p}, \tag{1.3}$$

and

$$f^{(-1)}(x) = \int_0^x f(y)\,dy, \qquad f^{(0)}(x) = f(x),$$

$$f^{(m)}(x) = \frac{d^m}{dx^m} f(x), \qquad m = 1, 2, \dots.$$

He also obtained an upper bound for $\zeta_s(\rho)$ as follows. Let

$$\kappa_s(\rho) = \kappa_s(P_X, P_Y) = s\int_{-\infty}^{\infty} |x|^{s-1} |F_\rho(x)|\,dx, \tag{1.4}$$

which is the so-called difference pseudomoment of order s and $b_s = \min(E|X|^s, E|Y|^s)$. If $s > 0$, $\kappa_s(\rho) < \infty$, $E(X^j - Y^j) = 0$, $j = 1, 2, \dots, m$ (when $s > 1$), and further $E(|X|^s + |Y|^s) < \infty$ when s is noninteger, then

$$\zeta_s(\rho) \leq \frac{\Gamma(1 + 1/p)}{\Gamma(1 + s)}\left[m\kappa_s(\rho) + \{\kappa_s(\rho)\}^{1/p} b_s^{1-1/p}\right].$$

He moreover showed in Zolotarev (1979) that if s is a positive integer, then ζ_s

can be represented by the following integral form:

$$\zeta_s(\rho) = \int_{-\infty}^{\infty} |F_{s,\rho}(x)|\,dx. \tag{1.5}$$

In this paper, first we introduce a new ideal metric of order $s > 0$ with two advantages:

(A.1) It has an integral form like (1.5) for any $s > 0$ (not necessarily integer).

(A.2) For any $s > 0$, there exists an upper bound of the ideal metric, which depends on the difference pseudomoments κ_s but not on b_s.

Second, we investigate rates of convergence to self-similar processes with this new metric.

Denote by $=_d$ and $\Rightarrow_d$ the equality and the convergence of all finite-dimensional distributions, respectively. Let $\{X(t),\ t \geq 0\}$ be a self-similar process with parameter $H > 0$ having stationary increments in the sense that $X(c\,\cdot\,) =_d c^H X(\cdot)$ for any $c > 0$ and $X(\cdot + b) - X(b) =_d X(\cdot) - X(0)$ for any $b > 0$. We refer to the recent paper by Vervaat (1985) for some basic properties of such processes. The stationary sequence $\{Y_j\}_{j=1}^{\infty}$ is said to belong to the domain of attraction of $X(t)$ if, for some slowly varying function $L(\cdot)$,

$$n^{-H} L(n)^{-1} \sum_{j=1}^{[nt]} Y_j \Rightarrow_d X(t). \tag{1.6}$$

Many limit theorems of type (1.6) for strongly dependent random variables $\{Y_j\}$ have been obtained [for example, Dobrushin and Major (1979), Kesten and Spitzer (1979), Taqqu (1979) and Maejima (1983)]. Our purpose here is to study the rate of convergence in (1.6), especially for $X(t)$ a fractional stable process as considered by Maejima (1983) (see our Section 4 for a definition).

In Section 2, we shall introduce the new metric and prove its properties. Section 3 will discuss some relations between some ideal metrics and fractional calculus. In Section 4, our metric will be applied to estimate the distance between two elements in the domain of attraction of the fractional stable process. In the final section, the distance between the fractional stable process and processes in its domain of attraction will be examined.

2. An ideal metric and its properties.

Denote

$$\|g\|_p = \left[\int_{-\infty}^{\infty} |g(x)|^p\,dx\right]^{1/p}, \qquad 1 \leq p < \infty,$$

$$\|g\|_\infty = \operatorname{ess\,sup} |g(x)|,$$

and for $r = 0, 1, 2, \ldots,$ and $p \in [1, \infty]$,

$$\mathscr{F}(r, p) = \left\{f \colon \|f^{(r)}\|_q \leq 1\right\},$$

where $1/p + 1/q = 1$. Let $s > 0$ and define m and p as in (1.3). Denote

$$\theta_s(\rho) \equiv \theta_s(P_X, P_Y) \equiv \sup\left\{|\rho(f)| \colon f \in \mathscr{F}(m+1, p)\right\},$$

with the same notation as in (1.1).

It is easily seen that the mapping θ_s is an ideal metric of order s. In the following we shall show some properties of θ_s.

THEOREM 1. *Let $s > 0$. If $\theta_s(\rho) < \infty$, then*

$$E(X^j - Y^j) = 0, \qquad j = 1,2,\dots,m, \tag{2.1}$$

when $s > 1$, and

$$\theta_s(\rho) = \|F_{m+1,\rho}\|_p. \tag{2.2}$$

PROOF. If $f \in \mathscr{F}(m+1, p)$, we have

$$\rho(f) = \sum_{j=1}^{m} \frac{f^{(j)}(0)}{j!} \int_{-\infty}^{\infty} x^j \, dF_\rho(x) + \int_{-\infty}^{\infty} dF_\rho(x) \int_0^x \frac{(x-t)^m}{m!} f^{(m+1)}(t)\, dt,$$

with the convention $\sum_{j=1}^{0} = 0$. Suppose that $m \ge 1$ (consequently $s > 1$) and (2.1) does not hold for some j_0, $1 \le j_0 \le m$, namely,

$$E(X^{j_0} - Y^{j_0}) = \int_{-\infty}^{\infty} x^{j_0}\, dF_\rho(x) \ne 0.$$

Then

$$\theta_s(\rho) \ge \sup_{c \in \mathbf{R}} \left| c \int_{-\infty}^{\infty} x^{j_0}\, dF_\rho(x) \right| = \infty,$$

which proves (2.1).

Therefore, if $\theta_s(\rho) < \infty$, for any $m \ge 0$

$$\begin{aligned} \rho(f) &= \int_{-\infty}^{\infty} dF_\rho(x) \int_0^x \frac{(x-t)^m}{m!} f^{(m+1)}(t)\, dt \\ &= (-1)^{m+1} \int_{-\infty}^{0} f^{(m+1)}(t) F_{m+1,\rho}(t)\, dt + \int_0^{\infty} f^{(m+1)}(t) \bar{F}_{m+1,\rho}(t)\, dt, \end{aligned}$$

where

$$\bar{F}_{m+1,\rho}(t) = \int_t^{\infty} \frac{(u-t)^m}{m!}\, dF_\rho(u). \tag{2.3}$$

Thus by (2.1) again

$$\rho(f) = (-1)^{m+1} \int_{-\infty}^{\infty} f^{(m+1)}(t) F_{m+1,\rho}(t)\, dt.$$

By the duality of the spaces $\mathscr{L}^p = \{f\colon \|f\|_p < \infty\}$ and $\mathscr{L}^q = \{f\colon \|f\|_q < \infty\}$ with $1/p + 1/q = 1$, we conclude (2.2). □

The integral representation (2.2) assures advantage (A.1) mentioned in Section 1 and statement (b) in the next theorem gives us advantage (A.2).

THEOREM 2. *Suppose* (2.1) *is satisfied.*

(a) *For any* $s > 0$,

$$\theta_s(\rho) \le \zeta_s(\rho), \tag{2.4}$$

where ζ_s *is the Zolotarev metric.*

(b) *If* $0 < s \le 1$,

$$\theta_s(\rho) \le \{\kappa_1(\rho)\}^{1/p}$$

and if $s > 1$,

$$\theta_s(\rho) \le \frac{\Gamma(1 + 1/p)}{\Gamma(1 + s)} \kappa_s(\rho), \tag{2.5}$$

where κ_s *is the difference pseudomoment defined in* (1.4).

(c) *For* $s > 1$,

$$\theta_s(\rho) \le ((m-1)!)^{-1} \eta_s(\rho), \tag{2.6}$$

where

$$\eta_s(\rho) = \int_{-\infty}^0 |x|^{m-1} \left(\int_{-\infty}^x |F_\rho(u)|^p \, du \right)^{1/p} dx + \int_0^\infty x^{m-1} \left(\int_x^\infty |F_\rho(u)|^p \, du \right)^{1/p} dx.$$

(d) *For* $s = m + 1/p$ *and* $s' = m + 1/p'$, *where* $m = 0, 1, \ldots$ *and* $1 \le p' \le p$,

$$\theta_s(\rho) \le \{\max(E|X|^m, E|Y|^m)\}^{1-p'/p} \{\theta_{s'}(\rho)\}^{p'/p}. \tag{2.7}$$

PROOF. (a) For every $x, y \in \mathbb{R}$ and f with $\|f^{(m+1)}\|_q \le 1$, we have

$$|f^{(m)}(x) - f^{(m)}(y)| \le \|f^{(m+1)}\|_q |x - y|^{1/p} \le |x - y|^{1/p}.$$

Thus $\mathscr{F}(m + 1, p) \subset \mathscr{F}_s$ and hence (2.4) holds.

(b) The first assertion can easily be deduced from (2.2). If $s > 1$, then (2.1) and (2.2) imply

$$\begin{aligned} \theta_s(\rho) &\le \left[\int_{-\infty}^0 |F_{m+1,\rho}(x)|^p \, dx \right]^{1/p} + \left[\int_0^\infty |\overline{F}_{m+1,\rho}(x)|^p \, dx \right]^{1/p} \\ &\equiv I_1 + I_2, \end{aligned} \tag{2.8}$$

where $\overline{F}_{m+1,\rho}(x)$ is defined in (2.3). Let $I[\cdot]$ be the indicator function. Then by the Minkowski inequality

$$\begin{aligned} I_1 &= \left[\int_{-\infty}^0 \left| \int_{-\infty}^0 F_{m,\rho}(u) I[u \le x] \, du \right|^p dx \right]^{1/p} \\ &\le \int_{-\infty}^0 |u|^{1/p} |F_{m,\rho}(u)| \, du \equiv J_{m,p}, \end{aligned}$$

where

$$J_{1,p} = \int_{-\infty}^0 |u|^{1/p} |F_\rho(u)| \, du$$

and if $m \geq 2$,

$$J_{m,p} = \int_{-\infty}^{0} |u|^{1/p} \left| \int_{-\infty}^{u} \frac{(u-t)^{m-2}}{(m-2)!} F_\rho(t)\, dt \right| du$$

$$\leq \left\{ \left(1 + \frac{1}{p}\right)\left(2 + \frac{1}{p}\right) \cdots \left(m - 1 + \frac{1}{p}\right) \right\}^{-1} \int_{-\infty}^{0} |t|^{m-1+1/p} |F_\rho(t)|\, dt.$$

In a similar way we can estimate I_2 and get (2.5).

(c) It is easy to see that

$$\text{(2.9)} \qquad \left[\int_A \left| \int_B g(u)\, du \right|^p dx \right]^{1/p} \leq \int_A \left[\int_B |g(u)|^p\, du \right]^{1/p} dx, \qquad p \geq 1,$$

where $A = (-\infty, t]$ and $B = (-\infty, x]$, $x \leq t \leq 0$, or $A = [t, \infty)$ and $B = [x, \infty)$, $0 \leq t \leq x$. Using (2.8) and (2.9), we have

$$I_1 \leq \int_{-\infty}^{0} \left[\int_{-\infty}^{t} |F_{m,\rho}(x)|^p\, dx \right]^{1/p} dt$$

$$= \int_{-\infty}^{0} \left[\int_{-\infty}^{t} \left| \int_{-\infty}^{x} F_{m-1,\rho}(u)\, du \right|^p dx \right]^{1/p} dt$$

$$\leq \int_{-\infty}^{0} \int_{-\infty}^{t} \left[\int_{-\infty}^{x} |F_{m-1,\rho}(u)|^p\, du \right]^{1/p} dx\, dt$$

$$\leq \int_{-\infty}^{0} \frac{|x|^{m-1}}{(m-1)!} \left[\int_{-\infty}^{x} |F_{1,\rho}(u)|^p\, du \right]^{1/p} dx,$$

and similarly

$$I_2 \leq \int_{0}^{\infty} \frac{x^{m-1}}{(m-1)!} \left[\int_{x}^{\infty} |\overline{F}_{1,\rho}(u)|^p\, du \right]^{1/p} dx.$$

As $F_{1,\rho}(u) = F_\rho(u)$ and $\overline{F}_{1,\rho}(u) = -F_\rho(u)$, (2.6) follows.

(d) Let $B_m = \max(E|X|^m, E|Y|^m)$. Then

$$\theta_s(\rho) \leq B_m \left[\int_{-\infty}^{0} |B_m^{-1} F_{m+1,\rho}(x)|^p\, dx + \int_{0}^{\infty} |B_m^{-1} \overline{F}_{m+1,\rho}(x)|^p\, dx \right]^{1/p}$$

$$\leq B_m \left[\int_{-\infty}^{0} |B_m^{-1} F_{m+1,\rho}(x)|^{p'}\, dx + \int_{0}^{\infty} |B_m^{-1} \overline{F}_{m+1,\rho}(x)|^{p'}\, dx \right]^{1/p}$$

$$= B_m^{1-p'/p} \{\theta_s(\rho)\}^{p'/p}. \qquad \square$$

REMARK 1. Note that $\theta_s(\rho) = \zeta_s(\rho)$ in (2.4) for positive integer s, as then $m = s - 1$ and $p = 1$ in (1.3). Hence $\theta_s(\rho) = \|F_{s,\rho}\|_1 = \zeta_s(\rho)$ for such s, by (2.2) and (1.5).

The next theorem gives us a lower bound for the metric θ_s. Let $L(P_X, P_Y)$ be the Lévy metric in $\mathscr{X}$, namely,

$$L(P_X, P_Y) = \inf\{\varepsilon > 0\colon F_X(x-\varepsilon) - \varepsilon \le F_Y(x) \le F_X(x+\varepsilon) + \varepsilon, \text{ for all } x \in \mathbb{R}\}.$$

Theorem 3. *For any $s > 0$*

$$\{L(P_X, P_Y)\}^{s+1} \le C(s)\theta_s(\rho), \tag{2.10}$$

where $C(s) = (2m+2)!(2m+3)^{1/2}/\{(m+1)!(2/q+1)^{1/2}\}$. *For* $0 < s \le 2$ *inequality* (2.10) *can be sharpened to*:

$$\{L(P_X, P_Y)\}^{s+1} \le \theta_s(\rho), \quad \text{if } 0 < s \le 1, \tag{2.11}$$

$$\{L(P_X, P_Y)\}^{s+1} \le 4\theta_s(\rho), \quad \text{if } 1 < s \le 2. \tag{2.12}$$

The exponent $s+1$ in (2.10) is the best possible, as Remark 2 below shows. It is well known that if X has a bounded density $p_X(x)$, then

$$u(\rho) \le \Big(1 + \sup_{x\in\mathbb{R}} p_X(x)\Big) L(P_X, P_Y), \tag{2.13}$$

where $u(\rho)$ is the ordinary uniform (or Kolmogorov) metric:

$$u(\rho) = \sup_{x\in\mathbb{R}} |F_\rho(x)|.$$

Using (2.10)–(2.13), we obtain a lower bound for $\theta_s(\rho)$ in terms of $u(\rho)$.

Proof of Theorem 3. We follow the idea of Yamukov (1977). Let $s > 0$ and choose ε such that $0 < \varepsilon < L(P_X, P_Y)$. Then for some $z \in \mathbb{R}$

$$F_X(z) - F_Y(z+\varepsilon) > \varepsilon \tag{2.14}$$

or $F_Y(z) - F_X(z+\varepsilon) > \varepsilon$. Suppose that (2.14) holds without loss of generality. For $m = 0, 1, \ldots,$ put

$$g_s(x) = \{(1-x^2)^+\}^{m+1},$$

with the notation $a^+ = \max\{0, a\}$. Then

$$V = \int_{-1}^{1} g_s(x)\,dx = 2^{2m+3}\frac{\{(m+1)!\}^2}{(2m+2)!(2m+3)}.$$

Put

$$f_s(x) = 2V^{-1}\int_{-1}^{1} I\left[x - \frac{\varepsilon y}{2} < z + \frac{\varepsilon}{2}\right] g_s(y)\,dy - 1.$$

It is easy to see that $f_s(x) = 1$ for $x \le z$, $f_s(x) = -1$ for $x \ge z + \varepsilon$ and $|f_s(x)| \le 1$ for all $x \in \mathbb{R}$. Noticing that

$$f_s(x) = 2V^{-1}\frac{2}{\varepsilon}\int_{-\infty}^{\infty} I\left[u < z + \frac{\varepsilon}{2}\right] g_s\left(\frac{2(x-u)}{\varepsilon}\right) du - 1,$$

we have for $z < x < z + \varepsilon$

$$f_s^{(m+1)}(x) = 2V^{-1}\left(\frac{2}{\varepsilon}\right)^{m+1} \int_{-\infty}^{\infty} I\left[u < z + \frac{\varepsilon}{2}\right] g_s^{(m+1)}\left(\frac{2(x-u)}{\varepsilon}\right) du,$$

which is equal to

$$2V^{-1}\left(\frac{2}{\varepsilon}\right)^{m+1} \int_{-1}^{1} I\left[x - \frac{\varepsilon y}{2} < z + \frac{\varepsilon}{2}\right] g_s^{(m+1)}(y)\, dy.$$

Hence by the Minkowski inequality,

$$\begin{aligned}
\| f_s^{(m+1)} \|_q &= 2V^{-1}\left(\frac{2}{\varepsilon}\right)^{m+1} \\
&\quad \times \left\{ \int_z^{z+\varepsilon} \left| \int_{-1}^{1} I\left[x - \frac{\varepsilon y}{2} < z + \frac{\varepsilon}{2}\right] g_s^{(m+1)}(y)\, dy \right|^q dx \right\}^{1/q} \\
&\le 2V^{-1}\left(\frac{2}{\varepsilon}\right)^{m+1} \int_{-1}^{1} |g_s^{(m+1)}(y)| \left\{ \int_z^{z+\varepsilon} I\left[x - \frac{\varepsilon y}{2} < z + \frac{\varepsilon}{2}\right] dx \right\}^{1/q} dy \qquad (2.15) \\
&= 2V^{-1}\left(\frac{2}{\varepsilon}\right)^{m+1-1/q} \int_{-1}^{1} |g_s^{(m+1)}(y)| (y+1)^{1/q}\, dy \\
&\le 2V^{-1}\left(\frac{2}{\varepsilon}\right)^{m+1-1/q} \left\{ \int_{-1}^{1} |g_s^{(m+1)}(y)|^2\, dy \right\}^{1/2} \left\{ \int_{-1}^{1} (y+1)^{2/q}\, dy \right\}^{1/q} \\
&= 2\varepsilon^{-s} C(s),
\end{aligned}$$

where we have used

$$\int_{-1}^{1} |g_s^{(m+1)}(y)|^2\, dy = (2m+2)! \int_{-1}^{1} (1 - x^2)^{m+1}\, dx = (2m+2)!\,V.$$

Since $f_s(x)/\| f_s^{(m+1)} \|_q \in \mathscr{F}(m+1, p)$, it follows from the definition of $\theta_s(\rho)$ and the properties of $f_s(x)$ that

$$\begin{aligned}
\theta_s(\rho) &\ge \| f_s^{(m+1)} \|_q^{-1} \left| \int_{-\infty}^{\infty} f_s(x)\, dF_\rho(x) \right| \\
&= \| f_s^{(m+1)} \|_q^{-1} \left| \int_{-\infty}^{\infty} (f_s(x) + 1)\, dF_\rho(x) \right| \\
&\ge \| f_s^{(m+1)} \|_q^{-1} \left[\int_{-\infty}^{z} (f_s(x) + 1)\, dF_X(x) - \int_{-\infty}^{z+\varepsilon} (f_s(x) + 1)\, dF_Y(x) \right] \qquad (2.16) \\
&\ge \varepsilon^s C(s)^{-1} [F_X(z) - F_Y(z + \varepsilon)] \quad \text{by (2.15)} \\
&> \varepsilon^{s+1} C(s)^{-1} \quad \text{by (2.14)}.
\end{aligned}$$

Letting $\varepsilon \uparrow L(P_X, P_Y)$ in (2.16), we get (2.10).

The improved inequality (2.11) for $0 < s \le 1$ follows from the integral representation (2.2) for $\theta_s(\rho)$ and the definition of the Lévy metric.

To prove (2.12), we define, as in Grigorevski and Shiganov (1976),

$$f_0\left(z+\frac{\varepsilon}{2}+h\right)=\left\{\left(\left(1-\frac{2|h|}{\varepsilon}\right)^+\right)^2-1\right\}\operatorname{sgn} h.$$

Since $|f_0(x)|\le 1$ and $\|f_0^{(2)}\|_q=8\varepsilon^{-2+1/q}$, by the same reasoning as in (2.16), we have $\theta_s(\rho)\ge 4^{-1}\varepsilon^{s+1}$, which gives us (2.12). □

REMARK 2. In the following, we show by examples that the exponents of L in (2.10) are the best possible, for instance, in the case $0<s\le 3$.

Let $0<s\le 1$ and let $P(X=0)=1$, $P(Y=0)=1-\varepsilon$ and $P(Y=\varepsilon)=\varepsilon$. Then $L(P_X,P_Y)=\varepsilon$ and $\theta_s(\rho)=\varepsilon^{s+1}$.

Let $1<s\le 2$ and let $P(X=0)=\varepsilon$, $P(X=2\varepsilon)=1-\varepsilon$, $P(Y=\varepsilon)=2\varepsilon$ and $P(Y=2\varepsilon)=1-2\varepsilon$. Then $L(P_X,P_Y)=\varepsilon$ and $\theta_s(\rho)=\{2/(p+1)\}^{1/p}\varepsilon^{s+1}$.

Let $2<s\le 3$ and let

$$P(X\le x)=\left(x+\tfrac{1}{2}\right)I[-\varepsilon\le X<\varepsilon]+I[X\ge\varepsilon],$$
$$P(Y=\varepsilon)=P(Y=-\varepsilon)=\tfrac{1}{2}-\varepsilon$$

and

$$P\left(Y=\varepsilon/\sqrt{3}\right)=P\left(Y=-\varepsilon/\sqrt{3}\right)=\varepsilon.$$

Then $L(P_X,P_Y)=\varepsilon/2\sqrt{3}$ and standard calculations give us $\theta_s(\rho)\le$ const. ε^{s+1}.

REMARK 3. In case s is a positive integer, we can improve our inequality to

$$C(s)\theta_s(\rho)\ge\{\pi(P_X,P_Y)\}^{s+1}, \tag{2.17}$$

where π is the Lévy–Prokhorov metric,

$$\pi(P_X,P_Y)\equiv\inf\{\varepsilon>0\colon P(X\in A)-P(Y\in A^\varepsilon)<\varepsilon,$$
$$P(Y\in A)-P(X\in A^\varepsilon)<\varepsilon,\text{ for all Borel sets }A\},$$

with $A^\varepsilon\equiv\bigcup_{y\in A}\{x\colon |x-y|<\varepsilon\}$. Since $\pi\ge L$, (2.17) is better than (2.10). The proof of (2.17) can be carried out the same way in Yamukov (1977). [See also Grigorevski and Shiganov (1976), but there is a gap in the proof of their Theorem 1.] The constants $C(s)$ in (2.17) are much better than those in Yamukov (1977). We also note that for noninteger s, it is impossible to find the estimation

$$\theta_s\ge\sigma(\pi),$$

for some nondecreasing function $\sigma(t)$ of $t\ge 0$ such that $\sigma(0)=0$ and $\sigma(t)>0$ for $t>0$. The following example explains this situation.

Define $\{X_n,Y_n,\ n=1,2,\dots\}$ by

$$P(X_n=2j)=P(Y_n=2j+1)=\frac{1}{n},\qquad j=1,\dots,n.$$

Then $\pi(P_{X_n},P_{Y_n})=1$, but when $1<s<2$ ($s=1+1/p$, $p>1$),

$$\theta_s\left(P_{X_n},P_{Y_n}\right)=\left[\int_{-\infty}^{\infty}\left|F_{X_n}(x)-F_{Y_n}(x)\right|^p dx\right]^{1/p}=n^{1/p-1}\to 0,$$

as $n\to\infty$.

3. Relations with fractional calculus. In this section, we comment briefly on relations between ideal metrics and fractional calculus.

Using the Weyl integral of order $\alpha > 0$,

$$K^{\alpha}F(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (y-x)^{\alpha-1}\, dF(y)$$

[see, for instance, (1.4) of Erdélyi (1975)], we can consider the metric

$$w_s(\rho) = \|K^s F_\rho\|_1 = \int_{-\infty}^{\infty} \left| \int_x^{\infty} \frac{(y-x)^{s-1}}{\Gamma(s)}\, dF_\rho(y) \right| dx,$$

as a possible generalization of the notion of ζ_s or θ_s. Actually, for integer s, we get under (2.1) that

$$w_s(\rho) = \int_{-\infty}^{\infty} |F_{s,\rho}(x)|\, dx = \zeta_s(\rho) = \theta_s(\rho).$$

Moreover, for any $s > 0$, w_s is regular and for any $c > 0$,

(3.1) $$w_s(\rho_c) = |c|^s w_s(\rho).$$

Furthermore, with $\kappa_s(\rho)$ as in (1.4) and

$$\gamma_s(\rho) = \int_{-\infty}^{\infty} |y|^s (F_\rho^+ + F_\rho^-)(dy),$$

where $F_\rho^+ = \max(0, F_\rho)$, $F_\rho^- = -\min(0, F_\rho)$, it can be shown that

(3.2) $$w_s(\rho) \le \begin{cases} \dfrac{\gamma_s(\rho)}{\Gamma(s+1)}, & \text{if } 0 < s < 1, \\ \dfrac{\kappa_s(\rho)}{\Gamma(s+1)}, & \text{if } s \ge 1, \end{cases}$$

for any ρ with support concentrated on the positive half line. [The proof of (3.2) will be given at the end of this section.]

However, it might be difficult to find a satisfactory class of distributions $F_\rho = F_X - F_Y$ for which (3.1) is true also for negative $c < 0$ and, in addition, the values of w_s are finite.

Next, we give another relation between the ideal metrics and fractional calculus. Note that $F_{r,\rho}$ can be regarded as the convolution of F_ρ with the function p_r defined by

$$p_r(x) = \frac{x^{r-1}}{\Gamma(r)}, \quad \text{if } x > 0,$$
$$= 0, \quad \text{if } x \le 0, \qquad r = 1, 2, \dots$$

[see (1.2) of Erdélyi (1975)]. Hence if $\theta_s < \infty$, then

$$\theta_s(\rho) = \|p_{m+1} * F_\rho\|_p,$$

by Theorem 1. This formula suggests another way of constructing ideal metrics, namely, by means of "smoothing" of the signed measure ρ. More precisely, let μ be an ideal metric of order $\tau \geq 0$ and define the function

$$M_{s,\rho}(h) = |h|^s \mu(\rho * \varphi_h), \qquad h \in \mathbb{R}, \quad s > 0,$$

where φ is a probability measure and $\varphi_h(\cdot) = \varphi(\cdot/h)$. Then the metric

$$m_{s,p}(\rho) = \|M_{s,\rho}\|_p, \qquad 1 \leq p \leq \infty,$$

is an ideal metric of order $\tau + s + 1/p$. As an example, the following ideal metric might be interesting.

Let $\mu = \zeta_0$ (the distance in variation and $\tau = 0$) and ψ be the stable measure with characteristic function $\exp\{-|z|^\alpha\}$. Then $\lambda_s \equiv m_{s,\infty}$ is an ideal metric of order $s > 0$ and

$$\lambda_s(\rho) \leq \theta_s(\rho) \|F_\varphi^{(s+1)}\|_1, \tag{3.3}$$

for any integer $s > 0$.

Proof of (3.2). If ρ has its support on the positive half line,

$$\begin{aligned} w_s(\rho) &\leq \int_0^\infty \int_x^\infty \frac{(y-x)^{s-1}}{\Gamma(s)} (F_\rho^+ + F_\rho^-)(dy)\, dx \\ &= \int_0^\infty \frac{y^s}{\Gamma(s+1)} (F_\rho^+ + F_\rho^-)(dy) \\ &= \frac{\gamma_s(\rho)}{\Gamma(s+1)}, \end{aligned}$$

for any $s > 0$, and further if $s \geq 1$,

$$\begin{aligned} w_s(\rho) &= \int_0^\infty \left| \int_x^\infty \frac{(y-x)^{s-2}}{\Gamma(s-1)} F_\rho(y)\, dy \right| dx \leq \int_0^\infty \int_x^\infty \frac{(y-x)^{s-2}}{\Gamma(s-1)} |F_\rho(y)|\, dy\, dx \\ &= \int_0^\infty \frac{y^{s-1}}{\Gamma(s)} |F_\rho(y)|\, dy = \frac{\kappa_s(\rho)}{\Gamma(s+1)}. \end{aligned}$$

□

Proof of (3.3). Let $s > 0$ be an integer. We have by the definition,

$$\begin{aligned} \lambda_s(\rho) &= \sup_h |h|^s \int_{-\infty}^\infty \left| \int_{-\infty}^\infty F'_{\varphi_h}(y)\, dy F_\rho(x-y) \right| dx \\ &= \sup_h |h|^s \int_{-\infty}^\infty \left| F_{\varphi_h}^{(s+1)}(y) F_{s,\rho}(x-y)\, dy \right| dx \\ &\leq \sup_h |h|^s \int_{-\infty}^\infty \int_{-\infty}^\infty |F_{\varphi_h}^{(s+1)}(y)| |F_{s,\rho}(x-y)|\, dy\, dx \\ &= \int_{-\infty}^\infty |F_{s,\rho}(t)|\, dt \cdot \sup_h |h|^s \int_{-\infty}^\infty |F_{\varphi_h}^{(s+1)}(y)|\, dy \\ &\leq \theta_s(\rho) \cdot \sup_h |h|^s \int_{-\infty}^\infty |F_{\varphi_h}^{(s+1)}(y)|\, dy. \end{aligned}$$

On the other hand, since φ is a stable law, we have

$$|h|^s\int_{-\infty}^{\infty}\left|F_{\varphi_h}^{(s+1)}(y)\right|dy=\int_{-\infty}^{\infty}\frac{1}{|h|}\left|F_{\varphi}^{(s+1)}\left(\frac{y}{h}\right)\right|dy=\int_{-\infty}^{\infty}\left|F_{\varphi}^{(s+1)}(t)\right|dt. \qquad \square$$

4. Distance between the elements in the domain of attraction of the fractional stable process. The metric μ was introduced as a distance between probability measures. However, for the convenience of notation, we shall write $\mu(X, Y)$ for $\mu(P_X, P_Y)$ in the rest of the paper.

The fractional stable process defined by Maejima (1983) is self-similar with parameter $H \in (0,1)$. We recall briefly its definition and the main result of Maejima (1983).

Let $\{X_j,\ j \in \mathbb{Z}\}$ be a sequence of independent and identically distributed random variables belonging to the domain of normal attraction of a strictly stable random variable of index $\alpha < 2$ with characteristic function

$$\exp\{-|z|^{\alpha}(A_1 + iA_2 \operatorname{sgn} z)\} \tag{4.1}$$

for some $A_1 > 0$, $A_2 \in \mathbb{R}$, with $|A_1^{-1}A_2| \le \tan(\alpha\pi/2)$, where $\operatorname{sgn} z = +1$, 0 or -1, according as $z > 0$, $= 0$ or < 0. Take $\beta \in (1/\alpha - 1, 1/\alpha)$ with $\beta \neq 0$ and consider the random variables

$$Y_k \equiv \sum_{j\in\mathbb{Z}} c_j X_{k-j}, \qquad k = 1, 2, \dots,$$

where $c_j = j^{\uparrow(-\beta-1)}$. [Here as in Vervaat (1985), $x^{\uparrow\gamma} = |x|^{\gamma}\operatorname{sgn} x$ for $x \neq 0$ and $= 0$ for $x = 0$.] For $t \in [0,1]$, we define

$$\Delta_n(t) \equiv |\beta| n^{-H}\left(\sum_{k=1}^{[nt]} Y_k + (nt - [nt])Y_{[nt]+1}\right),$$

where $[a]$ is the integer part of a, $\sum_{k=1}^{0}$ means 0 and $H = 1/\alpha - \beta$. Consider two independent stable processes $\{Z_+(t),\ t \ge 0\}$ and $\{Z_-(t),\ t \ge 0\}$, both having characteristic functions

$$E\{e^{izZ_{\pm}(t)}\} = \exp\{-t|z|^{\alpha}(A_1 + iA_2 \operatorname{sgn} z)\}. \tag{4.2}$$

Define the fractional stable process by

$$\Delta(t) = \int_{-\infty}^{\infty}\left(|t-s|^{-\beta} - |s|^{-\beta}\right) dZ(s), \qquad t \in [0,1],$$

where $\Delta(0) = 0$ a.s. and $Z(s) = Z_+(s)I[s \ge 0] - Z_-(-s+0)I[s < 0]$.

THEOREM A [Maejima (1983)]. *As* $n \to \infty$,

$$\Delta_n(t) \Rightarrow_d \Delta(t). \tag{4.3}$$

In the remainder of this paper, we shall apply our new metric θ_s to get the rate of convergence of (4.3) in terms of the uniform metric u. The program is as follows. Besides $\{X_j,\ j \in \mathbb{Z}\}$ in the generality introduced before (4.1) we consider a more special choice $\{X_j^*,\ j \in \mathbb{Z}\}$ with X_j^* itself already distributed as the attracting stable random variable with characteristic function (4.1). Denote $\Delta_n^*(t)$ corresponding to $\{X_j^*,\ j \in \mathbb{Z}\}$ in the same way as $\Delta_n(t)$. Then we have

$$u(\Delta_n(t), \Delta(t)) \le u(\Delta_n(t), \Delta_n^*(t)) + u(\Delta_n^*(t), \Delta(t)), \tag{4.4}$$

where u is the uniform metric as defined after (2.13). Since $\Delta_n(t)$ and $\Delta_n^*(t)$ can be expressed as the infinite weighted sums of $\{X_j\}$ and $\{X_j^*\}$, respectively, we can apply the method of probability metrics to estimate $u(\Delta_n(t), \Delta_n^*(t))$. As to $u(\Delta_n^*(t), \Delta(t))$, we shall use the Esseen inequality.

In this section, we examine $u(\Delta_n(t), \Delta_n^*(t))$. The other step will be dealt with in the next section. To estimate $u(\Delta_n(t), \Delta_n^*(t))$, it is enough to study $\theta_s(\Delta_n(t), \Delta_n^*(t))$, because of (2.10) and (2.13). For the latter $\{X_j^*\}$ need not be so special. So let $\{X_j',\ j \in \mathbb{Z}\}$ be another sequence with the same properties as $\{X_j,\ j \in \mathbb{Z}\}$ and define the process $\Delta_n'(t)$ corresponding to $\{X_j'\}$ in the same way. The main result in this section is the following.

THEOREM 4. *Consider the following cases*:

(a) $0 < \alpha < 1,\ 2\alpha/(\alpha + 1) < s \le 1,\ 0 < H < 1$;
(b) $0 < \alpha < 1,\ s > 1,\ 1 - 1/s < H < 1$;
(c) $1 \le \alpha < 2,\ s > \alpha,\ 1 - 1/s < H < 1\ (H \ne 1/\alpha)$.

Then we have for each $t \in [0,1]$,

$$\theta_s(\Delta_n(t), \Delta_n'(t)) \le Cn^Q D(n)\theta_s(X_0, X_0'), \tag{4.5}$$

where C is a positive constant depending on H, s and t,

$$Q = -Hs + (s-1)^+ + \left\{2 - \left(1 - H + \frac{1}{\alpha}\right)s\right\}^+$$

and

$$D(n) = \begin{cases} 1, & \text{if } H \ne 1/\alpha + 1 - 2/s, \\ \log n, & \text{if } H = 1/\alpha + 1 - 2/s. \end{cases}$$

The above restrictions on (α, H, s) have the effect that $C < \infty$ and that Q, the exponent of n in (4.5), is negative. The details will be explained after the proof of the theorem.

We also note here that $\theta_s(X_0, X_0')$ in (4.5) might be infinite. The conditions that guarantee the finiteness of $\theta_s(X_0, X_0')$ are given in Theorem 2.

PROOF OF THEOREM 4. Define for $t \ge 0$

$$S(t) \equiv \sum_{k=1}^{[t]} Y_k + (t - [t])Y_{[t]+1}$$

and

$$S'(t) \equiv \sum_{k=1}^{[t]} Y_k' + (t - [t])Y_{[t]+1}'.$$

Since θ_s is an ideal metric of order s,

$$\theta_s(\Delta_n(t), \Delta'_n(t)) = \theta_s(|\beta| n^{-H} S(nt), |\beta| n^{-H} S'(nt)) \tag{4.6}$$
$$= |\beta|^s n^{-Hs} \theta_s(S(nt), S'(nt)).$$

Note that

$$S(nt) = \sum_{k \in \mathbf{Z}} \xi_k(nt) X_k,$$

where

$$\xi_k(nt) = \sum_{j=1-k}^{[nt]-k} c_j + (nt - [nt]) c_{[nt]+1-k}. \tag{4.7}$$

By the homogeneity and regularity of θ_s, we obtain the following estimate:

$$\theta_s(S(nt), S'(nt)) \le \theta_s(X_0, X'_0) \sum_{k \in \mathbf{Z}} |\xi_k(nt)|^s \tag{4.8}$$

[cf. Zolotarev (1976a, 1979)]. By the definition of $\{c_j\}$ it follows that

$$\sum_{k \ge 0} \gamma_k(nt)^s \le \sum_{k \in \mathbf{Z}} |\xi_k(nt)|^s \le 4 \sum_{k \ge 0} \gamma_k(nt)^s, \tag{4.9}$$

where

$$\gamma_k(nt) = \sum_{j=1+k}^{[nt]+1+k} j^{-\beta-1}, \qquad k \ge 0.$$

The following estimates for $J_n(s, \beta) \equiv \sum_{k \ge 0} \gamma_k(nt)^s$ are essential in the proof:

$$J_n(s, \beta) = \infty, \quad \text{if } \beta \le \frac{1}{s} - 1, \tag{4.10}$$

$$J_n(s, \beta) \le ([nt] + 1)^{(s-1)^+} \Big[1 + \{(\beta + 1)s - 1\}^{-1} + ([nt] + 2)^{2-(\beta+1)s}$$
$$\times \{(\beta + 1)s - 1\}^{-1} \{2 - (\beta + 1)s\}^{-1}\Big], \tag{4.11}$$
$$\text{if } \frac{1}{s} - 1 < \beta < \frac{2}{s} - 1,$$

$$J_n(s, \beta) \le ([nt] + 1)^{(s-1)^+} \{3 + \log([nt] + 2)\}, \quad \text{if } \beta = \frac{2}{s} - 1 \tag{4.12}$$

and

$$J_n(s, \beta) \le ([nt] + 1)^{(s-1)^+} \big[1 + 3\{((\beta + 1)s - 2)\}^{-1}\big], \quad \text{if } \beta > \frac{2}{s} - 1. \tag{4.13}$$

These estimates can easily be proved by the standard calculus, so we omit their proofs. Then (4.5) follows from (4.6), (4.8), (4.9), (4.11), (4.12) and (4.13). □

REMARK 4. We explain the reason for the restrictions on (α, H, s) in Theorem 4.

(a) If $0 < \alpha < 1$ and $s \le 2\alpha/(\alpha + 1)$, then

$$Q = (\alpha + 1)\alpha^{-1}\left(\frac{2\alpha}{\alpha + 1} - s\right) \ge 0.$$

Therefore we have to assume $s > 2\alpha/(\alpha + 1)$.

(b) If $0 < \alpha < 1$, $s > 1$ and $H \le 1 - 1/s$, then

$$Q = -Hs + s - 1 \ge 0.$$

Hence we need $H > 1 - 1/s$.

(c) If $1 \le \alpha < 2$ and $s \le 1$,

$$Q = (\alpha + 1)\alpha^{-1}\left(\frac{2\alpha}{\alpha + 1} - s\right) > 0.$$

If $1 \le \alpha < 2$ and $1 \le s \le \alpha$, then

$$Q = -\frac{s}{\alpha} + 1 \ge 0, \quad \text{when } H \ge \frac{1}{\alpha} + 1 - \frac{2}{s},$$

$$Q = -Hs + s - 1 > 0, \quad \text{when } H < \frac{1}{\alpha} + 1 - \frac{2}{s}.$$

Hence we have to assume $s > \alpha$ in case $1 \le \alpha < 2$. In this case, if $H \le 1 - 1/s$, then

$$Q = -Hs + s - 1 \ge 0.$$

REMARK 5. We exhibit below the negativity of Q in all cases in Theorem 4. Recall that $H = 1/\alpha - \beta$.

(a) The case of $0 < \alpha < 1$, $2\alpha/(\alpha + 1) < s \le 1$, $0 < H < 1$.
 (i) If $1/\alpha + 1 - 2/s < H < 1$ (which is possible only when $s < 2\alpha$), then $Q = 2 - s(\alpha + 1)/\alpha < 0$.
 (ii) If $0 < H \le 1/\alpha + 1 - 2/s$, then $Q = -Hs < 0$.

(b) The case of $0 < \alpha < 1$, $s > 1$, $1 - 1/s < H < 1$.
 (i) If $1/\alpha + 1 - 2/s \le H < 1$ (which is possible only when $s < 2\alpha$), then $Q = -s/\alpha + 1 < 0$.
 (ii) If $1 - 1/s < H < 1/\alpha + 1 - 2/s$, then $Q = -Hs + s - 1 < 0$.

(c) The case of $1 \le \alpha < 2$, $s > \alpha$, $1 - 1/s < H < 1$ $(H \ne 1/\alpha)$.
 (i) If $1/\alpha + 1 - 2/s < H < 1$ (which is possible only when $s < 2\alpha$), then $Q = -s/\alpha + 1 < 0$.
 (ii) If $1 - 1/s < H \le 1/\alpha + 1 - 2/s$, then $Q = -Hs + s - 1 < 0$.

If we look at the definition of $J_n(s, \beta)$, we easily see that if $\beta > 1/s$, then

$$J_n(s, \beta) \le \beta^{-s}\{(\beta + 1)^s + 1 + (\beta s - 1)^{-1}\} < \infty.$$

Hence in this case

$$\theta_s(\Delta_n(t), \Delta'_n(t)) \le Cn^{-Hs}\theta_s(X_0, X'_0). \tag{4.14}$$

Therefore another set of combinations of (α, H, s) for getting the rate of convergence of $\theta_s(\Delta_n(t), \Delta'_n(t))$ is given as follows. The proof is obvious.

THEOREM 5. *If*

(a) $0 < \alpha < 1$, $\alpha < s < \alpha/(1-\alpha)$, $0 < H < 1/\alpha - 1/s$,
(b) $0 < \alpha < 1$, $s \geq \alpha/(1-\alpha)$, $0 < H < 1$

or

(c) $1 \leq \alpha < 2$, $s > \alpha$, $0 < H < 1/\alpha - 1/s$,

then (4.14) *holds.*

5. Distance between the fractional stable process and any element in its domain of attraction. For simplicity, we assume $A_1 = 1$ and $A_2 = 0$ in (4.1) and (4.2). A bound on $u(\Delta_n^*(t), \Delta(t))$ is given as follows.

THEOREM 6.

$$u(\Delta_n^*(t), \Delta(t)) \leq (K\pi)^{-1}\psi(n),$$

where

$$K \equiv K(\alpha, \beta) \equiv \int_{-\infty}^{\infty} \left| |t-s|^{-\beta} - |s|^{-\beta} \right|^{\alpha} ds < \infty$$

and

$$\psi(n) \equiv \begin{cases} C_1 n^{-1}, & \text{for } \beta < 0, \\ C_2(n,t) n^{-H\alpha}, & \text{for } \beta > 0. \end{cases}$$

Here C_1 depends only on α and β, $C_2(n,t) < C_3$ if $nt > C_4$, and C_3 and C_4 are constants depending on α and β.

From (2.10), (2.13), (4.4) and Theorem 6, we have the following.

THEOREM 7. *If $\theta_s(X_0, X_0^*) < \infty$, then we have in all cases considered in Theorems* 4 *and* 5,

$$u(\Delta_n(t), \Delta(t)) \leq (1 + C_5)\{C(s)\theta_s(\Delta_n(t), \Delta_n^*(t))\}^{(s+1)^{-1}} + (K\pi)^{-1}\psi(n), \tag{5.1}$$

for each $t \in [0,1]$, where C_5 is a positive constant depending on α and β.

To get the rate of convergence of $\Delta_n(t)$ to $\Delta(t)$ in Theorem 7, it is enough to apply Theorem 4 or 5 to $\theta_s(\Delta_n(t), \Delta_n^*(t))$ in the first term on the right-hand side of (5.1). We start with the proof of Theorem 7.

PROOF OF THEOREM 7. Recall (4.4) and note that Theorem 6 deals with the second terms on the right-hand sides of (4.4) and (5.1). It remains to compare the first terms. By (2.13),

$$u(\Delta_n(t), \Delta_n^*(t)) \le (1 + a_n)L(\Delta_n(t), \Delta_n^*(t)),$$

where $a_n = \sup_x|(d/dx)P(\Delta_n^*(t) \le x)|$. Since

$$h_n(z) \equiv E\{\exp(iz\Delta_n^*(t))\} = \exp\left\{-|z|^\alpha|\beta|^\alpha n^{-H\alpha} \sum_{k\in\mathbf{Z}} |\xi_k(nt)|^\alpha\right\},$$

where $\xi_k(nt)$ is the one defined in (4.7), we write

$$h_n(z) = \exp\{-K_n|z|^\alpha\},$$

where

$$K_n \equiv K_n(\alpha, \beta) \equiv |\beta|^\alpha n^{-H\alpha} \sum_{k\in\mathbf{Z}} |\xi_k(nt)|^\alpha.$$

In Maejima (1983), it was shown that $K_n \to K$ as $n \to \infty$. Therefore

$$\left|\frac{d}{dx}P(\Delta_n^*(t) \le x)\right| \le (2\pi)^{-1}\int_{-\infty}^{\infty} |h_n(z)|\,dz \le (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-2^{-1}K|z|^\alpha)\,dz,$$

for large n. Thus $a_n \le C_5$, some positive constant depending on α and β. On the other hand, we have by (2.10) in Theorem 3 that

$$L(\Delta_n(t), \Delta_n^*(t)) \le \{C(s)\theta_s(\Delta_n(t), \Delta_n^*(t))\}^{(s+1)^{-1}}.$$

Hence

$$u(\Delta_n(t), \Delta_n^*(t)) \le (1 + C_5)\{C(s)\theta_s(\Delta_n(t), \Delta_n^*(t))\}^{(s+1)^{-1}}. \qquad \square$$

PROOF OF THEOREM 6. Recall that

$$h_n(z) = E\{\exp(iz\Delta_n^*(t))\} = \exp(-K_n|z|^\alpha)$$

and

$$h(z) \equiv E\{\exp(iz\Delta(t))\} = \exp(-K|z|^\alpha)$$

[see Maejima (1983)]. Hence

$$|h_n(z) - h(z)| \le \exp(-K|z|^\alpha)|z|^\alpha|K_n - K|.$$

By the Esseen inequality [cf. Petrov (1975), page 109],

$$u(\Delta_n^*(t), \Delta(t)) \le b\int_{-T}^{T}\left|\frac{h_n(z) - h(z)}{z}\right|dz + r(b)CT^{-1},$$

where T is an arbitrary positive number, b is any positive number greater than $(2\pi)^{-1}$, $r(b)$ is a positive constant depending only on b and

$$C = \sup_x\left|\frac{d}{dx}P(\Delta(t) \le x)\right|.$$

Hence

$$u(\Delta_n^*(t), \Delta(t)) \le b|K_n - K|\int_{-T}^{T}\exp(-K|z|^\alpha)|z|^{\alpha-1}\,dz + r(b)CT^{-1}.$$

Since $\exp(-K|z|^{\alpha})|z|^{\alpha-1}$ is integrable over $(-\infty,\infty)$ for $0<\alpha<2$, we obtain by letting $T\to\infty$

$$u(\Delta_n^*(t),\Delta(t))\le (K\pi)^{-1}|K_n-K|.$$

Hence, to complete the proof of the theorem, it remains to show

$$|K_n-K|\le\psi(n). \tag{5.2}$$

The proof of (5.2) is split into the following four parts:

(a) $$\left|\sum_{k=-\infty}^{0}|\beta n^{-H}\xi_k(nt)|^{\alpha}-\int_{-\infty}^{0}||t-s|^{-\beta}-|s|^{-\beta}|^{\alpha}\,ds\right|\le\psi(n),$$

(b) $$\left|\sum_{k=1}^{[[nt]/2]}|\beta n^{-H}\xi_k(nt)|^{\alpha}-\int_{0}^{t/2}||t-s|^{-\beta}-|s|^{-\beta}|^{\alpha}\,ds\right|\le\psi(n),$$

(c) $$\left|\sum_{k=[[nt]/2]+1}^{[nt]-1}|\beta n^{-H}\xi_k(nt)|^{\alpha}-\int_{t/2}^{t}||t-s|^{-\beta}-|s|^{-\beta}|^{\alpha}\,ds\right|\le\psi(n)$$

and

(d) $$\left|\sum_{k=[nt]}^{\infty}|\beta n^{-H}\xi_k(nt)|^{\alpha}-\int_{t}^{\infty}||t-s|^{-\beta}-|s|^{-\beta}|^{\alpha}\,ds\right|\le\psi(n).$$

We only prove (a). The other inequalities can be handled similarly.

PROOF OF (a). Denote

$$\delta_n\equiv\left|\sum_{k=-\infty}^{-2}|\beta n^{-H}\xi_k(nt)|^{\alpha}-\int_{-\infty}^{-3/n}||t-s|^{-\beta}-|s|^{-\beta}|^{\alpha}\,ds\right|.$$

Since

$$\xi_k(nt)=\sum_{j=1-k}^{[nt]-k}j^{-\beta-1}+(nt-[nt])([nt]+1-k)^{-\beta-1},$$

for $k\le 0$ and $\beta+1>0$, we have

$$0\le\int_{1-k}^{nt+1-k}x^{-\beta-1}\,dx\le\xi_k(nt)\le\int_{-k}^{nt-k}x^{-\beta-1}\,dx.$$

Hence

$$\begin{aligned}\underline{\lambda}_n&\equiv\sum_{k=-\infty}^{-2}n^{-1}\left|\left(\frac{1-k}{n}\right)^{-\beta}-\left(t+\frac{1-k}{n}\right)^{-\beta}\right|^{\alpha}\\&\le\sum_{k=-\infty}^{-2}|\beta n^{-H}\xi_k(nt)|^{\alpha}\\&\le\sum_{k=-\infty}^{-2}n^{-1}\left|\left(-\frac{k}{n}\right)^{-\beta}-\left(t-\frac{k}{n}\right)^{-\beta}\right|^{\alpha}\equiv\bar{\lambda}_n.\end{aligned}$$

Denote

$$g(a) \equiv n^{-1}\left||t-a|^{-\beta} - |a|^{-\beta}\right|^{\alpha}, \qquad a \in \mathbb{R}.$$

Then

$$\underline{\lambda}_n \geq \int_{-\infty}^{-3/n} \left||t-s|^{-\beta} - |s|^{-\beta}\right|^{\alpha} ds \equiv L_n$$

and

$$\bar{\lambda}_n \leq g\left(\frac{3}{n}\right) + g\left(\frac{2}{n}\right) + L_n.$$

Hence

$$\delta_n \leq g\left(\frac{2}{n}\right) + g\left(\frac{3}{n}\right).$$

For any $a \in \mathbb{R}$,

$$g\left(\frac{a}{n}\right) \leq n^{-1}\max\left\{\left|t-\frac{a}{n}\right|^{-\alpha\beta}, \left|\frac{a}{n}\right|^{-\alpha\beta}\right\} \leq \psi(n).$$

Thus

(5.3) $$\delta_n \leq \psi(n).$$

Let $t > 0$ (if $t = 0$, $K_n = K = 0$) and choose n such that $1/n < t$. Since $t \in [0,1]$, we have for some $C > 0$

$$\int_0^{1/n} \left|u^{-\beta} - (t+u)^{-\beta}\right|^{\alpha} du \leq C\left\{\left(\frac{1}{n}\right)^{1-\alpha\beta} + \left(t+\frac{1}{n}\right)^{1-\alpha\beta} - t^{1-\alpha\beta}\right\} \leq \psi(n).$$

So

(5.4) $$\int_{-3/n}^{0} \left||t-s|^{-\beta} - |s|^{-\beta}\right|^{\alpha} ds \leq \psi(n).$$

If $k = 0, 1$,

$$\left|\beta n^{-H}\xi_{-1}(nt)\right|^{\alpha} = |\beta n^{-H}|^{\alpha}\left|\sum_{j=1}^{[nt]+1} j^{-\beta-1} + (nt-[nt])([nt]+2)^{-\beta-1}\right|^{\alpha}$$

(5.5) $$\leq n^{-1}\left|\left(t+\frac{1}{n}\right)^{-\beta} - \left(\frac{1}{n}\right)^{-\beta}\right|^{\alpha}$$

$$\leq n^{-1}\max\left\{\left(t+\frac{1}{n}\right)^{-\alpha\beta}, \left(\frac{1}{n}\right)^{-\alpha\beta}\right\} \leq \psi(n)$$

and similarly

(5.6) $$\left|\beta n^{-H}\xi_0(nt)\right|^{\alpha} \leq \psi(n).$$

By (5.3)–(5.6), we obtain (a).

A similar approach to (b)–(d) completes the proof of Theorem 6. □

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REFERENCES

DOBRUSHIN, R. L. and MAJOR, P. (1979). Non-central limit theorems for non-linear functionals of Gaussian fields. *Z. Wahrsch. verw. Gebiete* **50** 27–52.

ERDÉLYI, A. (1975). Fractional integrals of generalized functions. *Fractional Calculus and Its Applications. Lecture Notes in Math.* **457** 151–170. Springer, Berlin.

GRIGOREVSKI, N. N. and SHIGANOV, I. S. (1976). On certain modifications of Dudley's metric. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **41** 17–24.

KESTEN, H. and SPITZER, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. verw. Gebiete* **50** 5–25.

MAEJIMA, M. (1983). On a class of self-similar processes. *Z. Wahrsch. verw. Gebiete* **62** 235–245.

PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer, Berlin.

TAQQU, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. verw. Gebiete* **50** 53–83.

VERVAAT, W. (1985). Sample path properties of self-similar processes with stationary increments. *Ann. Probab.* **13** 1–27.

YAMUKOV, G. I. (1977). Estimates of generalized Dudley metrics in spaces of finite-dimensional distributions. *Theory Probab. Appl.* **22** 579–584.

ZOLOTAREV, V. M. (1976a). Metric distances in spaces of random variables and their distributions. *Math. USSR-Sb.* **30** 373–401.

ZOLOTAREV, V. M. (1976b). Approximation of distributions of sums of independent random variables with values in infinite-dimensional spaces. *Theory Probab. Appl.* **21** 721–737.

ZOLOTAREV, V. M. (1979). Ideal metrics in the problems of probability theory and mathematical statistics. *Austral. J. Statist.* **21** 193–208.

ZOLOTAREV, V. M. (1983). Probability metrics. *Theory Probab. Appl.* **28** 278–302.

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