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BROWNIAN EXCURSIONS AND MINIMAL THINNESS. I

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Integral-type criteria for thinness and minimal thinness are given for a class of sets. A Kolmogorov-type criterion for Brownian excursions is obtained as a corollary.

Introduction. The relationship between Brownian motion and potential theory has been studied for years and two recent books by Port and Stone (1978) and Doob (1984) have been devoted exclusively to this subject. More recently the excursion theory of Markov processes has been initiated by Itô (1972) and generalized by Maisonneuve (1975) to excursions starting at a random point. It has become clear that excursions are also closely connected with potential theory and some work in this direction has been done, e.g., Rogers (1983). The present paper will follow the path started in Burdzy (1986a) where the local properties of Brownian excursions have been related to the boundary behavior of the Green function.

The present article is the first in a series of three articles. The first (i.e., the present) one gives some criteria for minimal thinness and shows how to apply them to study the local properties of Brownian excursions. The second article [Burdzy (1986b)] is devoted to the boundary behavior of the Green function. The third one [Burdzy (1986c)] presents applications of probability in general and Brownian excursion laws in particular to the angular derivative problem. For ease of reference there is a continuous numbering of sections, formulae and theorems throughout all three articles of the series.

The methods used in the paper include Brownian motion and potential theory [see Port and Stone (1978) and Doob (1984)] and excursion theory [see Maisonneuve (1975) and Burdzy (1986a)].

The main results of Part I are:

(i) an estimate of capacity for a class of sets [Theorem 2.2, cf. Landkof (1972), II, Section 3.14, and Port and Stone (1978) Proposition 3.3.4];
(ii) an integral-type criterion for thinness for a class of sets (Theorems 2.3 and 2.4);
(iii) a lemma relating minimal thinness and thinness in a special case (Theorem 3.1);
(iv) an integral-type criterion for minimal thinness for a class of sets (Theorem 3.2);
(v) a Kolmogorov-type criterion for Brownian excursions (Remark 3.2).

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1. Brownian excursions. In order to keep this section short the reader is referred to Doob (1984) for definitions of Brownian motion in a region D, Martin boundary and topology, attainable Martin boundary points and related concepts.

The probability space Ω will be the set of all paths ω: $(0, \infty) \to \mathbb{R}^n \cup \{\delta\}$ which are continuous on $(0, R)$ and have a left limit at R if $R < \infty$. The lifetime R of a path ω is defined as the time of the jump to the isolated trap δ in $\mathbb{R}^n \cup \{\delta\}$ (possibly $R = \infty$).

The process X is the coordinate mapping, i.e.,

$$X_t(\omega) = \omega(t), \quad \text{for all } \omega \text{ and } t.$$

Let

$$\mathbb{F}_t^0 = \bigcap_{u>t} \sigma(X(s), 0 < s \le u)$$

and

$$\mathbb{F}^0 = \sigma(X(s), 0 < s \le \infty).$$

$\mathbb{F}_t$ and $\mathbb{F}$ are the universal completions of $\mathbb{F}_t^0$ and $\mathbb{F}^0$ in $\mathbb{F}^0$.

The hitting time T_A of an analytic set $A \subset \mathbb{R}^n$ is defined as $\inf\{t > 0: X(t) \in A\}$ and $T_{A-} = \inf\{t > 0: \lim_{s \to t-} X(s) \in A\}$ where $\inf \varnothing = \infty$.

Let $D \subset \mathbb{R}^n$, $n \ge 2$, be a Greenian domain. Its Euclidean (minimal Martin) boundary will be denoted $\partial D(\partial_1^M D)$. It will be understood that the convergence to a Euclidean (Martin) boundary point takes place in the Euclidean (Martin) topology.

The distribution of the Brownian motion (Brownian motion in D) starting from $z \in D$ will be denoted P^z (P_D^z). Analogously, P^μ (P_D^μ) will be the distribution of Brownian motion (Brownian motion in D) with the initial distribution μ. "Brownian motion in D" will be abbreviated to BM_D. In the present setting $R = T_{\partial D-}$, P_D^z-a.s. for every $z \in D$.

Definition 1.1. A σ-finite measure H^x, $x \in \partial_1^M D$, on $(\Omega, \mathbb{F})$ is called an excursion law in D (or equivalently an excursion law from K, $K = \mathbb{R}^n \setminus D$) if

$$H^x\Big(\lim_{t \to 0+} X(t) \neq x\Big) = 0$$

and H^x is strong Markov for BM_D transition probabilities in the following sense. For all everywhere positive $\mathbb{F}_t$-stopping times T, $\mathbb{F}_T^0$-measurable a and $\mathbb{F}^0$-measurable b

$$H^x(a \cdot b(\theta_T)) = H^x\big(a \cdot P_D^{X_T}(b)\big),$$

where θ is the usual shift operator.

There is no natural normalization of infinite measures, so "unique excursion law" will mean "unique up to a multiplicative constant excursion law." "Distinct" should be understood in the same spirit.

DEFINITION 1.2. An excursion law H^x is called standard if $H^x(T_B < \infty) < \infty$ for all compact sets $B \subset D$. An excursion law H^x is called null if $H^x(\Omega \setminus \{\omega_\delta\}) = 0$, where $\omega_\delta(t) \equiv \delta$.

THEOREM 1.1. *If $x \in \partial_1^M D$ is attainable, then there exists a unique nonnull standard excursion law H^x.*

REMARK 1.1. (i) The above result has been given in Burdzy (1986a) as Theorem 4.1.

(ii) If D is a bounded Lipschitz domain, then the minimal Martin boundary and topology coincide with the Euclidean boundary and topology [see Hunt and Wheeden (1970)]. In such a case all boundary points are attainable by the results of Cranston (1985). This can be extended to unbounded Lipschitz domains by the last exit decomposition, i.e., even if D is unbounded, then every point $z \in \partial D$ corresponds to a unique attainable minimal Martin boundary point (but not vice versa).

(iii) If $n = 2$, i.e., D is a plane domain and if D is bounded and simply connected, then the minimal Martin boundary points are represented by the prime ends [see Doob (1984), page 199, and Ohtsuka (1970), Section 3.2]. If a prime end is accessible [see Ohtsuka (1970), page 253, for definition], then it is attainable, as has been proved in Burdzy (1986d).

An element $x \in \mathbb{R}^n$ will be also denoted $(x_1, x_2, \dots, x_n)$ and the symbol 0 will be used instead of $(0, 0, \dots, 0)$. The Euclidean norm of $x \in \mathbb{R}^n$ will be denoted $|x|$. A set A will be called regular for a measure P if $P(T_A \neq 0) = 0$.

THEOREM 1.2. *Let $D = \{x_1 > 0\} \subset \mathbb{R}^n$. Then there exists a unique excursion law H^0 in D.*

The H^0-density of $X(R-)$ (wrt surface area measure) is proportional to $|x|^{-n}$, $x \in \partial D$.

A set $A \subset D$ is regular for H^0 if and only if the set

$$A_1 \overset{\text{df}}{=} \left\{x \in \mathbb{R}^{n+2}\colon \left(\sqrt{x_1^2 + x_2^2 + x_3^2}, x_4, \dots, x_{n+2}\right) \in A\right\}$$

is regular for P^0 in $\mathbb{R}^{n+2}$.

See Theorems 3.3(iv) and 3.4 of Burdzy (1986a) for a proof of the above result.

2. Wiener's test and its applications. Theorem 1.2 can be successfully applied only if one has a usable test for regularity. The well-known Wiener's test (Theorem 2.1 below) is rather abstract. The rest of this section is devoted to deriving more applicable forms of this test for special sets. Attention will be focused on n-dimensional spaces, $n \geq 4$, due to a particularly simple form of the results. Also, Theorem 1.2 requires a regularity criterion only for $n \geq 4$.

Let $A \subset \mathbb{R}^n$, $n \geq 4$. Choose λ, $0 < \lambda < 1$, and define

$$A_k = \{z \in A\colon \lambda^{k+1} \leq |z| < \lambda^k\}, \qquad k \geq 1.$$

Let $C(B)$ denote the Newtonian capacity of a set B.

THEOREM 2.1 (Wiener's test). *A set A is regular for P^0 if and only if*

$$\sum_{k=1}^{\infty} \lambda^{k(2-n)} \cdot C(A_k) = \infty.$$

Various proofs of the above theorem may be found in Doob (1984), Landkof (1972) and Port and Stone (1978).

The next theorem will be used to obtain special forms of Wiener's test.

Fix an $n \geq 4$ and denote

$$L \overset{\text{df}}{=} \{x \in \mathbb{R}^n : x_1 = x_2 = x_3 = 0\}.$$

The orthogonal projection on a k-dimensional hyperplane N, $k \leq n-1$, will be denoted proj_N. Let $h: L \to \mathbb{R}$ be a nonnegative Lipschitz function with a constant K, $K < \infty$, i.e.,

$$|h(x) - h(y)| \leq K|x - y|, \quad \text{for all } x, y \in L.$$

Define

$$A = \left\{x \in \mathbb{R}^n : \sqrt{x_1^2 + x_2^2 + x_3^2} < h(\text{proj}_L x)\right\}.$$

Throughout this section "constants" $c, c_0, c_1, \ldots$ are strictly positive and finite numbers which may depend on n and K only.

THEOREM 2.2. *There exists a constant c such that*

$$c \int_L h(x)\, dx \leq C(A) \leq c^{-1} \int_L h(x)\, dx,$$

where the integral is taken wrt surface area measure.

PROOF. Suppose first that $\int_L h(x)\, dx < \infty$ and therefore h is bounded. Assume also that $h \leq 1$.

Some more notation will be now introduced.

$$B \overset{\text{df}}{=} \text{proj}_L(A).$$

For each integer k, $k \geq 1$, let $Q_k^1, Q_k^2, \ldots$ be the sequence of all sets of the form

$$\left\{x \in L : \frac{k_4}{2^k} < x_4 < \frac{k_4+1}{2^k}, \frac{k_5}{2^k} < x_5 < \frac{k_5+1}{2^k}, \ldots, \frac{k_n}{2^k} < x_n < \frac{k_n+1}{2^k}\right\},$$

for some integers $k_4, k_5, \ldots, k_n$.

A set $E \subset \mathbb{R}^n$ will be called a set of class 1 if there exists a set Q_1^j such that

$$E = \left\{x \in A : \text{proj}_L x \in Q_1^j\right\}$$

and $h(x) \geq 2^{-1}$ for some $x \in Q_1^j$. Call $\mathbb{E}_1$ the union of all sets of class 1.

Sets of class k, $k \geq 2$, will be defined inductively.

A set $E \subset \mathbb{R}^n$ will be called a set of class k if there exists a set Q_k^j such that

$$E = \left\{ x \in A \setminus \bigcup_{m=1}^{k-1} \mathbb{E}_m : \operatorname{proj}_L x \in Q_k^j \right\}$$

and $h(x) \geq 2^{-k}$ for some $x \in Q_k^j$. $\mathbb{E}_k$ is the union of all sets of class k.

Let $A_1, A_2, \ldots$ be a sequence of all sets belonging to any class defined above. The class of the set A_k will be denoted $\chi(k)$.

$$B_k \stackrel{\text{df}}{=} \operatorname{proj}_L A_k, \qquad k \geq 1.$$

There exists a constant c_0 such that each set A_k is contained in an n-dimensional cube with side length $2^{-\chi(k)} c_0$. By scaling, the capacity of a cube with side a is $c_1 a^{n-2}$ [see Port and Stone (1978), page 60]. Therefore

$$C(A_k) \leq c_1 \cdot \left(c_0 \cdot 2^{-\chi(k)}\right)^{n-2} = c_2 \cdot 2^{-\chi(k)(n-2)}.$$

It follows easily from the definition of the class and the Lipschitz property of h that for some constant c_3

$$\begin{aligned} 2 \cdot 2^{-\chi(k)(n-2)} \geq \int_{B_k} h(x)\, dx &\geq c_3 \cdot \left(2^{-\chi(k)}\right)^{n-3} \cdot 2^{-\chi(k)} \\ &= c_3 \cdot 2^{-\chi(k)(n-2)}. \end{aligned} \tag{2.1}$$

Thus

$$C(A_k) \leq c_2 \cdot c_3^{-1} \int_{B_k} h(x)\, dx.$$

It is easy to see that $C(A_k) = C(\overline{A_k})$ for all $k \geq 1$ where $\overline{A_k}$ is the closure of A_k. Since $h(x) = 0$ for almost all (wrt surface area measure) $x \in B \setminus \bigcup_{k=1}^\infty B_k$, it follows that

$$\begin{aligned} C(A) \leq \sum_{k=1}^\infty C(\overline{A_k}) &= \sum_{k=1}^\infty C(A_k) \\ &\leq \sum_{k=1}^\infty c_2 \cdot c_3^{-1} \int_{B_k} h(x)\, dx \\ &= c_2 \cdot c_3^{-1} \int_B h(x)\, dx. \end{aligned}$$

It remains to prove the first inequality of the theorem. Assume that A is bounded. Let r_0 be the diameter of A and assume WLOG that $0 \in A$.

Define an order $\gg$ among positive integers by declaring that $j \gg k$ if

$$\chi(j) > \chi(k)$$

or

$$\chi(j) = \chi(k) \quad \text{and} \quad j > k.$$

Choose c_4 so small that for each $k \geq 1$ there is a sphere $S_k^0 = S_k^0(x^k, r_k)$, $r_k = c_4 \cdot 2^{-\chi(k)}$ such that

$$S_k^0 \subset A_k, \qquad \operatorname{dist}\left(S_k^0, A \setminus A_k\right) \geq c_4 \cdot 2^{-\chi(k)}$$

and

$$\operatorname{dist}\left(\operatorname{proj}_L x^k, A \setminus A_k\right) \geq c_4 \cdot 2^{-\chi(k)}.$$

Such a choice is possible due to the Lipschitz property of h. For each $k \geq 1$ a new sphere S_k will be defined such that S_k is concentric with S_k^0 and has radius $c_5 \cdot 2^{-\chi(k)}$, $c_5 < c_4$. The constant c_5 will be chosen later. Let $S = \bigcup_{k=1}^{\infty} S_k$. If

$$\sum_{k=1}^{\infty} P^x\left(T_{S_k} < \infty\right) < \infty, \tag{2.2}$$

then

$$\begin{aligned} P^x(T_S < \infty) &= P^x\left(\bigcup_{k=1}^{\infty} \{T_{S_k} < \infty\}\right) \\ &= \sum_{k=1}^{\infty} P^x\left(T_{S_k} < \infty\right) - \sum_{j \gg k} P^x\left(T_{S_k} < \infty \text{ and } T_{S_j} < \infty\right). \end{aligned} \tag{2.3}$$

If $S^0 = S^0(y, r)$ is a sphere and $|x - y| \geq r$, then

$$P^x(T_{S^0} < \infty) = \frac{r^{n-2}}{|x - y|^{n-2}}.$$

Suppose that $|x| > 2r_0$. Since $A \subset \{|x| \leq r_0\}$, then for each $y \in A$

$$|x|/2 \leq |x - y| \leq 2|x|.$$

Therefore if $|x| > 2r_0$, then

$$\frac{\left(c_5 \cdot 2^{-\chi(k)}\right)^{n-2}}{\left(2|x|\right)^{n-2}} \leq P^x\left(T_{S_k} < \infty\right) \leq \frac{\left(c_5 \cdot 2^{-\chi(k)}\right)^{n-2}}{\left(|x|/2\right)^{n-2}}. \tag{2.4}$$

In view of (2.1)

$$P^x\left(T_{S_k} < \infty\right) \leq \frac{c_5^{n-2}}{\left(|x|/2\right)^{n-2}} c_3^{-1} \int_{B_k} h(x)\, dx$$

and

$$\sum_{k=1}^{\infty} P^x\left(T_{S_k} < \infty\right) \leq \frac{c_5^{n-2}}{\left(|x|/2\right)^{n-2}} c_3^{-1} \int_B h(x)\, dx < \infty.$$

Thus (2.2) and therefore (2.3) are valid for $|x| > 2r_0$.

Let d_{jk} be the distance between S_k and S_j. For every $x \in S^k$

$$P^x\left(T_{S_j} < \infty\right) \leq \frac{\left(c_5 \cdot 2^{-\chi(j)}\right)^{n-2}}{\left(d_{jk}\right)^{n-2}}.$$

The strong Markov property applied at T_{S_k} and (2.4) imply that for $|x| > 2r_0$

$$P^x\left(T_{S_k} < \infty \text{ and } \inf\left\{t > T_{S_k}: X(t) \in S_j\right\} < \infty\right)$$

$$\leq \frac{\left(c_5 \cdot 2^{-\chi(k)}\right)^{n-2}}{\left(|x|/2\right)^{n-2}} \frac{\left(c_5 \cdot 2^{-\chi(j)}\right)^{n-2}}{\left(d_{jk}\right)^{n-2}}$$

$$= c_6 \cdot c_5^{2(n-2)} \frac{2^{-\chi(k)(n-2)} 2^{-\chi(j)(n-2)}}{|x|^{n-2} d_{jk}^{n-2}}.$$

It follows that for $|x| > 2r_0$

$$P^x\left(T_{S_k} < \infty \text{ and } T_{S_j} < \infty\right) \leq P^x\left(T_{S_k} < \infty \text{ and } \inf\left\{t > T_{S_k}: X(t) \in S_j\right\} < \infty\right)$$

$$+ P^x\left(T_{S_j} < \infty \text{ and } \inf\left\{t > T_{S_j}: X(t) \in S_k\right\} < \infty\right)$$

$$\leq 2 \cdot c_6 \cdot c_5^{2(n-2)} \frac{2^{-\chi(k)(n-2)} 2^{-\chi(j)(n-2)}}{|x|^{n-2} d_{jk}^{n-2}}.$$

By (2.1)

$$2^{-\chi(j)(n-2)} \leq c_3^{-1} \int_{B_j} h(y)\, dy \leq c_3^{-1} \int_{B_j} 2^{-\chi(j)+1}\, dy.$$

It follows from the definition of c_4 that for some c_7 and all $j \gg k$

$$d_{jk} \geq c_7 \sup_{y \in A_j} |\text{proj}_L x^k - y|.$$

Then

$$\frac{2^{-\chi(j)(n-2)}}{\left(d_{jk}\right)^{n-2}} \leq \frac{1}{c_3\left(d_{jk}\right)^{n-2}} \int_{B_j} 2^{-\chi(j)+1}\, dy$$

$$\leq \frac{2^{-\chi(j)+1}}{c_3 c_7^{n-2}} \int_{B_j} \frac{1}{|\text{proj}_L x^k - y|^{n-2}}\, dy$$

and for $|x| > 2r_0$

$$P^x\left(T_{S_k} < \infty \text{ and } T_{S_j} < \infty\right)$$

$$\leq 2 \cdot c_6 \cdot c_5^{2(n-2)} \frac{2^{-\chi(k)(n-2)}}{|x|^{n-2}} \frac{2^{-\chi(j)+1}}{c_3 c_7^{n-2}} \int_{B_j} \frac{1}{|\text{proj}_L x^k - y|^{n-2}}\, dy$$

$$= c_8 \cdot c_5^{2(n-2)} \frac{2^{-\chi(k)(n-2)}}{|x|^{n-2}} \cdot 2^{-\chi(j)} \int_{B_j} \frac{1}{|\text{proj}_L x^k - y|^{n-2}}\, dy.$$

Since $\chi(j) \geq \chi(k)$ for all $j \gg k$, then for $|x| > 2r_0$

$$\sum_{j \gg k} P^x\big(T_{S_k} < \infty \text{ and } T_{S_j} < \infty\big)$$

$$\text{(2.5)}\qquad \begin{aligned} &\leq \sum_{j \gg k} c_8 \cdot c_5^{2(n-2)} \frac{2^{-\chi(k)(n-2)}}{|x|^{n-2}} \cdot 2^{-\chi(k)} \int_{B_j} \frac{1}{|\operatorname{proj}_L x^k - y|^{n-2}}\, d \\ &\leq c_8 \cdot c_5^{2(n-2)} \frac{2^{-\chi(k)(n-2)}}{|x|^{n-2}} \\ &\quad \times 2^{-\chi(k)} \int_{\{|\operatorname{proj}_L x^k - y| \geq c_4 2^{-\chi(k)}\}} \frac{1}{|\operatorname{proj}_L x^k - y|^{n-2}}\, dy < \infty. \end{aligned}$$

Note that the value of

$$2^{-\chi(k)} \int_{\{|\operatorname{proj}_L x^k - y| \geq c_4 2^{-\chi(k)}\}} \frac{1}{|\operatorname{proj}_L x^k - y|^{n-2}}\, dy$$

does not depend on k. Therefore it is possible to choose c_5 so small that (2.5) is less than

$$\text{(2.6)}\qquad \frac{1}{2} \cdot \frac{\big(c_5 \cdot 2^{-\chi(k)}\big)^{n-2}}{(2|x|)^{n-2}}.$$

The constant c_5 chosen in this way depends on c_7, but c_7 may be chosen independently of c_5. Thus the definitions of the constants involved are not circular.

By (2.4) and (2.6) the left-hand side of (2.3) is not less than

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\frac{\big(c_5 \cdot 2^{-\chi(k)}\big)^{n-2}}{(2|x|)^{n-2}} - \frac{1}{2} \frac{\big(c_5 \cdot 2^{-\chi(k)}\big)^{n-2}}{(2|x|)^{n-2}} \right) &\geq \frac{c_5^{n-2}}{2^{n-1}|x|^{n-2}} \sum_{k=1}^{\infty} 2^{-\chi(k)(n-2)} \\ &\geq \frac{c_5^{n-2}}{2^n |x|^{n-2}} \int_B h(x)\, dx. \end{aligned}$$

The last inequality follows from (2.1) and implies that

$$\liminf_{|x| \to \infty} P^x(T_S < \infty)|x|^{n-2} \geq \frac{c_5^{n-2}}{2^n} \int_B h(x)\, dx.$$

By (6) of Port and Stone [(1978), page 59]

$$C(A) \geq C(S) \geq \frac{c_5^{n-2}}{2^n} \int_B h(x)\, dx.$$

Thus the theorem is proved in the special case when $\int_B h(x)\, dx < \infty$, $h(x) \leq 1$, and A is bounded. These additional assumptions will be now removed.

Suppose that A is unbounded, but $\int_B h(x)\, dx < \infty$ and $h \leq 1$. Then there exists a Lipschitz function h_1, $h_1 \leq h$, with the same constant as h such that the corresponding set A_1 is bounded and

$$\int_B h_1(x)\,dx > \tfrac{1}{2}\int_B h(x)\,dx.$$

It follows that

$$C(A) \geq C(A_1) \geq c_9 \int_B h_1(x)\,dx \geq \tfrac{1}{2}c_9 \int_B h(x)\,dx$$

and therefore the theorem holds for unbounded sets A (with a different constant c).

If $\int_B h(x)\,dx < \infty$ but h is not necessarily bounded by 1, the result is obtained by scaling.

It remains to treat the case when $\int_B h(x)\,dx = \infty$. In this case h is an increasing limit of a sequence of Lipschitz functions $h_1, h_2, \dots$ with the same constant as h and such that $\int_B h_k(x)\,dx = k$. Thus A is the union of a family of sets A_k (corresponding to h_k's) which have capacities greater than $c_{11}k$. □

Let L, h and A be defined as in Theorem 2.2. Recall that it is assumed that $n \geq 4$.

THEOREM 2.3. *If h is Lipschitz then $A \subset \mathbb{R}^n$ is regular for P^0 if and only if*

$$\int_{L \cap \{|x| \leq 1\}} \frac{h(x)}{|x|^{n-2}}\,dx = \infty.$$

PROOF. Let $\lambda = \frac{1}{2}$ and A_k be defined as in Theorem 2.1.

$$B_k = \overline{A}_k \cap L.$$

Consider two cases. First assume that $\limsup_{|x|\to 0} h(x)/|x| > 0$. Then there exists a sequence of balls M_k, $k \geq 1$, in A with a constant ratio of the radius to the distance from 0 and such that 0 is a cluster point of their union. Let $N_k = \bigcup_{m=k}^{\infty} M_m$. Elementary properties of Brownian motion (scaling, continuity of paths, etc.) imply that

$$\begin{aligned} P^0(T_A = 0) &\geq P^0\left(\bigcap_{k=1}^{\infty} \{T_{N_k} < \infty\}\right) = \lim_{k\to\infty} P^0(T_{N_k} < \infty) \\ &\geq \lim_{k\to\infty} P^0(T_{M_k} < \infty) = c_1 > 0. \end{aligned}$$

By the 0–1 law, $c_1 = 1$. It is easy to see that

$$\int_{L \cap \{|x| \leq 1\}} \frac{h(x)}{|x|^{n-2}}\,dx = \infty$$

and the theorem follows.

Now suppose that $\lim_{|x|\to 0} h(x)/|x| = 0$. This and the Lipschitz property imply that there exist a constant c_2, $0 < c_2 < 1$, and sets A_k^1, A_k^2, $k \geq 1$, such that for all k larger than some k_0, A_k^1 and A_k^2 correspond to Lipschitz functions

h_k^1 and h_k^2 with the same constant as h,

$$A_k^1 \subset A_k \subset A_k^2$$

and

$$c_2 \int_{B_k} h(x)\,dx \le \int_L h_k^1(x)\,dx \le \int_{B_k} h(x)\,dx \le \int_L h_k^2(x)\,dx \le c_2^{-1} \int_{B_k} h(x)\,dx.$$

The existence of sets A_k^1 and A_k^2 may be shown, for example, using families of sets analogous to $\mathbb{E}_k$'s defined in the proof of Theorem 2.2. Thus by Theorem 2.2

$$C(A_k) \ge C(A_k^1) \ge c \int_L h_k^1(x)\,dx \ge c \cdot c_2 \int_{B_k} h(x)\,dx$$

and

$$C(A_k) \le C(A_k^2) \le c^{-1} \int_L h_k^2(x)\,dx \le c^{-1} c_2^{-1} \int_{B_k} h(x)\,dx,$$

for $k > k_0$. For $x \in B_k$

$$\frac{2^{2-n}}{|x|^{n-2}} \le \left(\frac{1}{2}\right)^{k(2-n)} \le \frac{1}{|x|^{n-2}},$$

so

$$\left(\frac{1}{2}\right)^{k(2-n)} \cdot C(A_k) \ge \left(\frac{1}{2}\right)^{k(2-n)} c \cdot c_2 \int_{B_k} h(x)\,dx$$
$$\ge 2^{2-n} c \cdot c_2 \int_{B_k} \frac{h(x)}{|x|^{n-2}}\,dx$$

and

$$\left(\frac{1}{2}\right)^{k(2-n)} \cdot C(A_k) \le \left(\frac{1}{2}\right)^{k(2-n)} c^{-1} c_2^{-1} \int_{B_k} h(x)\,dx$$
$$\le c^{-1} c_2^{-1} \int_{B_k} \frac{h(x)}{|x|^{n-2}}\,dx,$$

for $k > k_0$. It follows that $\int_B (h(x)/|x|^{n-2})\,dx < \infty$ if and only if $\sum_{k=1}^{\infty} (\frac{1}{2})^{k(2-n)} C(A_k) < \infty$ and this completes the proof in view of Theorem 2.1. □

Let L, h and A be as in Theorem 2.2 with the exception that h need not be Lipschitz but $h(x) = h_1(|x|)$ for some nonnegative function h_1: $[0, \infty) \to \mathbb{R}$.

THEOREM 2.4. *Suppose that h_1 is nondecreasing in some neighborhood of* 0. *Then $A \subset \mathbb{R}^n$ is regular for P^0 if and only if*

$$\int_{L \cap \{|x| \le 1\}} \frac{h(x)}{|x|^{n-2}}\,dx = \infty. \tag{2.7}$$

REMARK 2.1. The condition (2.7) is equivalent to

$$\int_{0+} \frac{h_1(r)}{r^2}\, dr = \infty.$$

PROOF. If $\limsup_{r \to 0} h_1(r)/r > 0$, then

$$\int_{L \cap \{|x| \le 1\}} \frac{h(x)}{|x|^{n-2}}\, dx = \infty$$

and it also easily follows from the Wiener's test that A is regular for P^0.

Suppose now that $h_1(r)/r \to_{r \to 0} 0$ and assume that

$$\frac{h_1(r)}{r} < \frac{1}{2}, \quad \text{for } r < 1. \tag{2.8}$$

The proof is analogous when $h_1(r)/r$ is bounded by another constant. Define

$$B_k = L \cap \{2^{-k-1} < |x| < 2^{-k}\}.$$

Let g_1 be the least Lipschitz majorant with the constant 1 of h_1 on $(0,1)$ and let g_2 be the greatest Lipschitz minorant with the constant 1 of h_1 on $(0,1)$. It follows from (2.8) that for all $x \in B_k$ and for all $y \in B_{k-2}$

$$g_1(|x|) \le \sup_{z \in B_{k-1}} h_1(|z|) \le h_1(|y|),$$

and therefore

$$\int_{B_k} \frac{g_1(|x|)}{|x|^{n-2}}\, dx \le \int_{B_k} \frac{\sup_{z \in B_{k-1}} h_1(|z|)}{|x|^{n-2}}\, dx$$

$$\le 2^{2(n-2)} \int_{B_{k-2}} \frac{h_1(|x|)}{|x|^{n-2}}\, dx.$$

Analogously, for all $x \in B_k$ and $y \in B_{k+2}$

$$g_2(|x|) \ge \inf_{z \in B_{k+1}} h_1(|z|) \ge h_1(|y|)$$

and

$$\int_{B_k} \frac{g_2(|x|)}{|x|^{n-2}}\, dx \ge \int_{B_k} \frac{\inf_{z \in B_{k+1}} h_1(|z|)}{|x|^{n-2}}\, dx$$

$$\ge 2^{-2(n-2)} \int_{B_{k+2}} \frac{h_1(|x|)}{|x|^{n-2}}\, dx.$$

It follows that

$$\int_{|x| \le 1/4} \frac{g_1(|x|)}{|x|^{n-2}}\, dx \le \int_{|x| \le 1} \frac{h_1(|x|)}{|x|^{n-2}}\, dx \cdot 2^{2(n-2)}$$

and

$$\int_{|x| \le 1} \frac{g_2(|x|)}{|x|^{n-2}}\, dx \ge \int_{|x| \le 1/4} \frac{h_1(|x|)}{|x|^{n-2}}\, dx \cdot 2^{-2(n-2)}.$$

Obviously,

$$\int_{|x|\le 1/4}\frac{g_2(|x|)}{|x|^{n-2}}\,dx \le \int_{|x|\le 1/4}\frac{h_1(|x|)}{|x|^{n-2}}\,dx \le \int_{|x|\le 1/4}\frac{g_1(|x|)}{|x|^{n-2}}\,dx.$$

Thus (2.7) is equivalent to either of the conditions

$$\int_{|x|\le 1/4}\frac{g_1(|x|)}{|x|^{n-2}}\,dx = \infty, \qquad \int_{|x|\le 1/4}\frac{g_2(|x|)}{|x|^{n-2}}\,dx = \infty.$$

Let A^1 and A^2 correspond to g_1 and g_2 in the same way as h_1 corresponds to A. Then

$$A^1 \text{ is regular for } P^0$$

implies

$$A \text{ is regular for } P^0,$$

which in turn implies

$$A^2 \text{ is regular for } P^0.$$

The first and last conditions are equivalent to (2.7) by Theorem 2.3 and previous remarks. Therefore the middle condition is also equivalent to (2.7). □

REMARK 2.2. For $n = 4$ and suitable h the set A considered in Theorems 2.3 and 2.4 reduces to a thorn [see Port and Stone (1978), page 68]. Thus the results of this section extend Proposition 3.3.5 of Port and Stone (1978) in the case $n = 4$.

Note that the original proof of Port and Stone does not require (at least in the case $n \ge 4$) the assumption of monotonicity of $h(r)/r$; the monotonicity of $h(r)$ is sufficient.

It seems that the just mentioned proposition of Port and Stone remains true for $n \ge 3$ and h monotone or Lipschitz, and might be proved like Theorems 2.3 and 2.4 above.

3. Minimal thinness and excursion laws. The concepts of thinness and minimal thinness [see Doob (1984), 1 XI 1 and 1 XII 11] are equivalent to regularity, as indicated below.

A set $A \subset \mathbb{R}^n$, $n \ge 2$, is thin at $x \in \mathbb{R}^n$ if and only if A is not regular for P^x [see Doob (1984), 1 XII 12].

Let P_h^x denote the distribution of an h-process in D starting from $x \in D \cup \partial_1^M D$ [for definition see Doob (1984), 2 X 1 and 3 III 2]. Fix an arbitrary $x^0 \in D$. A set $A \subset D$ is minimal thin at an attainable minimal boundary point x if and only if A is not regular for P_h^x where $h(\cdot) = G_D(x^0, \cdot)$ [see Doob (1984), 3 III 2 and 3 III 3].

Suppose $x \in \partial_1^M D$ is attainable and let H^x be a standard, nonnull excursion law in D.

LEMMA 3.1. *A set $A \subset D$ is minimal thin at x if and only if A is not regular for H^x.*

PROOF. Fix a point $x^0 \in D$. Let B_1 and B_2 be two spheres such that

$$x^0 \in B_1 = B_1(x^0, r) \subset B_2 = B_2(x^0, 2r) \subset D.$$

Let $h(\cdot) = G_D(x^0, \cdot)$.

The H^x-distribution of $\{X(t),\ 0 < t < T_{B_1}\}$ conditioned to hit B_1 is equal to $P^x_{h_1}$ for some harmonic function h_1 on $D \setminus B_1$ [see proof of Theorem 4.1 in Burdzy (1986a)].

It is easy to see using formula 2 X (2.1) of Doob (1984) that the P^x_h- and $P^x_{h_1}$-distributions of $\{X(t),\ 0 < t < T_{B_2}\}$ are mutually absolutely continuous. Therefore A is regular for P^x_h if and only if A is regular for $P^x_{h_1}$. By the 0–0 law for excursion laws [Lemma 4.1 of Burdzy (1986a)] the last property is equivalent to A being regular for H^x which completes the proof. □

For the rest of this section D will denote the halfspace $\{x_1 > 0\} \subset \mathbb{R}^n$, $n \geq 2$.

THEOREM 3.1. *A set $A \subset D$ is minimal thin at $0 \in \partial D$ if and only if the set*

$$A_1 = \left\{x \in \mathbb{R}^{n+2}: \left(\sqrt{x_1^2 + x_2^2 + x_3^2}, x_4, \dots, x_{n+2}\right) \in A\right\}$$

is thin at $0 \in \mathbb{R}^{n+2}$.

PROOF. The minimal thinness and thinness are equivalent to regularity as shown at the beginning of this section. It remains to use Theorem 1.2. □

REMARK 3.1. Naim (1957) gave a test for minimal thinness similar in spirit to Wiener's test for thinness. The above theorem establishes a link between the two tests in the special case of halfspace.

Denote $L = \partial D$. Let $h: L \to \mathbb{R}$ be a nonnegative function and

$$A = \{x \in D: 0 < x_1 \leq h(\mathrm{proj}_L x)\}.$$

THEOREM 3.2. *Suppose h is Lipschitz or $h(x) = h_1(|x|)$ for some monotone function $h_1: [0, \infty) \to \mathbb{R}$. Then A is minimal thin at 0 if and only if*

$$\int_{L \cap \{|x| \leq 1\}} \frac{h(x)}{|x|^n}\, dx < \infty.$$

PROOF. Use Theorems 3.1, 2.3 and 2.4. □

REMARK 3.2. (i) In view of Lemma 3.1 the above theorem gives a criterion for H^0-regularity where H^0 is the standard excursion law in D. This extends a two-dimensional Lemma 3.1 of Shimura (1984) and a multidimensional Corollary 3.1 of Burdzy (1986a). (ii) See Essen and Jackson (1980) for related results on minimal thinness.

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