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JOINT CONTINUITY OF THE INTERSECTION LOCAL TIMES OF MARKOV PROCESSES[1]

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We describe simple conditions on the transition density functions of two independent Markov processes X and Y which guarantee the existence of a continuous version for the intersection local time, formally given by

$$\alpha(z, H) = \int_H\int \delta_z(Y_t - X_s)\, ds\, dt.$$

In the analogous case of self-intersections α can be discontinuous at $z = 0$. We develop a Tanaka-like formula for α and use this to show that the singular part of $\alpha(z, [0, T]^2)$ as $z \to 0$ is given by

$$2\int_0^T U(X_t - z, X_t)\, dt, \quad \text{a.s.},$$

where U is the 1-potential of X.

1. Introduction. In two and three dimensions, but not in four, a pair of Brownian paths will intersect, and each individual path will intersect itself [Dvoretzky, Erdös and Kakutani (1950)].

The purely formal expression

$$\int_H\int \delta(Y_t - X_s)\, ds\, dt, \tag{1.1}$$

where δ is the delta "function" can be thought of as quantifying the intersections of two independent Brownian paths X and Y over the time set H. For self-intersections we take $Y \equiv X$.

In previous papers [Rosen (1983), and Geman, Horowitz and Rosen (1984)] we have shown how to give meaning to (1.1) as the $z = 0$ value of an intersection local time $\alpha(z, H)$ which is jointly continuous in (z, a, b, c, d) where $H = [a, b] \times [c, d]$. (As we will explain shortly, for self-intersections we must require $c > b$.) The main goal of this paper is to generalize this and related results to general Markov processes.

Let us first recall the definition of local time. Let X and Y be two Markov processes in $\mathbb{R}^d$. The process

$$Z_{s,t} = Y_t - X_s \tag{1.2}$$

defines, for each $H \subseteq \mathbb{R}^2_+$ a measure μ_H on $\mathbb{R}^d$:

$$\mu_H(A) = \lambda_2\big(Z^{-1}(A) \cap H\big). \tag{1.3}$$

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λ_n denotes Lebesgue measure on $\mathbb{R}^n$. If $\mu_H \ll \lambda_d$ we say that Z has a local time on H, and

$$\alpha(z, H) \doteq \frac{d\mu_H}{d\lambda_d}(z) \tag{1.4}$$

is called the local time of Z on H, or the intersection local time of X and Y on H. Note that

$$\int_H \int f(Y_t - X_s)\, ds\, dt = \int f(z)\alpha(z, H)\, d\lambda_d(z), \tag{1.5}$$

for all bounded Borel functions f, so that formally taking $f = \delta$ we can identify (1.1) with $\alpha(0, H)$. Of course, (1.4) is only defined for a.e. z, so that we must first produce a continuous version of $\alpha(z, H)$ before we can even begin to talk about $\alpha(0, H)$.

Sufficient conditions for the existence of the intersection local time for independent Markov processes are contained in Dynkin (1981). In Theorem 1 we describe simple conditions on the transition density functions which guarantee the existence of a local time $\alpha(z, H)$ which is jointly continuous in (z, a, b, c, d).

For applications, in the Brownian case such a jointly continuous local time leads to an easy derivation of the Hausdorff dimension of $\{s, t | X_s = Y_t\}$, and has allowed Le Gall (1984, 1986) to prove a conjecture of Taylor on the more detailed Hausdorff measure of the intersection points. In the general Markov case we will see that the small time asymptotics of the transition density functions provide a lower bound on the Hausdorff dimension of $\{s, t | X_s = Y_t\}$. For details on such asymptotics see Azencott et al. (1981). (A complementary upper bound can sometimes be derived from the continuity properties of the paths.)

For Brownian intersections we developed a Tanaka-like formula for $\alpha(0, H)$ [Rosen (1985)], which has been refined and extended in Yor (1985a). Our Theorem 3 presents such a formula for general diffusions. Assume that X satisfies the stochastic differential equation

$$dX_s = \sigma(X_s)\, dW_s + r(X_s)\, ds. \tag{1.6}$$

If $p_t(x, y)$ is the transition density function of X, then

$$U(x, y) = \int_0^\infty e^{-s} p_s(x, y)\, ds \tag{1.7}$$

defines the 1-potential of X.

Our Tanaka-like formula reads

$$\begin{aligned} -\alpha(0, [a, b] \times [c, d]) &= \int_c^d U(X_b, Y_t)\, dt - \int_c^d U(X_a, Y_t)\, dt \\ &\quad - \int_a^b \sigma(X_s) \int_c^d \nabla U(X_s, Y_t)\, dt\, dW_s - \int_a^b \int_c^d U(X_s, Y_t)\, dt\, ds, \end{aligned} \tag{1.8}$$

where the various terms are defined more carefully in Section 3.

In studying self-intersections, $Y \equiv X$, we must require that H not contain the diagonal (e.g., $c > b$). Our work on Brownian motion shows that without this condition the local time $\alpha(z, H)$ is discontinuous at the origin [Rosen (1986) and Le Gall (1986)]. Moreover, we are not really interested in the diagonal, which causes the tautological $X_s = X_s$ to register as true self-intersections. In Theorem 2 we obtain a jointly continuous (self)-intersection local time off the diagonal for general Markov processes.

For planar Brownian motion Varadhan (1969) has shown how to renormalize (1.1), so that it can be given meaning even for H containing the diagonal. It is not hard to show (see Section 4) that $\alpha(z,[0,T]^2)$ is jointly continuous in T, z for $z \neq 0$. Renormalization says that

$$\alpha(z,[0,T]^2) - \frac{2T}{\pi}\lg\left(\frac{1}{|z|}\right) \tag{1.9}$$

has a continuous extension to all z. This extension, evaluated at $z = 0$, is the "renormalized" version of (1.1). Since

$$U(z,0) - \frac{1}{\pi}\lg\left(\frac{1}{|z|}\right)$$

is continuous, renormalization tells us that the singular behavior of $\alpha(z,[0,T]^2)$ is given by $2TU(z,0)$.

In Rosen (1985) we presented one approach to this renormalization using our Tanaka-like formula for $\alpha(z, H)$. Additional approaches are in Dynkin (1985) and Le Gall (1985), while Yor (1985b) describes a renormalization for Brownian intersections in three dimensions. In Section 4 we generalize our approach, via Takana's formula, to a class of diffusions which includes planar Brownian motion. We will see that the singular behavior of $\alpha(z,[0,T]^2)$ as $z \to 0$ is given by

$$2\int_0^T U(X_t - z, X_t)\,dt.$$

For results on the continuity of local times for (one-dimensional) Markov processes themselves, i.e., not intersections, see Getoor and Kesten (1972) and Berman (1985). It is especially to the latter that we are indebted for certain ideas in this paper.

2. Joint continuity.

We consider first the intersections of two independent time-homogeneous Markov processes X and Y, with respective transition densities p and q. For simplicity, we assume X and Y start at the origin.

Δ_i^h will denote the h-difference operator in the ith argument, e.g.,

$$\Delta_2^h f(x,y) = f(x, y+h) - f(x,y).$$

THEOREM 1. *If for some $\beta > 0$,*

$$\int_0^h \sup_x \|p_s(x,\cdot)\|_2\,ds = O(h^\beta), \tag{2.1}$$

$$\int_0^T \sup_x \|\Delta_i^h p_s(x,\cdot)\|_2\,ds = O(|h|^\beta), \qquad i = 1,2, \tag{2.2}$$

as $h \to 0$, and similarly for q, then $Z_{s,t} = Y_t - X_s$ has a jointly continuous local time on $[0, T]^2$.

REMARKS. (i) Since for any $t > 0$,

$$\begin{aligned}\int_t^{t+h} \sup_x \| p_s(x, \cdot) \|_2 \, ds &= \int_0^h \sup_x \left\| \int p_t(x, y) p_s(y, \cdot) \, dy \right\|_2 ds \\ &\le \int_0^h \sup_x \int p_t(x, y) \| p_s(y, \cdot) \|_2 \, dy \, ds \\ &\le \int_0^h \sup_y \| p_s(y, \cdot) \|_2 \, ds,\end{aligned}$$

we see that (2.1) implies

$$\int_0^T \sup_x \| p_s(x, \cdot) \|_2 \, ds < \infty. \tag{2.3}$$

(ii) Our proof will show that (2.3) for both p and q, and (2.2) for one of them suffices to guarantee a local time continuous in the space variable. (2.1) for p and q then provides continuity in the respective time variables.

(iii) For p symmetric (2.1) is

$$\int_0^h \sup_x \sqrt{p_s(x, x)} \, ds = O(h^\beta).$$

(iv) It is easy to check that the conditions of Theorem 1 are satisfied by d-dimensional Brownian motion if and only if $d < 4$.

More generally, let X be a diffusion satisfying the stochastic differential equation

$$dX_s = \sigma(X_s) \, dW_s + r(X_s) \, ds, \tag{2.4}$$

where σ, r are smooth and bounded together with their derivatives, and where

$$\sigma(x)\sigma^*(x) \ge \lambda I,$$

for some $\lambda > 0$ independent of x. We will refer to such a process as a smooth uniformly elliptic diffusion. The transition density $p_t(x, y)$ is smooth for $t > 0$ and satisfies the following bounds:

$$p_t(x, y) \le M t^{-d/2} e^{-\alpha|x-y|^2/t}, \tag{2.4a}$$

$$\left| \frac{\partial p_t}{\partial x_i}(x, y) \right| \le M t^{-(d+1)/2} e^{-\alpha|x-y|^2/t}, \tag{2.4b}$$

$$\left| \frac{\partial p_t}{\partial y_j}(x, y) \right| \le M t^{-(d+1)/2} e^{-\alpha|x-y|^2/t}, \tag{2.4c}$$

$$\left| \frac{\partial p_t}{\partial x_i \, \partial x_j}(x, y) \right| \le M t^{-(d/2)-1} e^{-\alpha|x-y|^2/t}, \tag{2.4d}$$

for some positive constants α, M [Dynkin (1965), page 229]. It follows easily from this, arguing separately for $|h|^2 < t$ and $h^2 \geq t$, that for any $0 \leq \delta \leq 1$,

$$(2.4e)\quad |p_t(x+h, y) - p_t(x, y)| \leq M|h|^\delta t^{-(d+\delta)/2}\left(e^{-\alpha|x-y|^2/t} + e^{-\alpha|x+h-y|^2/t}\right),$$

with a similar inequality for differences in the second coordinate, and

$$(2.4f)\quad \begin{aligned} &\left|\frac{\partial p_t}{\partial x_i}(x+h, y) - \frac{\partial p_t}{\partial x_i}(x, y)\right| \\ &\qquad \leq M|h|^\delta t^{-(d+1+\delta)/2}\left(e^{-\alpha|x-y|^2/t} + e^{-\alpha|x+h-y|^2/t}\right). \end{aligned}$$

It follows easily from these bounds that the conditions of our theorem are satisfied if X, Y are smooth uniformly elliptic diffusions in $d < 4$ dimensions.

PROOF OF THEOREM 1. Let $f \in C_0^\infty(\mathbb{R}^d)$ be an even positive function normalized so that $\int f(x)\,dx = 1$. Set $f_\varepsilon(x) = \varepsilon^{-d} f(x/\varepsilon)$ and for $\varepsilon > 0$ define

$$\alpha_\varepsilon(z, H) = \int_H \int f_\varepsilon(Y_t - X_s - z)\, ds\, dt,$$

where $H = [a, b] \times [c, d] \subseteq [0, T]^2$. We will prove that locally

$$(2.5)\quad |\alpha_\varepsilon(z, H) - \alpha_{\varepsilon'}(z', H')| \leq c_\omega |(\varepsilon, z, H) - (\varepsilon', z', H')|^\gamma,$$

for some $\gamma > 0$, where we have used the abbreviation (ε, z, H) for $(\varepsilon, z, a, b, c, d)$. This will insure the existence and joint continuity of

$$(2.6)\quad \alpha(z, H) \doteq \lim_{\varepsilon \to 0} \alpha_\varepsilon(z, H),$$

with convergence (locally) uniform in z. Consequently, for any continuous compactly supported g we have

$$\begin{aligned} \int g(z)\alpha(z, H)\, dz &= \lim_{\varepsilon \to 0} \int g(z)\alpha_\varepsilon(z, H)\, dz \\ &= \lim_{\varepsilon \to 0} \int_H \int \int g(z) f_\varepsilon(Y_t - X_s - z)\, dz\, ds\, dt \\ &= \lim_{\varepsilon \to 0} \int_H \int f_\varepsilon * g(Y_t - X_s)\, ds\, dt \\ &= \int_H \int g(Y_t - X_s)\, ds\, dt. \end{aligned}$$

This identifies $\alpha(z, H)$ as the local time of $Z_{s,t} = Y_t - X_s$ on H.
To prove (2.5) we will show

$$(2.7)\quad \mathbb{E}\{\alpha_\varepsilon(z, H) - \alpha_{\varepsilon'}(z', H')\}^m \leq c_m |(\varepsilon, z, H) - (\varepsilon', z', H')|^{\gamma m},$$

for all even m. The multiparameter version of Kolmogorov's lemma [Meyer (1980)] then yields (2.5) locally, first for rational arguments, but consequently for all arguments since $\alpha_\varepsilon(z, H)$ is clearly continuous when $\varepsilon > 0$. [Locally means regions of (ε, z, H) space of the form $\{0 < \varepsilon < 1\} \times \{\|(z, H)\| \leq n\}$ for each n.]

(2.7) will be proven by establishing the following three bounds:

$$\mathbb{E}\{\alpha_\varepsilon(z,H)-\alpha_\varepsilon(z,H')\}^m \le c_m|(H)-(H')|^{\gamma m}, \tag{2.8}$$

$$\mathbb{E}\{\alpha_\varepsilon(z,H)-\alpha_\varepsilon(z',H)\}^m \le c_m|z-z'|^{\gamma m}, \tag{2.9}$$

$$\mathbb{E}\{\alpha_\varepsilon(z,H)-\alpha_{\varepsilon'}(z,H)\}^m \le c_m|\varepsilon-\varepsilon'|^{\gamma m}. \tag{2.10}$$

We first turn to (2.8), and note that

$$|\alpha_\varepsilon(z,H)-\alpha_\varepsilon(z,H')| \le \alpha_\varepsilon(z,H\Delta H'),$$

where $H\Delta H'$, the symmetric difference of H and H', consists of at most eight rectangles, each of which has (at least) one side of length $\le |(H)-(H')|$. It therefore suffices to show

$$\mathbb{E}\{\alpha_\varepsilon(z,H)\}^m \le c_m[(b-a)(d-c)]^{\beta m}, \tag{2.11}$$

for all rectangles $H=[a,b]\times[c,d]$.

We now compute

$$\begin{aligned}
&\mathbb{E}\{\alpha_\varepsilon(z,H)\}^m\\
&= \int_{H^m}\!\!\int \mathbb{E}\left(\prod_{i=1}^m f_\varepsilon(Y_{t_i}-X_{s_i}-z)\right)ds\,dt\\
&= m!\int\cdots\int_{\left\{\substack{a<s_1<\cdots<s_m<b\\ [c,d]^m}\right\}} \mathbb{E}\left(\prod_{i=1}^m f_\varepsilon(Y_{t_i}-X_{s_i}-z)\right)ds\,dt\\
&= m!\sum_\pi \int_{H^m_<}\!\!\int \mathbb{E}\left(\prod_{i=1}^m f_\varepsilon(Y_{t_{\pi_i}}-X_{s_i}-z)\right)ds\,dt\\
&= m!\sum_\pi \int_{H^m_<}\!\!\int\left(\int\!\!\int Q_T(Y)F_\varepsilon(Y_\pi-X-z)P_S(X)\,dX\,dY\right)ds\,dt\\
&= m!\sum_\pi \int F_1(X)\int_{H^m_<}\!\!\int\left(\int Q_T(Y)P_S(Y_\pi+\varepsilon X-z)\,dY\right)ds\,dt\,dX,
\end{aligned} \tag{2.12}$$

where

$$P_S(X)=p_{s_1}(0,x_1)\cdots p_{s_m-s_{m-1}}(x_{m-1},x_m),$$

$$Q_T(Y)=q_{t_1}(0,y_1)\cdots q_{t_m-t_{m-1}}(y_{m-1},y_m),$$

$$F_\varepsilon(X)=\prod_{i=1}^m f_\varepsilon(x_i),$$

$$Y_\pi=(y_{\pi_1},\dots,y_{\pi_m}),$$

$$H^m_< = \{a<s_1<\cdots<s_m<b,\ c<t_1<\cdots<t_m<d\},$$

and the sum Σ_π extends over all permutations π of $\{1,\dots,m\}$.

Now using (2.1) and (2.3) we have

$$\int_{H_<^m}\int\left(\int Q_T(Y)P_S(Y_\pi+\varepsilon X-z)\,dY\right)ds\,dt$$

$$\le \int_{H_<^m}\int \|Q_T\|_2\|P_S\|_2\,ds\,dt$$

$$\le \int_{H_<^m}\int\prod_{i=1}^m\left(\sup_{y_{i-1}}\left\|q_{t_i-t_{i-1}}(y_{i-1},\cdot)\right\|_2\right)\left(\sup_{x_{i-1}}\left\|p_{s_i-s_{i-1}}(x_{i-1},\cdot)\right\|_2\right)ds\,dt$$

$$\le k\left[(b-a)(d-c)\right]^{m\beta},$$

which establishes (2.11)—and hence (2.8).

We now turn to (2.9). We compute as in (2.12), with $\delta = z'-z$,

$$\mathbb{E}\{\alpha_\varepsilon(z',H)-\alpha_\varepsilon(z,H)\}^m$$

$$= m!\sum_\pi\int_{H_<^m}\int\left(\int\int Q_T(Y)\left(\left(\prod_{i=1}^m\Delta_i^\delta\right)F_\varepsilon\right)\right.$$

$$\left.\times(Y_\pi-X-z)P_S(X)\,dX\,dY\right)ds\,dt \qquad (2.13)$$

$$= m!\sum_\pi\int F_1(X)\int_{H_<^m}\int\left(\int Q_T(Y)\left(\left(\prod_{i=1}^m\Delta_i^\delta\right)P_S\right)\right.$$

$$\left.\times(Y_\pi+\varepsilon X-z)\,dY\right)ds\,dt\,dX.$$

Once again we have

$$(2.14)\qquad \left|\int_{H_<^m}\int\left(\int Q_T(Y)\left(\left(\prod_{i=1}^m\Delta_i^\delta\right)P_S\right)(Y_\pi+\varepsilon X-z)\,dY\right)ds\,dt\right|$$

$$\le\int_{H_<^m}\int\|Q_T\|_2\left\|\prod_{i=1}^m\Delta_i^\delta P_S\right\|_2 ds\,dt.$$

We now use a device contained in Berman (1985).

$$\prod_{i=1}^m\Delta_i^\delta P_S(X)$$

can be written as a sum of $2^{m/2}$ terms of the form

$$\prod_{i=1}^{m/2}\Delta_{2i}^\delta P_S(\bar{x}_1,x_2,\bar{x}_3,x_4,\dots,\bar{x}_{m-1},x_m),$$

where for odd indices i, $\bar{x}_i$ can be either x_i or $x_i + \delta$. Hence

$$\begin{aligned}
\left\| \prod_{i=1}^{m} \Delta_i^{\delta} P_S \right\|_2 &\le k \| p_{s_1}(0, \cdot) \|_2 \\
&\quad \times \prod_{i=1}^{m/2} \sup_x \| p_{\bar{s}_{2i}}(x, x_{2i} + \delta) p_{\bar{s}_{2i+1}}(x_{2i} + \delta, x_{2i+1}) \\
&\qquad - p_{\bar{s}_{2i}}(x, x_{2i}) p_{\bar{s}_{2i+1}}(x_{2i}, x_{2i+1}) \|_2 \\
&\qquad \text{(where the } L^2 \text{ norm is with respect to } x_{2i}, x_{2i+1}) \\
&\le k \| p_{s_1}(0, \cdot) \|_2 \prod_{i=1}^{m/2} \sup_{x, y} \Big[\| \Delta_2^{\delta} p_{\bar{s}_{2i}}(x, \cdot) \|_2 \| p_{\bar{s}_{2i+1}}(y, \cdot) \|_2 \\
&\qquad + \| p_{\bar{s}_{2i}}(x, \cdot) \|_2 \| \Delta_1^{\delta} p_{\bar{s}_{2i+1}}(y, \cdot) \|_2 \Big],
\end{aligned} \tag{2.15}$$

where $\bar{s}_j = s_j - s_{j-1}$, and there is an obvious modification in our formula for $2i = m$.

Combining this with (2.14), and using (2.2) and (2.3) yields (2.9).

Finally, we turn to (2.10). Once again we can compute

$$\begin{aligned}
&\mathbb{E}(\alpha_\varepsilon(z, H) - \alpha_{\varepsilon'}(z, H))^m \\
&= m! \sum_\pi \int_{H_<^m} \int \bigg(\int \int Q_T(Y) \bigg(\prod_{i=1}^{m} (f_\varepsilon(\cdot) - f_{\varepsilon'}(\cdot)) \bigg) \\
&\qquad \times (Y_\pi - X - z) P_S(X) \, dX \, dY \bigg) \, ds \, dt \\
&= m! \sum_\pi \int_{H_<^m} \int \bigg(\int \int Q_T(Y) \prod_{i=1}^{m} (f_\varepsilon(x_i) - f_{\varepsilon'}(x_i)) \\
&\qquad \times P_S(Y_\pi + X - z) \, dX \, dY \bigg) \, ds \, dt \\
&= m! \sum_\pi \int F_1(X) \int_{H_<^m} \int \bigg(\int Q_T(Y) \bigg(\prod_{i=1}^{m} \Delta^{\delta x_i} P_S \bigg) (Y_\pi + \varepsilon X - z) \, dY \bigg) \, ds \, dt \, dx,
\end{aligned} \tag{2.16}$$

where $\delta = \varepsilon' - \varepsilon$. Since F_1 has compact support we can use (2.14), (2.15), (2.2) and (2.3) to bound (2.16) by

$$k \int F_1(X) \prod_{i=1}^{m/2} |\delta x_{2i}|^\beta \, dX \le k |\varepsilon' - \varepsilon|^{m\beta/2}.$$

This completes the proof of Theorem 1. □

REMARKS. (i) Let $\bar{\beta}$ be the smallest β which works in (2.1) for both $p_.$ and $q_.$. Considerations such as (2.11) and (2.5) lead to the fact that (locally) $\alpha(0, H) \le (\lambda_2(H))^{\tilde{\beta}}$ for any $\tilde{\beta} < \beta$.

If X and Y have continuous sample paths, and

$$\alpha(0, \mathbb{R}_+^2) > 0,$$

then [see Adler (1981), page 231], our bound on $\alpha(0, H)$ implies

$$\dim\{(s, t) | X_s = Y_t\} \geq 2\beta,$$

where dim denotes Hausdorff dimension. Since $\alpha(0, H)$ has positive expectation, the condition $\alpha(0, \mathbb{R}_2^+) > 0$ will certainly hold at least with positive probability. (Proving this occurs with probability one generally requires some form of ergodicity.)

Complementarily, if X and Y are Hölder continuous of order $\hat{\beta}$, then we have $\dim\{(s, t)|X_s = Y_t\} \leq 2 - \hat{\beta}d$ [see Adler (1981), page 230]. Together, these show that for smooth uniformly elliptic diffusions in $d = 2$ or 3 dimensions $\dim\{(s, t)|X_s = Y_t\} = 2 - d/2$ with positive probability.

In Shieh (1985) this is shown to hold with probability 1.

(ii) If we assume the stronger condition $\sup\|p_s(x, \cdot)\|_2 \leq s^{-(1-\beta)}$, then as in Rosen (1983) we can show that $\alpha(0, H) \leq c(\lambda_2(H))\beta|\log|\lambda_2(H)||^{2-2\beta}$ for small squares H.

Let $p_t^*(x, y) = p_t(y, x)$.

THEOREM 2. *Assume that p and p^* satisfy the conditions of Theorem* 1, *and in addition*,

$$\sup_{s \geq \rho} \| p_s(\ , \ ) \|_\infty = M_\rho < \infty, \quad \textit{for each } \rho > 0. \tag{2.17}$$

Then $Z_{s,t} = X_t - X_s$ has a jointly continuous local time away from the diagonal in $[0, T]^2$.

PROOF. The proof is similar to that of Theorem 1. We define

$$\alpha_\varepsilon(z, H) = \int_H\int f_\varepsilon(X_t - X_s)\, ds\, dt,$$

for $H = [a, b] \times [c, d]$ with $c - b \doteq \rho > 0$. We must establish (2.8)–(2.10). We will show in detail how to obtain (2.11), and the other bounds can be established similarly.

We have

$$\begin{aligned}
&\mathbb{E}(\alpha_\varepsilon(z, H))^m \\
&\quad = m! \sum_\pi \int_{H_<^m}\int \left(\int\int \overline{P}_T(Y) p_{t_1 - s_m}(x_m, y_1) \right. \\
&\qquad\qquad \left. \times F_\varepsilon(Y_\pi - X - z) P_S(X)\, dX\, dY \right) ds\, dt \qquad (2.18) \\
&\quad = m! \sum_\pi \int F_1(X) \int_{H_<^m}\int \left(\int \overline{P}_T(Y) p_{t_1 - s_m}(y_{\pi_m} + \varepsilon x_m - z, y_1) \right. \\
&\qquad\qquad \left. \times P_S(Y_\pi + \varepsilon X - z)\, dY \right) ds\, dt\, dX,
\end{aligned}$$

where

$$\bar{P}_T(Y) = p_{t_2-t_1}(y_1, y_2) \cdots p_{t_m-t_{m-1}}(y_{m-1}, y_m).$$

Let $l = \pi_1$. Since $t_1 - s_m \ge \rho$ we can use (2.17) to obtain the bound

$$(2.19)\qquad \begin{aligned}
&\int_{H^m_<}\int\left(\int \bar{P}_T(Y)p_{t_1-s_m}(y_{\pi_m} + \varepsilon x_m - z, y_1)P_S(Y_\pi + \varepsilon X - z)\,dY\right) ds\,dt \\
&\quad\le M_\rho \int_{H^m_<}\int\left(\int \bar{P}_T(Y)p_{s_1}(0, y_{\pi_1} + \varepsilon x_1 - z)\bar{P}_S(Y_\pi + \varepsilon X - z)\,dY\right) ds\,dt \\
&\quad\le M_\rho \int_{H^m_<}\int\left(\int p_{s_1}(0, y_l + \varepsilon x_1 - z)\right. \\
&\qquad\qquad \left.\times\left(\int \bar{P}_T(Y)\bar{P}_S(y_\pi + \varepsilon X - z)\,d\bar{Y}\right)\right) dy_l\,ds\,dt \\
&\quad\le M_\rho \int_{H^m_<}\int \sup_{y_l}\left(\int \bar{P}_T(Y)\bar{P}_S(Y_\pi + \varepsilon X - z)\,d\bar{Y}\right) ds\,dt \\
&\quad\le M_\rho \int_{H^m_<}\int \left(\sup_{y_l}\|\bar{P}_T\|_2\right)\left(\sup_{x_1}\|\bar{P}_S\|_2\right) ds\,dt,
\end{aligned}$$

where $d\bar{Y} = dy_1\,dy_2 \cdots \widehat{dy_l} \cdots dy_m$, and L^2 norms preceded by a "sup" are in the remaining variables. Thus in $\sup_{y_l}\|\bar{P}_T\|_2$, $\bar{P}_T$ is a function of $Y = (y_1, \dots, y_m)$ and the norm is in all variables excluding y_l.

We can write

$$\bar{P}_T(Y) = P_T^{*(l)}(Y)P_T^{(l)}(Y),$$

where

$$\begin{aligned}
P_T^{(l)}(Y) &= p_{t_{l+1}-t_l}(y_l, y_{l+1}) \cdots p_{t_m-t_{m-1}}(y_{m-1}, y_m), \\
P_T^{*(l)}(Y) &= p_{t_2-t_1}(y_1, y_2) \cdots p_{t_l-t_{l+1}}(y_{l-1}, y_l) \\
&= p^*_{t_l-t_{l-1}}(y_l, y_{l-1}) \cdots p^*_{t_2-t_1}(y_2, y_1).
\end{aligned}$$

It is obvious that

$$\begin{aligned}
\sup_{y_l}\|\bar{P}_T\|_2 &= \sup_{y_l}\|P_T^{*(l)}\|_2\|P_T^{(l)}\|_2 \\
&\le \left(\prod_{i=1}^{l}\sup_{y_i}\left\|p^*_{t_i-t_{i-1}}(y_i, \cdot)'\right\|_2\right)\left(\prod_{i=l+1}^{m}\sup_{y_{i-1}}\left\|p_{t_i-t_{i-1}}(y_{i-1}, \cdot)\right\|_2\right).
\end{aligned}$$

The rest of the proof is completed as in Theorem 1. □

3. Tanaka's formula. From now on we will assume that X is a diffusion satisfying the stochastic differential equation

$$(3.1)\qquad dX_s = \sigma(X_s)\,dW_s + r(X_s)\,ds.$$

Let

$$U(x, y) = \int_0^\infty e^{-s} p_s(x, y)\, ds \tag{3.2}$$

be the 1-potential of X. Our goal is to prove the following Tanaka-like formula

$$\begin{aligned} -\alpha(0, [a, b] \times [c, d]) = \int_a^b U(X_d, X_s)\, ds - \int_0^b U(X_c, X_s)\, ds \\ - \int_c^d \sigma(X_t) \int_a^b \nabla U(X_t, X_s)\, ds\, dW_t \\ - \int_c^d \int_a^b U(X_s, X_t)\, ds\, dt, \end{aligned} \tag{3.3}$$

for the local time of rectangles away from the diagonal. We will use a related formula, in the next section, to obtain the asymptotics of $\alpha(z, [0, T]^2)$ as $z \to 0$. The terms in (3.3) will be defined in the course of the proof.

To simplify our presentation we will assume that the stochastic differential equation (3.1) is of the form described in Remark (ii) following Theorem 1, i.e., smooth and uniformly elliptic. It will be clear from our proofs that our results apply to a much larger class of diffusions—it is the lack of concrete examples outside the class we consider which cautions us against formulating our hypothesis in more general terms.

We start with an easy result.

PROPOSITION 1. *In two or three dimensions*

$$G(x) = \int_a^b U(x, X_s)\, ds \tag{3.4}$$

is continuous.

PROOF. Let

$$U_\varepsilon(x, y) = \int_\varepsilon^\infty e^{-s} p_s(x, y)\, ds. \tag{3.5}$$

Under the conditions of this section, by (2.4e), for any $\varepsilon > 0$, $U_\varepsilon(x, y)$ is uniformly continuous (this is true in any dimension). Hence

$$G_\varepsilon(x) = \int_a^b U_\varepsilon(x, X_s)\, ds \tag{3.6}$$

is also continuous. We will show that (locally),

$$|G_\varepsilon(x) - G_\varepsilon(x')| \le c|x - x'|^\gamma \tag{3.7}$$

uniformly in rational $\varepsilon > 0$.

The monotone convergence theorem then shows (3.7) holds for G, proving our proposition. As usual (3.7) will follow from the bound

$$\mathbb{E}(G_\varepsilon(x) - G_{\varepsilon'}(x'))^m \le c_m |(\varepsilon, x) - (\varepsilon', x')|^{\gamma m}, \tag{3.8}$$

which we break up into two parts. First with $\delta = x' - x$,

$$\mathbb{E}\big(\Delta^{\delta} G_{\varepsilon}(x)\big)^m = m! \int \cdots \int_{\{a < s_1 < \cdots < s_m < b\}} \int \bigg(\prod_{i=1}^{m} \Delta_1^{\delta} U_{\varepsilon}(x, y_i) P_s(Y) \bigg) \, dY \, ds$$

$$\leq m! \big(\| \Delta_1^{\delta} U_{\varepsilon}(x, \cdot) \|_2 \big)^m \int \cdots \int_{\{a < s_1 < \cdots < s_m < b\}} \| P_s \|_2 \, ds$$

$$\leq k |x - x'|^{m\beta},$$

using the bounds (2.4e) [see (3.16c)].

Similarly,

$$\mathbb{E}\big(G_{\varepsilon}(x) - G_{\varepsilon'}(x)\big)^m = m! \int \cdots \int_{\{a < s_1 < \cdots < s_m < b\}} \mathbb{E} \bigg(\prod_{i=1}^{m} \int_{\varepsilon}^{\varepsilon'} e^{-t} p_t(x, Y_{s_i}) \, dt \bigg) \, ds$$

$$\leq k \bigg(\int_{\varepsilon}^{\varepsilon'} \| p_t(x, \cdot) \|_2 \, dt \bigg)^m \int \cdots \int_{\{a < s_1 < \cdots < s_m < b\}} \| P_S \|_2 \, ds$$

$$\leq k |\varepsilon - \varepsilon'|^{m\beta},$$

as before. This completes the proof of Proposition 1. □

THEOREM 3. *In two and three dimensions Tanaka's formula* (3.3) *holds if* $c > b$, *with the stochastic integral defined below.*

PROOF. Set $U^{\varepsilon}(x, y) = \int f_{\varepsilon}(x - z) U(z, y) \, dz$ and $G^{\varepsilon}(x) = f_{\varepsilon} * G(x) = \int_a^b U^{\varepsilon}(x, X_s) \, ds$. We apply Itô's formula to the smooth nonanticipating functional G^{ε} on the interval $c \leq t \leq d$ to obtain

$$G^{\varepsilon}(X_d) = G^{\varepsilon}(X_c) + \int_c^d \sigma(X_t) \nabla G^{\varepsilon}(X_t) \, dW_t + \int_c^d \mathbb{L} G^{\varepsilon}(X_t) \, dt, \tag{3.9}$$

where

$$\mathbb{L} = \frac{1}{2} \sum_{i,j} (\sigma \sigma^*)_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_k r_k \frac{\partial}{\partial x_k}.$$

Using

$$(-\mathbb{L} + 1) U(\cdot, y) = \delta_{(y)}(\cdot),$$

we have

$$-\int_c^d \mathbb{L} G^{\varepsilon}(X_t) \, dt = \int_H \int f_{\varepsilon}(X_t - X_s) \, ds \, dt - \int_c^d G^{\varepsilon}(X_t) \, dt, \tag{3.10}$$

so that (3.9) becomes

$$\begin{aligned} &-\alpha_{\varepsilon}(0, [a, b] \times [c, d]) \\ &\quad = G^{\varepsilon}(X_d) - G^{\varepsilon}(X_c) - \int_c^d \sigma(X_t) \nabla G^{\varepsilon}(X_t) \, dW_t - \int_c^d G^{\varepsilon}(X_t) \, dt. \end{aligned} \tag{3.11}$$

Using Theorem 2 and Proposition 1, we take the $\varepsilon \to 0$ limit in (3.11) to find

$$\begin{aligned} &-\alpha(0, [a, b] \times [c, d]) \\ &\quad = G(X_d) - G(X_c) - \lim_{\varepsilon \to 0} \int_c^d \sigma(X_t) \nabla G^{\varepsilon}(X_t) \, dW_t - \int_c^d G(X_t) \, dt, \end{aligned} \tag{3.12}$$

and we will now show that

$$\lim_{\varepsilon\to 0}\int_c^d \sigma(X_t)\nabla G^\varepsilon(X_t)\,dW_t \tag{3.13}$$

exists and is given by a stochastic integral, completing the proof of Theorem 3. For this it suffices to show that the integrals in (3.13) form a Cauchy sequence in L^2 of the underlying probability space. With $U^{\varepsilon,\varepsilon'} = U^\varepsilon - U^{\varepsilon'}$, we have

$$\begin{aligned}
&\mathbb{E}\Big(\int_c^d \sigma(X_t)(\nabla G^\varepsilon(X_t) - \nabla G^{\varepsilon'}(X_t))\,dW_t\Big)^2\\
&\quad = \int_c^d \mathbb{E}\Big(|\sigma(X_t)(\nabla G^\varepsilon(X_t) - \nabla G^{\varepsilon'}(X_t))|^2\Big)\,dt\\
&\quad = 2\int_c^d\int_{a\le s_1<s_2\le b}\int\Big(\int\langle \sigma(x)\nabla U^{\varepsilon,\varepsilon'}(x,y_1), \sigma(x)\nabla U^{\varepsilon,\varepsilon'}(x,y_2)\rangle\\
&\qquad\qquad \times p_{s_1}(0,y_1)p_{s_2-s_1}(y_1,y_2)p_{t-s_2}(y_2,x)\,dY\,dX\Big)\,ds\,dt \qquad (3.14)\\
&\quad \le k\int|\nabla U^{\varepsilon,\varepsilon'}(x,y_1)||\nabla U^{\varepsilon,\varepsilon'}(x,y_2)|U(0,y_1)U(y_1,y_2)\,dx\,dy,
\end{aligned}$$

using (2.17), which we write as

$$\int U(0,y_1)U(y_1,y_2)\Big(\int|\nabla U^{\varepsilon,\varepsilon'}(x,y_1)||\nabla U^{\varepsilon,\varepsilon'}(x,y_2)|\,dx\Big)\,dy. \tag{3.15}$$

It follows from (2.4a)–(2.4f), by scaling out $|x-y|$ in the region $0\le t\le 1$, that

$$U(x,y) \le \begin{cases} M|\lg(|x-y|)|, & d=2,\ |x-y|<\frac{1}{2},\\ M\dfrac{1}{|x-y|^{d-2}}, & d\ge 3,\end{cases} \tag{3.16a}$$

$$\left|\frac{\partial U(x,y)}{\partial x_j}\right| \le M\frac{1}{|x-y|^{d-1}}, \tag{3.16b}$$

$$|U(x+h,y)-U(x,y)| \le M|h|^\delta\left(\frac{1}{|x-y|^{d-2+\delta}} + \frac{1}{|x+h-y|^{d-2+\delta}}\right), \tag{3.16c}$$

$$\begin{aligned}&\left|\frac{\partial}{\partial x_j}U(x+h,y) - \frac{\partial}{\partial x_i}U(x,y)\right|\\ &\quad\le M|h|^\delta\left(\frac{1}{|x-y|^{d-1+\delta}} + \frac{1}{|x+h-y|^{d-1+\delta}}\right),\end{aligned} \tag{3.16d}$$

while $U(x,y)$ and $\nabla U(x,y)$ fall off exponentially in $|x-y|$, as follows easily by arguing separately for $t\le|x-y|$ and $t\ge|x-y|$. These estimates allow the integral in (3.15) to be bounded, with $\gamma = \varepsilon - \varepsilon'$, by

$$\int f(z_1)f(z_2)\int U(0, y_1)U(y_1, y_2)\left(\int |\Delta_1^{\gamma z_1}\nabla U(x, y_1)||\Delta_1^{\gamma z_2}\nabla U(x, y_2)|\,dx\right) dy\,dz$$

$$\text{(3.17)}\qquad \le M\gamma^{\delta}\sup_z \int U(0, y_1)U(y_1, y_2)\frac{1}{|y_1 - y_2 - z|^{d-2+2\delta}}\,dy_1\,dy_2$$

$$\le M(\varepsilon - \varepsilon')^{\delta}\sup_z \int U^0(y)\frac{1}{|y - z|^{d-2+2\delta}}\,dy \le M(\varepsilon - \varepsilon')^{\delta},$$

where U^0 is the Brownian 1-potential. □

4. Renormalization and asymptotics of the intersection local time. In this section we continue with the assumptions of Section 3: X is a smooth, uniformly elliptic diffusion and we shall see that our results require $d = 2$, as is known to be the case for Brownian motion.

Let $U_z^{\varepsilon}(x, y) = U^{\varepsilon}(x - z, y)$ and apply Itô's formula to the nonanticipating functional of t and x,

$$\text{(4.1)}\qquad \int_0^t U_z^{\varepsilon}(x, X_s)\,ds,$$

on the interval $0 \le t \le T$. We obtain

$$\text{(4.2)}\qquad \begin{aligned}\int_0^T U_z^{\varepsilon}(X_T, X_s)\,ds = 0 &+ \int_0^T U_z^{\varepsilon}(X_t, X_t)\,dt\\ &+ \int_0^T \sigma(X_t)\int_0^t \nabla U_z^{\varepsilon}(X_t, X_s)\,ds\,dW_t\\ &+ \int_0^T\int_0^t \mathbb{L}U_z^{\varepsilon}(X_t, X_s)\,ds\,dt.\end{aligned}$$

As in Section 3 this leads to

$$\text{(4.3)}\qquad \begin{aligned}&\alpha_{\varepsilon}(z, D_T) - \int_0^T U_z^{\varepsilon}(X_t, X_t)\,dt\\ &\quad = -\int_0^T U_z^{\varepsilon}(X_T, X_s)\,ds + \int_0^T \sigma(X_t)\int_0^t \nabla U_z^{\varepsilon}(X_t, X_s)\,ds\,dW_t\\ &\qquad + \int_{D_T}\int U_z^{\varepsilon}(X_t, X_s)\,ds\,dt,\end{aligned}$$

where $D_T = \{(s, t) | 0 \le s \le t \le T\}$.

We will show that as $\varepsilon \to 0$, the right-hand side of (4.3) converges to a random variable jointly continuous in z, T. However, if $z \ne 0$, the path continuity of X assures us that in D_T $|X_t - X_s - z| \ge \bar{\varepsilon}$ off $D_{T,\rho} = \{(s, t) | 0 \le s \le t - \rho, t \le T\}$ for some ρ small. Thus the limit

$$\begin{aligned}\alpha(z, D_T) &= \lim_{\varepsilon \to 0}\int_{D_T}\int f_{\varepsilon}(X_t - X_s - z)\,ds\,dt\\ &= \lim_{\varepsilon \to 0}\int_{D_{T,\rho}}\int f_{\varepsilon}(X_t - X_s - z)\,ds\,dt = \alpha(z, D_{T,\rho})\end{aligned}$$

exists and is jointly continuous in T, $z \neq 0$. Since the same is true, using (3.16), for

$$\int_0^T U_z(X_t, X_t)\,dt = \lim_{\varepsilon\to 0}\int_0^T U_z^{\varepsilon}(X_t, X_t)\,dt, \qquad z \neq 0,$$

a jointly continuous limit for the right-hand side of (4.3) will show that

$$\alpha(z, D_T) - \int_0^T U_z(X_t, X_t),$$

continuous for $z \neq 0$ has a continuous extension for all z. Thus the singular part of $\alpha(z,[0,T]^2) = 2\alpha(z, D_T)$ as $z \to 0$ is given by $2\int_0^T U_z(X_t, X_t)\,dt$.

It remains to show that the three terms in (4.3) have $\varepsilon \to 0$ limits which are jointly continuous in z, T. We will consider only the stochastic integral term—the others being similar, and simpler.

As in the proof of Theorem 1, it will suffice to show that

$$\begin{aligned}(4.4)\quad &\mathbb{E}\left(\int_0^T \sigma(X_t)\int_0^t \nabla U_z^{\varepsilon}(X_t, X_s)\,ds\,dW_t - \int_0^{T'}\int_0^t \nabla U_{z'}^{\varepsilon'}(X_t, X_s)\,ds\,dW_t\right)^{2m}\\ &\quad\le k_m\left|(T,\varepsilon,z) - (T',\varepsilon',z')\right|^{m\delta},\end{aligned}$$

for all m and some $\delta > 0$. Again as in the proof of that theorem it will suffice to consider separately the variation in T, ε, z.

We begin with bounding

$$\begin{aligned}&\mathbb{E}\left(\int_0^T \sigma(X_t)\int_0^t \nabla U_z^{\varepsilon}(X_t, X_s)\,ds\,dW_t\right)^{2m}\\ &\quad\le k\mathbb{E}\left(\int_0^T\int_0^t\int_0^t \langle \sigma(X_t)\nabla U_z^{\varepsilon}(X_t, X_r), \sigma(X_t)\nabla U_z^{\varepsilon}(X_t, X_s)\rangle\,dr\,ds\,dt\right)^m\\ (4.5)\quad &\quad\le k\int\int\int_{r_i\le s_i\le t_i}\mathbb{E}\left(\prod_{i=1}^m \left|\nabla U_z^{\varepsilon}(X_{t_i}, X_{r_i})\right|\left|\nabla U_z^{\varepsilon}(X_{t_i}, X_{s_i})\right|\right)dr\,ds\,dt\\ &\quad\le k\int_{\bar t_1<\cdots<\bar t_{3m}}\left(\int\prod_{i=1}^m \left|\nabla U_z^{\varepsilon}(x_{\pi_i}, x_{\bar\pi_i})\right|\left|\nabla U_z^{\varepsilon}(x_{\pi_i}, x_{\tilde\pi_i})\right| P_{\bar T}(X)\,dx\right)d\bar t\\ &\quad\le k\int\prod_{i=1}^m \left|\nabla U_z^{\varepsilon}(x_{\pi_i}, x_{\bar\pi_i})\right|\left|\nabla U_z^{\varepsilon}(x_{\pi_i}, x_{\tilde\pi_i})\right| U(0,x_1)\cdots U(x_{3m-1}, x_{3m})\,dX,\end{aligned}$$

where π, $\bar\pi$, $\tilde\pi$ are three complementary injections of $\{1,\dots,m\}$ into $\{1,2,\dots,3m\}$ such that for each i, $\bar\pi_i < \tilde\pi_i < \pi_i$. This follows from $r_i \le s_i \le t_i$.

As in (3.17) this is bounded by

$$\begin{aligned}(4.6)\quad &\sup_{z_{i,j}}\int\prod_{i=1}^m \left|\nabla U_{z_{i,1}}(x_{\pi_i}, x_{\bar\pi_i})\right|\left|\nabla U_{z_{i,2}}(x_{\pi_i}, x_{\tilde\pi_i})\right|\\ &\qquad\times U(0,x_1)\cdots U(x_{3m-1}, x_{3m})\,dX.\end{aligned}$$

We can bound (4.6) by integrating successively starting from x_{3m}. We encounter three types of integrals.

CASE I ($i \in$ range π).

$$\int |\nabla U_a(x_i, x_j)| |\nabla U_b(x_i, x_k)| U(x_{i-1}, x_i)\, dx_i$$
$$\leq M\left(\int |\nabla U_a(x_i, x_j)|^{3/2} |\nabla U_b(x_i, x_k)|^{3/2}\, dx_i\right)^{2/3}$$
$$\leq M \frac{1}{|x_j + a - x_k - b|^{2/3}},$$

where the first inequality comes from Hölder's inequality and the second from (3.16).

In the above integral we necessarily have $j, k \leq i$. It is possible that j or k equals $i - 1$. Then $d = 2$ is crucial. Note that $i = 3m$ is of this type.

CASE II ($i \in$ range $\tilde{\pi}$).

$$\int \frac{1}{|x_j - x_i + a|^{2/3}} U(x_{i-1}, x_i)\, dx_i \leq M,$$

by (3.16).

CASE III ($i \in$ range $\bar{\pi}$).

$$\int U(x_{i-1}, x_i)\, dx_i = 1.$$

Note that $i = 1$ is of this type. Also note that we have actually bounded the integral in (4.6) independent of the $z_{i,j}$'s. This completes the proof that (4.5) is bounded.

Returning to (4.4), we see that given the bounds (3.16), the proof of Theorem 3 and the above method of bounding (4.5), there will be no problem in handling the variation in z and ε. We now discuss the variation in T.

Let $V = \{0 \leq r_i \leq s_i \leq t_i;\ T \leq t_i \leq T'\}$. In the third line of (4.5) we use Hölder's inequality in the time variables $dr\,ds\,dt$,

$$\int_V P_{\bar{T}}(X)\, dr\, ds\, dt \leq |V|^{1-1/\gamma}\left(\int_{\bar{t}_1 \leq \cdots \leq \bar{t}_{3m}} P_{\bar{T}}^{\gamma}(X)\right)^{1/\gamma}, \tag{4.7}$$

and we take $\gamma > 1$, but close to 1. Using (2.4a) we see that

$$p_t^{\gamma}(x, y) \leq M \frac{e^{-\alpha\gamma|x-y|^2/t}}{t^{\gamma}},$$

which leads as before to

$$\int_0^T p_t^{\gamma}(x, y)\, dt \leq M \frac{1}{|x - y|^{2(\gamma-1)}}$$

and exponential falloff in $|x - y|$ large. With γ close to 1, we can bound the integrals as before, and are left with the bound

$$M|V|^{1-1/\gamma} \leq M|T' - T|^{m\delta}.$$

This completes the proof of (4.4).

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